4.7 Suggested Problems

In Problems 1-4 find (a) $A+B$, (b) $A-3B$, (c) AB, (d) BA, (e) $A\mathbf{x}$, and (f) $B\mathbf{x}$ whenever the indicated combination is defined. If the combination is *not* defined, briefly explain why.

1. $A=\begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix}, \quad B=\begin{bmatrix} 4 & 7 \\ -5 & -1 \end{bmatrix}, \quad \mathbf{x}=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

2. $A=\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B=\begin{bmatrix} -2 & 4 & 3 \\ 6 & 1 & -5 \end{bmatrix}, \quad \mathbf{x}=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

3. $A=\begin{bmatrix} 1 & 4 & 0 \\ 2 & -4 & 3 \\ 1 & 5 & 2 \end{bmatrix}, \quad B=\begin{bmatrix} 1 & -1 & 3 \\ 1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix}, \quad \mathbf{x}=\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$

4. $A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B=\begin{bmatrix} -2 & 1 \\ 0 & 4 \\ 3 & -5 \end{bmatrix}, \quad \mathbf{x}=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

5. Verify $(AB)^T = B^T A^T$ for the matrices in Problem 4.

6. Verify that $(AB)\,C = A\,(BC)$ when

$$A=\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, \quad B=\begin{bmatrix} 2 & 3 \\ -3 & 4 \end{bmatrix}, \quad C=\begin{bmatrix} 1 & 0 \\ 5 & 2 \end{bmatrix}$$

7. Verify that $A\,(B+C) = AB + AC$ when

$$A=\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix}, \quad B=\begin{bmatrix} 2 & 3 & 5 \\ -3 & 4 & 1 \end{bmatrix}, \quad C=\begin{bmatrix} 1 & 0 & -3 \\ 5 & 2 & 1 \end{bmatrix}$$

8. Are the following assertions true or false? Explain.

$$\begin{bmatrix} 1 & 2 \\ -1 & 5 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 10 \\ 19 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 7 \\ 2 & -1 & 6 \\ 1 & 0 & 1 \end{bmatrix} - 2\begin{bmatrix} 1 & -2 & 0 \\ 3 & 2 & -1 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 7 & 7 \\ -4 & -5 & 8 \\ 5 & -2 & -5 \end{bmatrix}$$

9. Solve for x, y, z, and w:

$$\begin{bmatrix} x & -7 \\ z & w \end{bmatrix} - 2\begin{bmatrix} y & x \\ w & 4-z \end{bmatrix} = \begin{bmatrix} -1 & -9 \\ -5 & 2 \end{bmatrix}$$

In Problems 10-13, express each system of linear equations in the matrix form $A\mathbf{x} = \mathbf{b}$.

10. $\begin{cases} 3x - 2y = 8 \\ 4x + 5y = -10 \end{cases}$

11. $\begin{cases} 4x - 2y + z = 6 \\ -5x + 7y + 4z = 0 \\ 3x - y = 5 \end{cases}$

12. $\begin{cases} 4x - 2y + z - 3w = -3 \\ x + y - 4z + 2w = 6 \\ 2x + 3y - 5z - w = 4 \end{cases}$

13. $\begin{cases} 4x - 2y + z = 6 \\ -5x + 7y + 4z = 0 \\ 3x - y = 5 \\ x - 2y + z = 3 \end{cases}$

In Problems 14-17 determine if the matrix has an inverse. If it does, find the inverse. If it doesn't explain briefly why not.

14. $\begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

16. $\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

17. $\begin{bmatrix} 1 & -2 & 3 \\ 3 & 0 & -3 \\ -1 & 2 & 3 \end{bmatrix}$

18. Let
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
with $\det A \neq 0$. Use systematic elimination of unknowns to show that
$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Hint. First assume $a \neq 0$. Then do the case $a = 0$.

19. The inverse of a 3×3 matrix can be expressed in terms of cross products and scalar triple products: Let A be a 3×3 matrix and think of A as composed of three row vectors

$$A = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \mathbf{r}_3 \end{bmatrix}$$

Show that

$$A^{-1} = \frac{1}{\mathbf{r}_1 \cdot \mathbf{r}_2 \times \mathbf{r}_3} \begin{bmatrix} \mathbf{r}_2 \times \mathbf{r}_3 \\ \mathbf{r}_3 \times \mathbf{r}_1 \\ \mathbf{r}_1 \times \mathbf{r}_2 \end{bmatrix}^T$$

Hint. Let B be the matrix on the right. Calculate AB by expressing each entry of the matrix product as a scalar triple product. Then use what you know about the geometric interpretation of the scalar triple product.

20. Verify: If A and B are invertible matrices of the same size, then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$. *Hint.* Don't make this hard! Let $C = AB$. By definition C is invertible if there is a matrix D so that $CD = I = DC$. Show by a simple one line calculation that $D = B^{-1}A^{-1}$ has the required property.

21. Find a matrix A such that AA^T and A^TA are defined and $AA^T \neq A^TA$.