Problem 1 (30 points) True or False (3 points for correct answer, 1 point for blank):

1. A matrix in RREF has the same solution set as the original matrix.

2. If V is a vector space, then every basis of V has the same size.

3. If $\mathbf{v}, \mathbf{w}$ are in $\mathbb{R}^3$, then the Span$(\mathbf{v}, \mathbf{w})$ is a subspace of $\mathbb{R}^3$.

4. The dimension of a vector space is the number of elements in the vector space.

5. If the matrix product AB is defined then the number of rows of A is equal to the number of columns of B.

6. An eigenvalue can have more than one eigenvector.

7. The multiplicity of an eigenvalue cannot be greater than the dimension of the corresponding eigenspace.

8. The eigenspaces of two distinct eigenvalues are orthogonal to eachother.

9. If $T : \mathbb{R}^5 \to \mathbb{R}^4$ is a linear transformation, then T is not 1-1.

10. There exist $\mathbf{v_1}, \mathbf{v_2}$ and $\mathbf{v_3}$ in $\mathbb{R}^4$, which are linearly independent.

Problem 2 (20 pts)

Consider the following matrices:

$$A = \begin{pmatrix} 2 & 0 & -1 \\ 4 & 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 \\ -1 & 5 \\ 0 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -3 & 2 \\ -2 & 1 & 2 \\ 5 & 0 & 3 \end{pmatrix} \text{ and } D = \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}.$$

If possible, compute the following:

(i) BAD

(ii) DB

(iii) $C^TC + B^T$

(iv) CBA

Problem 3 (15 pts)

(i) Let W be a subspace of $\mathbb{R}^3$. Show that $W^{\perp} = \{v \in \mathbb{R}^3 |\ v \cdot w = 0 \text{ for every } w \in W\}$ is also a subspace.

(ii) Find the dimension of the suspace $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \middle|\ x_1 + x_2 = 0 \text{ and } x_1 + x_3 = 0 \right\}$ of $\mathbb{R}^3$.

Problem 4 (20 pts)

Let A be the following 3×3 matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -3 & 4 & -6 \\ -1 & 1 & -1 \end{pmatrix}.$$

Test A for diagonalizability. If A is diagonalizable, find a 3×3 invertible matrix B and a 3×3 diagonal matrix D such that $B^{-1}AB = D$.

Problem 5 (20 pts)

(i) Let L be the transformation from P_3 to P_3 defined by

$$L(f(x)) = f'(x) + f(1).$$

Show that it is L is linear.

(ii) Write the matrix of L with respect to the standard bases on P_3.

(iii) Write the matrix of L with respect to the basis $\{1 + x, 2 + x, 1 + x^2\}$

Problem 6 (10 pts)

Suppose

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 8.$$

Find the determinants of the following matrices:

(a) $\begin{pmatrix} 2a & g & d \\ 2b & h & e \\ 2c & i & f \end{pmatrix}$.

(b) $\begin{pmatrix} -a & -b & -c \\ d & e & f \\ 2g+d & 2h+e & 2i+f \end{pmatrix}$.

Problem 7 (15 pts)

Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 - x_3 \\ x_1 + 3x_2 - 4x_3 \end{pmatrix}$$

(i) Find the matrix of T

(ii) Does T have any eigenvectors? Why or why not?

(iii) Is T one-to-one? onto?

Problem 8 (10 pts)

(i) Determine whether the set of vectors $\{1+x+x^3, 2+x+x^2+2x^3, 2-x+2x^2\}$ in P_3 is linearly independent. Justify your answer.

(i) Does this set form a basis? Why or why not?

Problem 9 (10 pts)

Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ be an orthonormal basis for a vector space V. If $\mathbf{x} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$ is a vector orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$ with the properties that $||\mathbf{x}|| = 10$ and $\mathbf{x} \cdot \mathbf{u}_3 = 6$, then what are the possible values of c_1, c_2, c_3, and c_4?