2.7 Suggested Problems

In Problems 1-6 use the algebraic properties DP1-DP5 whenever possible. Each problem has a short easy solution using these properties. No extensive calculation are needed nor is it helpful to express vectors in term of components.

1. Use dot product calculations to verify the identity

$$|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2\,|\mathbf{a}|^2 + 2\,|\mathbf{b}|^2\,.$$

   Then give a geometric interpretation of this result for vectors in $\mathbb{R}^2$.

2. Let $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ be mutually orthogonal unit vectors. If $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$ show that $c_1 = \mathbf{v}\cdot\mathbf{u}_1$, $c_2 = \mathbf{v}\cdot\mathbf{u}_2$, and $c_3 = \mathbf{v}\cdot\mathbf{u}_3$.

3. Let **a**, **b**, and **c** be vectors. Show, with the aid of the dot product, that:

   (a) $\mathbf{a} \perp \mathbf{b}$ and $\mathbf{a} \perp \mathbf{c}$ implies that $\mathbf{a} \perp \beta\mathbf{b}$ for any scalar β and $\mathbf{a} \perp (\mathbf{b}+\mathbf{c})$.

   (b) $\mathbf{a} \perp \mathbf{b}$ and $\mathbf{a} \perp \mathbf{c}$ implies that $\mathbf{a} \perp (\beta\mathbf{b}+\gamma\mathbf{c})$ for any scalars β and γ.

   (c) Interpret (b) geometrically for vectors in $\mathbb{R}^3$.

4. Let **a** and **b** be vectors.

   (a) Use the dot product to show that $\mathbf{a} \perp (\mathbf{b} - \text{proj}_{\mathbf{a}}\,\mathbf{b})$.

   (b) $\text{proj}_{\mathbf{a}}\,\mathbf{b}$ is a scalar multiple of which vector **a** or **b**? Explain briefly.

5. A rhombus is a parallelogram with four equal sides. Use dot products to establish the following.

   (a) The diagonals of a rhombus are perpendicular to each other.

   (b) If the diagonals of a parallelogram are perpendicular, then the parallelogram is a rhombus.

6. Let **a** be a nonzero vector and let **u** be a unit vector in the direction of **a**. For any vector **b** show:

   (a) $\text{comp}_{\mathbf{a}}\,\mathbf{b} = \mathbf{b}\cdot\mathbf{u}$;

   (b) $\text{proj}_{\mathbf{a}}\,\mathbf{b} = (\mathbf{b}\cdot\mathbf{u})\,\mathbf{u}$.

7. Use the Pythagorean theorem to verify the formulas for the length of a vector given in this lesson. *Hint.* In the 3-dimensional case use the figure for the position vector $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and append to the rectangle in the figure its diagonal extending from the origin to the point $(a_1, a_2, 0)$.

8. Derive the vector and scalar parametric equations for a line in 2-space.

9. You may remember from geometry the statement: "Two points determine a line." Find scalar parametric equations for the line determined by the two points $(-2, 0, 3)$ and $(5, 4, -1)$.

10. A line in space passes through the point $(1, -2, 3)$ and is parallel to the line through the two points $(-2, 0, 3)$ and $(5, 4, -1)$. Find scalar parametric equations for the line.

11. You may remember from geometry the statement: "Three points determine a plane." Find an equation for the plane determined by the three points $(-2, 0, 3)$, $(6, -8, 10)$ and $(5, 4, -1)$. *Hint.* This is easy if you know about cross products. It is still not hard if you don't. In that case, start by finding two vectors in the plane. A normal vector $\mathbf{N} = \langle a, b, c \rangle$ to the plane must be perpendicular to both. Take dot products with $\mathbf{N}$ and you are on your way to finding the components of a normal to the plane.

12. A plane in space contains the point $(5, 4, 1)$ and has normal direction parallel to the line through the points $(0, -2, 0)$ and $(11, 7, -5)$. Find an equation for the plane.