Factorials occur frequently in probability theory.

EXAMPLE 7 Genetics

Suppose two black guinea pigs are bred and produce exactly five offspring. Under certain conditions, it can be shown that the probability P that exactly r of the offspring will be brown and the others black is a function of r, $P = P(r)$, where

$$P(r) = \frac{5!\left(\frac{1}{4}\right)^r\left(\frac{3}{4}\right)^{5-r}}{r!(5-r)!} \qquad r = 0, 1, 2, \ldots, 5$$

The letter P in $P = P(r)$ is used in two ways. On the right side, P represents the function rule. On the left side, P represents the dependent variable. The domain of P is all integers from 0 to 5, inclusive. Find the probability that exactly three guinea pigs will be brown.

Solution: We want to find $P(3)$. We have

$$P(3) = \frac{5!\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2}{3!2!} = \frac{120\left(\frac{1}{64}\right)\left(\frac{9}{16}\right)}{6(2)} = \frac{45}{512}$$

NOW WORK PROBLEM 35

Problems 2.2

In Problems 1–4, determine whether the given function is a polynomial function.

1. $f(x) = x^2 - x^4 + 4$

2. $f(x) = \frac{x^3 + 7x - 3}{3}$

*3. $g(x) = \frac{1}{x^2 + 2x + 1}$

4. $g(x) = 3^{-2}x^2$

In Problems 5–8, determine whether the given function is a rational function.

*5. $f(x) = \frac{x^2 + x}{x^3 + 4}$

6. $f(x) = \frac{3}{2x + 1}$

7. $g(x) = \begin{cases} 1 & \text{if } x < 5 \\ 4 & \text{if } x \geq 5 \end{cases}$

8. $g(x) = 4x^{-4}$

In Problems 9–12, find the domain of each function.

9. $h(z) = 19$

10. $f(x) = \sqrt{\pi}$

11. $f(x) = \begin{cases} 5x & \text{if } x > 1 \\ 4 & \text{if } x \leq 1 \end{cases}$

12. $f(x) = \begin{cases} 4 & \text{if } x = 3 \\ x^2 & \text{if } 1 \leq x < 3 \end{cases}$

In Problems 13–16, state (a) the degree and (b) the leading coefficient of the given polynomial function.

13. $F(x) = 7x^3 - 2x^2 +$ [illegible]

14. $g(x) = 7x$

15. $f(x) = \frac{1}{\pi} - 3x^5 +$ [illegible]

16. $f(x) = 9$

In Problems 17–22, find [illegible] *function values for each function.*

17. $f(x) = 8$; $f(2)$, $f($[illegible] $-\sqrt{17})$

18. $g(x) = |x - 3|$; $g(10)$, [illegible] $g(-3)$

*19. $F(t) = \begin{cases} 1 & \text{if } t > \text{[illegible]} \\ 0 & \text{if } t = \text{[illegible]} \\ -1 & \text{if } t < \text{[illegible]} \end{cases}$

$F(10)$, $F(-\sqrt{3})$, $F(0)$, [illegible] $\frac{18}{5})$

20. $f(x) = \begin{cases} 4 & \text{if } x \geq 0 \\ 3 & \text{if } x < 0 \end{cases}$

$f(3)$, $f(-4)$, $f(0)$

*21. $G(x) = \begin{cases} x - 1 & \text{if } x \geq 3 \\ 3 - x^2 & \text{if } x < 3 \end{cases}$;

$G(8)$, $G(3)$, $G(-1)$, $G(1)$

22. $F(\theta) = \begin{cases} 2\theta - 5 & \text{if } \theta < 2 \\ \theta^2 - 3\theta + 1 & \text{if } \theta > 2 \end{cases}$;

$F(3)$, $F(-3)$, $F(2)$

In Problems 23–28, determine the value of each expression.

23. $6!$

24. $0!$

25. $(4 - 2)!$

26. $6! \cdot 2!$

*27. $\frac{n!}{(n-1)!}$

28. $\frac{8!}{5!(8-5)!}$

29. **Train Trip** A daily round-trip train ticket to the city costs \$4.50. Write the cost of a daily round-trip ticket as a function of a passenger's income. What kind of function is this?

30. **Geometry** A rectangular prism has length three more than its width and height one less than twice the width. Write the volume of the rectangular prism as a function of the width. What kind of function is this?

31. **Cost Function** In manufacturing a component for a machine, the initial cost of a die is \$850 and all other additional costs are \$3 per unit produced. (a) Express the total cost C (in dollars) as a linear function of the number q of units produced. (b) How many units are produced if the total cost is \$1600?

32. **Investment** If a principal of P dollars is invested at a simple annual interest rate of r for t years, express the total accumulated amount of the principal and interest as a function of t. Is your result a linear function of t?

33. **Sales** To encourage large group sales, a theater charges two rates. If your group is less than 12, each ticket costs \$9.50. If your group is 12 or more, each ticket costs \$8.75. Write a case-defined function to represent the cost of buying n tickets.

34. **Factorials** The business mathematics class has elected a grievance committee of four to complain to the faculty

Problems 2.5

In Problems 1 and 2, locate and label each of the points, and give the quadrant, if possible, in which each point lies.

1. $(2, 7), (8, -3), (-\frac{1}{2}, -2), (0, 0)$

2. $(-4, 5), (3, 0), (1, 1), (0, -6)$

3. Figure 2.27(a) shows the graph of $y = f(x)$.
  (a) Estimate $f(0)$, $f(2)$, $f(4)$, and $f(-2)$.
  (b) What is the domain of f?
  (c) What is the range of f?
  (d) What is a real zero of f?

4. Figure 2.27(b) shows the graph of $y = f(x)$.
  (a) Estimate $f(0)$ and $f(2)$.
  (b) What is the domain of f?
  (c) What is the range of f?
  (d) What is a real zero of f?

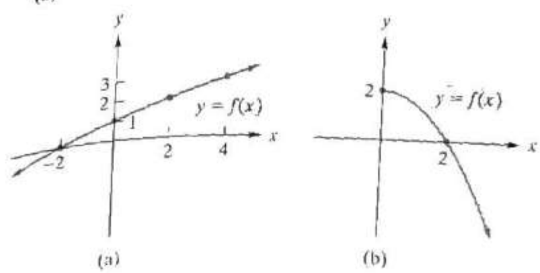

FIGURE 2.27 Diagram for Problems 3 and 4.

5. Figure 2.28(a) shows the graph of $y = f(x)$.
  (a) Estimate $f(0)$, $f(1)$, and $f(-1)$.
  (b) What is the domain of f?
  (c) What is the range of f?
  (d) What is a real zero of f?

6. Figure 2.28(b) shows the graph of $y = f(x)$.
  (a) Estimate $f(0)$, $f(2)$, $f(3)$, and $f(4)$.
  (b) What is the domain of f?
  (c) What is the range of f?
  (d) What is a real zero of f?

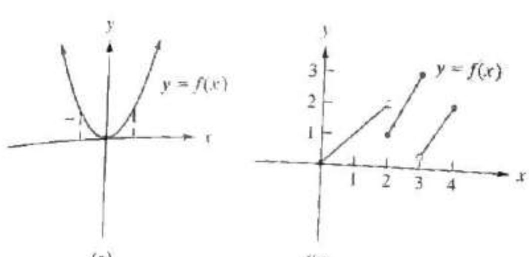

FIGURE 2.28 Diagram for Problems 5 and 6.

In Problems 7–20, determine the intercepts of the graph of each equation, and sketch the graph. Based on your graph, is y a function of x, and, if so, is it one-to-one and what are the domain and range?

7. $y = 2x$

8. $y = x + 1$

9. $y = 3x - 5$

10. $y = 3 - 2x$

11. $y = x^4$

12. $y = \frac{1}{x}$

13. $x = 0$

14. $y = 4x^2 - 16$

15. $y = x^3$

16. $x = -9$

17. $x = -|y|$

18. $x^2 = y^2$

19. $2x + y - 2 = 0$

20. $x + y = 1$

In Problems 21–34, graph each function and give the domain and range. Also, determine the intercepts.

21. $s = f(t) = 4 - t^2$

22. $f(x) = 5 - 2x^2$

23. $y = h(x) = 3$

24. $g(s) = -17$

25. $y = h(x) = x^2 - 4x + 1$

26. $y = f(x) = x^2 + 2x - 8$

27. $f(t) = -t^3$

28. $p = h(q) = 1 + 2q + q^2$

29. $s = f(t) = \sqrt{t^2 - 9}$

30. $F(r) = -\frac{1}{r}$

31. $f(x) = |2x - 1|$

32. $v = H(u) = |u - 3|$

33. $F(t) = \frac{16}{t^2}$

34. $y = f(x) = \frac{2}{x - 4}$

In Problems 35–38, graph each case-defined function and give the domain and range.

35. $c = g(p) = \begin{cases} p + 1 & \text{if } 0 \le p < 7 \\ 5 & \text{if } p \ge 7 \end{cases}$

36. $\phi(x) = \begin{cases} 3x + 2 & \text{if } -1 \le x < 3 \\ 20 - x^2 & \text{if } x \ge 3 \end{cases}$

37. $g(x) = \begin{cases} x + 6 & \text{if } x \ge 3 \\ x^2 & \text{if } x < 3 \end{cases}$

38. $f(x) = \begin{cases} x + 1 & \text{if } 0 < x \le 3 \\ 4 & \text{if } 3 < x \le 5 \\ x - 1 & \text{if } x > 5 \end{cases}$

39. Which of the graphs in Figure 2.29 represent functions of x?

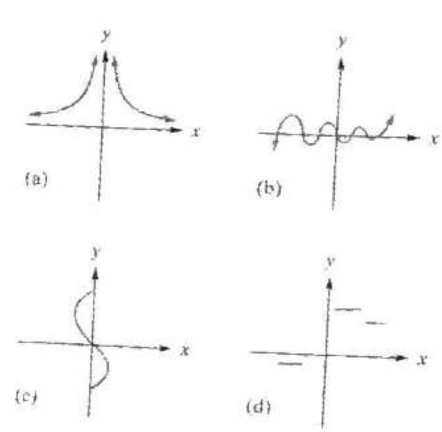

FIGURE 2.29 Diagram for Problem 39.

We have

$$r = pq$$
$$= (1000 - 2q)q$$
$$r = 1000q - 2q^2$$

Note that r is a quadratic function of q, with $a = -2$, [...]000, and $c = 0$. Si[...] $a < 0$ (the parabola opens downward), r is maximum a[t the v]ertex (q, r), where

$$q = -\frac{b}{2a} = -\frac{1000}{2(-2)} = 250$$

The maximum value of r is given by

$$r = 1000(250) - 2(250)^2$$
$$= 250{,}000 - 125{,}000 = 125{,}000$$

Thus, the maximum revenue that the manufacturer can receive is \$125,000, wh[ich] occurs at a production level of 250 units. Figure 3.25(a) shows the graph of the reve[nue] function. Only that portion for which $q \geq 0$ and $r \geq 0$ is drawn, since quantity [and] revenue cannot be negative.

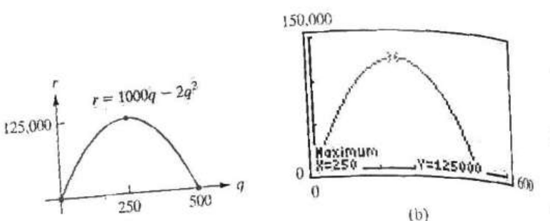

FIGURE 3.25 Graph of revenue function.

NOW WORK PROBLEM 29

TECHNOLOGY

The maximum (or minimum) value of a function can be conveniently found with a graphing calculator either by using trace and zoom or by using the "maximum" (or "minimum") feature. Figure 3.25(b) shows the display [for] the revenue function of Example 6, namely, the graph [of] $y = 1000x - 2x^2$. Note that we replaced r by y and q [by x].

Problems 3.3

In Problems 1–8, state whether the function is quadratic.

1. $f(x) = 5x^2$
2. $g(x) = \frac{1}{2x^2 - 4}$
3. $g(x) = 7 - 6x$
4. $k(v) = 3v^2(v^2 + 2)$
5. $h(q) = (3 - q)^2$
6. $f(t) = 2t(3 - t) + 4t$
7. $f(s) = \frac{s^2 - 9}{2}$
8. $g(t) = (t^2 - 1)^2$

In Problems 9–12, do not include a graph.

9. (a) For the parabola $y = f(x) = -4x^2 + 8x + 7$, find the vertex. (b) Does the vertex correspond to the highest point or the lowest point on the graph?
10. Repeat Problem 9 if $y = f(x) = 8x^2 + 4x - 1$.
11. For the parabola $y = f(x) = x^2 + x - 6$, find (a) the y-intercept, (b) the x-intercepts, and (c) the vertex.
12. Repeat Problem 11 if $y = f(x) = 5 - x - 3x^2$.

In Problems 13–22, graph each function. Give the vertex and intercepts, and state the range.

*13. $y = f(x) = x^2 - 6x + 5$
14. $y = f(x) = -4x^2$
*15. $y = g(x) = -2x^2 - 6x$
16. $y = f(x) = x^2 - 4$
*17. $s = h(t) = t^2 + 6t + 9$
18. $s = h(t) = 2t^2 + 3t$[...]
19. $y = f(x) = -9 + 8x - 2x^2$
20. $y = H(x) = 1 - x - x^2$
*21. $t = f(s) = s^2 - 8s + 14$
22. $t = f(s) = s^2 + 6s + 11$

In Problems 23–26, state whether $f(x)$ has a maximum value or a minimum value, and find that value.

23. $f(x) = 49x^2 - 10x + 17$
24. $f(x) = -3x^2 - 18x + 7$
25. $f(x) = 4x - 50 - 0.1x^2$
26. $f(x) = x(x + 3) - 12$

In Problems 27 and 28, restrict the quadratic function to those x satisfying $x \geq v$, where v is the x-coordinate of the vertex of the parabola. Determine the inverse of the restricted function. Graph the restricted function and its inverse in the same plane.

*27. $f(x) = x^2 - 2x + 4$
28. $f(x) = -x^2 + 4x - 3$

*29. **Revenue** The demand function for a manufacturer's product is $p = f(q) = 200 - 5q$, where p is the price (in dollars) per unit when q units are demanded (per day). Find the level of production that maximizes the manufacturer's total revenue and determine this revenue.

30. **Revenue** The demand function for an office supply company's line of plastic rulers is $p = 0.85 - 0.00045q$, where p is the price (in dollars) per unit when q units are demanded (per day) by consumers. Find the level of production that will maximize the manufacturer's total revenue, and determine this revenue.

31. **Revenue** The demand function for an electronics company's laptop computer line is $p = 2400 - 6q$, where p is the price (in dollars) per unit when q units are demanded (per week) by consumers. Find the level of production that will maximize the manufacturer's total revenue, and determine this revenue.

32. **Marketing** A marketing firm estimates that n months after the introduction of a client's new product, $f(n)$ thousand households will use it, where

$$f(n) = \frac{10}{9}n(12 - n), \quad 0 \leq n \leq 12$$

Estimate the maximum number of households that will use the product.

33. **Profit** The daily profit for the garden department of a store from the sale of trees is given by $P(x) = -x^2 + 18x + 144$, where x is the number of trees sold. Find the function's vertex and intercepts, and graph the function.

34. **Psychology** A prediction made by early psychology relating the magnitude of a stimulus, x, to the magnitude of a response, y, is expressed by the equation $y = kx^2$, where k is a constant of the experiment. In an experiment on pattern recognition, $k = 2$. Find the function's vertex and graph the equation. (Assume no restriction on x.)

35. **Biology** Biologists studied the nutritional effects on rats that were fed a diet containing 10% protein.[5] The protein consisted of yeast and corn flour. By varying the percentage, P, of yeast in the protein mix, the group estimated that the average weight gain (in grams) of a rat over a period of time was

$$f(P) = -\frac{1}{50}P^2 + 2P + 20, \quad 0 \leq P \leq 100$$

Find the maximum weight gain.

[5] Adapted from R. Bressani, "The Use of Yeast in Human Foods," in *Single-Cell Protein*, ed. R. I. Mateles and S. R. Tannenbaum (Cambridge, MA: MIT Press, 1968).

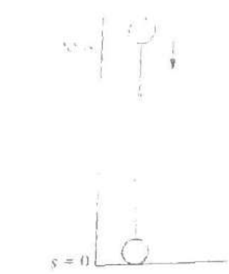

FIGURE 3.26 Ball thrown upward (Problem 36).

37. **Archery** A boy standing on a hill shoots an arrow straight up with an initial velocity of 85 feet per second. The height, h, of the arrow in feet, t seconds after it was released, is described by the function $h(t) = -16t^2 + 85t + 22$. What is the maximum height reached by the arrow? How many seconds after release does it take to reach this height?

38. **Toy Toss** A 6-year-old girl standing on a toy chest throws a doll straight up with an initial velocity of 16 feet per second. The height h of the doll in feet t seconds after it was released is described by the function $h(t) = -16t^2 + 16t + 4$. How long does it take the doll to reach its maximum height? What is the maximum height?

39. **Rocket Launch** A toy rocket is launched straight up from the roof of a garage with an initial velocity of 80 feet per second. The height, h, of the rocket in feet, t seconds after it was released, is described by the function $h(t) = -16t^2 + 80t + 16$. Find the function's vertex and intercepts, and graph the function.

40. **Area** Express the area of the rectangle shown in Figure 3.27 as a quadratic function of x. For what value of x will the area be a maximum?

FIGURE 3.27 Diagram for Problem 40.

41. **Enclosing Plot** A building contractor wants to fence in a rectangular plot adjacent to a straight highway using the highway for one side, which will be left unfenced. (See Figure 3.28.) If the contractor has 500 feet of fence, find the dimensions of the maximum enclosed area.

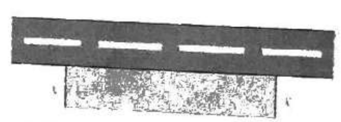

FIGURE 3.28 Diagram for Problem 41.

For example, this means that in 400 squares we would *expect* $400(0.072) \approx 29$ squares to contain exactly 4 cells. (In the experiment, in 400 squares the actual number observed was 30.)

Radioactive Decay

Radioactive elements are such that the amount of the element decreases with respect to time. We say that the element *decays*. It can be shown that, if N is the amount at time t, then

$$N = N_0 e^{-\lambda t} \tag{3}$$

where N_0 and λ (a Greek letter read "lambda") are positive constants. Notice that N involves an exponential function of t. We say that N follows an **exponential law of decay.** If $t = 0$, then $N = N_0 e^0 = N_0 \cdot 1 = N_0$. Thus, the constant N_0 represents the amount of the element present at time $t = 0$ and is called the **initial amount.** The constant λ depends on the particular element involved and is called the **decay constant.**

Because N decreases as time progresses, suppose we let T be the length of time it takes for the element to decrease to half of the initial amount. Then at time $t = T$, we have $N = N_0/2$. Equation (3) implies that

$$\frac{N_0}{2} = N_0 e^{-\lambda T}$$

We will now use this fact to show that over *any* time interval of length T, half of the amount of the element decays. Consider the interval from time t to $t + T$, which has length T. At time t, the amount of the element is $N_0 e^{-\lambda t}$, and at time $t + T$ it is

$$N_0 e^{-\lambda(t+T)} = N_0 e^{-\lambda t} e^{-\lambda T} = (N_0 e^{-\lambda T}) e^{-\lambda t}$$

$$= \frac{N_0}{2} e^{-\lambda t} = \frac{1}{2}(N_0 e^{-\lambda t})$$

which is half of the amount at time t. This means that if the initial amount present, N_0, were 1 gram, then at time T, $\frac{1}{2}$ gram would remain; at time $2T$, $\frac{1}{4}$ gram would remain; and so on. The value of T is called the **half-life** of the radioactive element. Figure 4.13 shows a graph of radioactive decay.

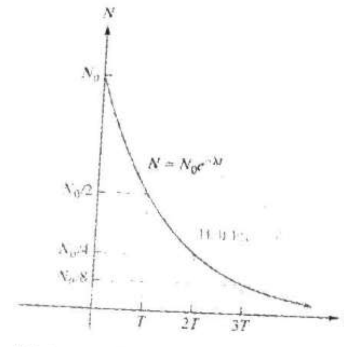

FIGURE 4.13 Radioactive decay.

● EXAMPLE 11 Radioactive Decay

A radioactive element decays such that after t days the number of milligrams present is given by

$$N = 100e^{-0.062t}$$

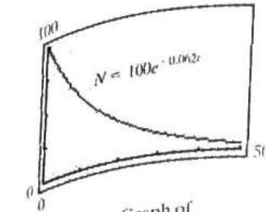

FIGURE 4.14 Graph of radioactive decay function $N = 100e^{-0.062t}$.

a. *How many milligrams are initially present?*

Solution: This equation has the form of Equation (3), $N = N_0 e^{-\lambda t}$, where $N_0 = 100$ and $\lambda = 0.062$. N_0 is the initial amount and corresponds to $t = 0$. Thus, 100 milligrams are initially present. (See Figure 4.14.)

b. *How many milligrams are present after 10 days?*

Solution: When $t = 10$,

$$N = 100e^{-0.062(10)} = 100e^{-0.62} \approx 53.8$$

Therefore, approximately 53.8 milligrams are present after 10 days.

NOW WORK PROBLEM 47 ●

Problems 4.1

In Problems 1–12, graph each function.

*1. $y = f(x) = 4^x$
2. $y = f(x) = 3^x$
*3. $y = f(x) = \left(\frac{1}{3}\right)^x$
4. $y = f(x) = \left(\frac{1}{8}\right)^x$
*5. $y = f(x) = 2^{(x-1)^2}$
6. $y = f(x) = 3(2)^x$
*7. $y = f(x) = 3^{x+2}$
8. $y = f(x) = 2^{x-1}$
9. $y = f(x) = 2^x - 1$
10. $y = f(x) = 3^{x-1} - 1$
11. $y = f(x) = 3^{-x}$
12. $y = f(x) = \frac{1}{2}(2^{x/2})$

Problems 13 and 14 refer to Figure 4.15, which shows the graphs of $y = 0.4^x$, $y = 2^x$, and $y = 5^x$.

13. Of the curves A, B, and C, which is the graph of $y = 5^x$?
14. Of the curves A, B, and C, which is the graph of $y = 0.4^x$?

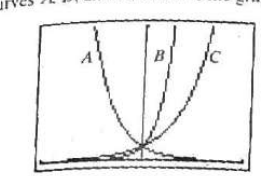

FIGURE 4.15 Diagram for Problems 13 and 14.

*15. **Population** The projected population of a city is given by $P = 125{,}000(1.11)^{t/20}$, where t is the number of years after 1995. What is the projected population in 2015?

16. **Population** For a certain city, the population P grows at the rate of 1.5% per year. The formula $P = 1{,}527{,}000(1.015)^t$ gives the population t years after 1998. Find the population in (a) 1999 and (b) 2000.

17. **Paired-Associate Learning** In a psychological experiment involving learning,[4] subjects were asked to give particular responses after being shown certain stimuli. Each stimulus was a pair of letters, and each response was either the digit 1 or 2. After each response the subject was told the correct answer. In this so-called *paired-associate* learning experiment, the theoretical probability P that a subject makes a correct response on the nth trial is given by

$$P = 1 - \frac{1}{2}(1 - c)^{n-1}, \quad n \geq 1,\ 0 < c < 1$$

where c is a constant. Take $c = \frac{1}{2}$ and find P when $n = 1$, $n = 2$, and $n = 3$.

18. Express $y = 2^{3x}$ as an exponential function in base 8.

In Problems 19–27, find (a) the compound amount and (b) the compound interest for the given investment and annual rate.

*19. \$4000 for 7 years at 6% compounded annually
20. \$5000 for 20 years at 5% compounded annually
21. \$700 for 15 years at 7% compounded semiannually
22. \$4000 for 12 years at $7\frac{1}{2}$% compounded semiannually
23. \$3000 for 16 years at $8\frac{3}{4}$% compounded quarterly
24. \$2000 for 12 years at 7% compounded quarterly
25. \$5000 for $2\frac{1}{2}$ years at 9% compounded monthly
26. \$500 for 5 years at 11% compounded semiannually
27. \$8000 for 3 years at $6\frac{1}{4}$% compounded daily. (Assume that there are 365 days in a year.)

28. **Investment** Suppose \$900 is placed in a savings account that earns interest at the rate of 4.5% compounded semiannually. (a) What is the value of the account at the end of five years? (b) If the account had earned interest at the rate of 4.5% compounded annually, what would be the value after five years?

29. **Investment** A certificate of deposit is purchased for \$6500 and is held for six years. If the certificate earns 4% compounded quarterly, what is it worth at the end of six years?

30. **Population Growth** The population of a town of 5000 grows at the rate of 3% per year. (a) Determine an equation that gives the population t years from now. (b) Find the population three years from now. Give your answer to (b) to the nearest integer.

*31. **Bacteria Growth** Bacteria are growing in a culture, and their number is increasing at the rate of 5% an hour. Initially, 400 bacteria are present. (a) Determine an equation that gives the number, N, of bacteria present after t hours. (b) How many bacteria are present after one hour? (c) After four hours? Give your answers to (b) and (c) to the nearest integer.

[4] D. Laming, *Mathematical Psychology* (New York: Academic Press, Inc., 1973).

Problems 4.2

In Problems 1–8, express each logarithmic form exponentially and each exponential form logarithmically.

1. $10^4 = 10{,}000$
2. $2 = \log_{12} 144$
3. $\log_2 64 = 6$
4. $8^{2/3} = 4$
5. $e^3 = 20.0855$
6. $e^{0.33647} = 1.4$
7. $\ln 3 = 1.09861$
8. $\log 5 = 0.6990$

In Problems 9–16, graph the functions.

9. $y = f(x) = \log_3 x$
10. $y = f(x) = \log_4 2x$
11. $y = f(x) = \log_{1/4} x$
12. $y = f(x) = \log_{1/5} x$
13. $y = f(x) = \log_2 (x - 4)$
14. $y = f(x) = \log_2 (-x)$
15. $y = f(x) = -2 \ln x$
16. $y = f(x) = \ln(x + 2)$

In Problems 17–28, evaluate the expression.

17. $\log_6 36$
18. $\log_2 64$
19. $\log_3 27$
20. $\log_{16} 4$
21. $\log_7 7$
22. $\log 10{,}000$
23. $\log 0.01$
24. $\log_2 \sqrt[5]{2}$
25. $\log_5 1$
26. $\log_5 \frac{1}{25}$
27. $\log_2 \frac{1}{8}$
28. $\log_4 \sqrt[5]{4}$

In Problems 29–48, find x.

29. $\log_3 x = 4$
30. $\log_2 x = 8$
31. $\log_5 x = 3$
32. $\log_4 x = 0$
33. $\log x = -1$
34. $\ln x = 1$
35. $\ln x = -3$
36. $\log_x 25 = 2$
37. $\log_x 8 = 3$
38. $\log_x 3 = \frac{1}{2}$
39. $\log_x \frac{1}{6} = -1$
40. $\log_x y = 1$
41. $\log_3 x = -3$
42. $\log_x (2x - 3) = 1$
43. $\log_x (6 - x) = 2$
44. $\log_8 64 = x - 1$
45. $2 + \log_2 4 = 3x - 1$
46. $\log_3(x + 2) = -2$
47. $\log_x(2x + 8) = 2$
48. $\log_x(6 + 4x - x^2) = 2$

In Problems 49–52, find x and express your answer in terms of natural logarithms.

49. $e^{3x} = 2$
50. $0.1e^{0.1x} = 0.5$
51. $e^{2x-5} + 1 = 4$
52. $6e^{2x} - 1 = \frac{1}{2}$

In Problems 53–56, use your calculator to find the approximate value of each expression. Round your answer to five decimal places.

53. $\ln 5$
54. $\ln 4.27$
55. $\ln 7.39$
56. $\ln 9.98$

57. **Appreciation** Suppose an antique gains 10% in value every year. Graph the number of years it is owned as a function of the multiplicative increase in its original value. Label the graph with the name of the function.

58. **Cost Equation** The cost for a firm producing q units of a product is given by the cost equation

$$c = (3q \ln q) + 12$$

Evaluate the cost when $q = 6$. (Round your answer to two decimal places.)

59. **Supply Equation** A manufacturer's supply equation is

$$p = \log\left(10 + \frac{q}{2}\right)$$

where q is the number of units sup[...]d at a price p per [...]. At what price will the manufactur[...] supply 1980 units?

60. **Earthquake** The magnitude, M, [...]n earthquake and [...] energy, E, are related by the equ[...][5]

$$1.5M = \log\left(\frac{E}{2.5 \times 10^{11}}\right)$$

where M is given in terms of Ri[...] preferred scale of 1958 and E is in ergs. Solve the [...]on for E.

61. **Biology** For a certain populati[...] cells, the number of cells at time t is given by $N = N_0(\ldots)$, where N_0 is the number of cells at $t = 0$ and k is [...]sitive constant. (a) F[...] N when $t = k$. (b) What is the si[...]cance of k? (c) Show that the time it takes to have po[...]tion N_1 can be written

$$t = k \log \frac{N_1}{N_0}$$

62. **Inferior Good** In a discussion of an inferior good, Pers[...][6] solves an equation of the form

$$u_0 = A \ln(x_1) + \frac{x_2^2}{2}$$

for x_1, where x_1 and x_2 are quantities of two products, [...] measure of utility, and A is a positive constant. Determine [...]

63. **Radioactive Decay** A 1-gram sample of radioactive lead 211 (^{211}Pb) decays according to the equation $N = e^{-0.0\ldots}$, where N is the number of grams present after t minutes. Find the half-life of ^{211}Pb to the nearest tenth of a minute.

64. **Radioactive Decay** A 100-milligram sample of radioactive actinium 227 (^{227}Ac) decays according to the equation

$$N = 100e^{-0.03194t}$$

where N is the number of milligrams present after t years. Find the half-life of ^{227}Ac to the nearest tenth of a year.

65. If $\log_y x = 3$ and $\log_z x = 2$, find a formula for z as an explicit function of y only.

66. Solve for y as an explicit function of x if

$$x + 3e^{2y} - 8 = 0$$

67. Suppose $y = f(x) = x \ln x$. (a) For what values of x is y [...] (*Hint:* Determine when the graph is below the x-axis.) (b) Determine the range of f.

68. Find the x-intercept of $y = x^2 \ln x$.

69. Use the graph of $y = e^x$ to estimate $\ln 3$. Round your ans[...] to two decimal places.

70. Use the graph of $y = \ln x$ to estimate e^2. Round your an[...] to two decimal places.

71. Determine the x-values of points of intersection of the graphs of $y = (x - 2)^2$ and $y = \ln x$. Round your answ[...] two decimal places.

[5] K. E. Bullen, *An Introduction to the Theory of Seismology* (Camb[...] U.K.: Cambridge at the University Press, 1963).

[6] A. L. Persky, "An Inferior Good and a Novel Indifference Map," *American Economist*, XXIX, no. 1 (Spring 1985).

OBJECTIVE

To study basic pr[...] [...]garithmic functions.

CAU[...]N

Make sure th[...] [...]rly understand [...] [...]1–3. They do not apply [...] [...] of a sum $[\log_b(m+n)]$, [...] [...] of a difference [...] [...] to the product of two logs [...] $[(\log_b m)(\log_b n)]$, or the quotient of two logs $\left[\frac{\log_b m}{\log_b n}\right]$.

4.3 Properties of Logarithms

The logarithmic function has many important properties. For example,

1. $\log_b(mn) = \log_b m + \log_b n$

which says that the logarithm of a product of two numbers is the sum of the logarithms of the numbers. We can prove this property by deriving the exponential form of the equation:

$$b^{\log_b m + \log_b n} = mn$$

Using first a familiar rule for exponents, we have

$$b^{\log_b m + \log_b n} = b^{\log_b m} b^{\log_b n}$$
$$= mn$$

where the second equality uses two instances of the fundamental equation (2) of Section 4.2. We will not prove the next two properties, since their proofs are similar to that of Property 1.

2. $\log_b \dfrac{m}{n} = \log_b m - \log_b n$

That is, the logarithm of a quotient is the difference of the logarithm of the numerator and the logarithm of the denominator.

3. $\log_b m^r = r \log_b m$

That is, the logarithm of a power of a number is the exponent times the logarithm of the number.

TABLE 4.4 Common Logarithms

x	$\log x$	x	$\log x$
2	0.3010	7	0.8451
3	0.4771	8	0.9031
4	0.6021	9	0.9542
5	0.6990	10	1.0000
6	0.7782	e	0.4343

Table 4.4 gives the values of a few common logarithms. Most entries are approximate. For example, $\log 4 \approx 0.6021$, which means $10^{0.6021} \approx 4$. To illustrate the use of properties of logarithms, we shall use this table in some of the examples that follow.

Although the logarithms in Example 1 can be found with a calculator, we will make use of properties of logarithms.

EXAMPLE 1 Finding Logarithms by Using Table 4.4

a. *Find* $\log 56$.

Solution: Log 56 is not in the table. But we can write 56 as the product $8 \cdot 7$. Thus, by Property 1,

$$\log 56 = \log(8 \cdot 7) = \log 8 + \log 7 \approx 0.9031 + 0.8451 = 1.7482$$

b. *Find* $\log \frac{9}{2}$.

Solution: By Property 2,

$$\log \frac{9}{2} = \log 9 - \log 2 \approx 0.9542 - 0.3010 = 0.6532$$

c. *Find* $\log 64$.

Solution: Since $64 = 8^2$, by Property 3,

$$\log 64 = \log 8^2 = 2 \log 8 \approx 2(0.9031) = 1.8062$$

d. *Find* $\log \sqrt{5}$.

Solution: By Property 3, we have

$$\log \sqrt{5} = \log 5^{1/2} = \frac{1}{2} \log 5 \approx \frac{1}{2}(0.6990) = 0.3495$$

In exponential form, we have

$$x(x+4) = 2^5$$
$$x^2 + 4x = 32$$
$$x^2 + 4x - 32 = 0 \quad \text{...atic equation)}$$
$$(x-4)(x+8) = 0$$
$$x = 4 \quad \text{or} \quad x = -8$$

Because we must have $x > 0$, the only sol... is 4, as can be verified by sub... into the original equation. Indeed, replacin... by 4 in $\log_2 x$ gives $\log_2 4 = \log$... while replacing x by 4 in $5 - \log_2(x+4)$ gives $5 - \log_2(4+4) = 5 - \log$... $5 - \log_2 2^3 = 5 - 3 = 2$. Since the results are the same, 4 is a solution of the e...

In solving a logarithmic equation, it is a good idea to check for ex... solutions.

NOW WORK PROBLE...

Problems 4.4

In Problems 1–36, find x. Round your answers to three decimal places.

*1. $\log(3x+2) = \log(2x+5)$
2. $\log x - \log 5 = \log 7$
3. $\log 7 - \log(x-1) = \log 4$
4. $\log_2 x + 3\log_2 2 = \log_2 \frac{2}{x}$
*5. $\ln(-x) = \ln(x^2 - 6)$
6. $\ln(4-x) + \ln 2 = 2\ln x$
*7. $e^{2x} \cdot e^{5x} = e^{14}$
8. $(e^{3x-2})^3 = e^3$
9. $(81)^{4x} = 9$
10. $(27)^{2x+1} = \frac{1}{3}$
11. $e^{2x} = 9$
12. $e^{4x} = \frac{3}{4}$
*13. $2e^{5x+2} = 17$
14. $5e^{2x-1} - 2 = 23$
15. $10^{4/x} = 6$
16. $\frac{4(10)^{0.2x}}{5} = 3$
17. $\frac{5}{10^{2x}} = 7$
18. $2(10)^x + (10)^{x+1} = 4$
19. $2^x = 5$
20. $7^{2x+3} = 9$
21. $7^{3x-2} = 5$
22. $4^{x/2} = 20$
23. $2^{-2x/3} = \frac{4}{5}$
24. $5(3^x - 6) = 10$
25. $(4)5^{3-x} - 7 = 2$
26. $\frac{7}{3^x} = 13$
27. $\log(x-3) = 3$
28. $\log_2(x+1) = 4$
29. $\log_4(9x-4) = 2$
30. $\log_4(2x+4) - 3 = \log_4 3$
31. $\log(3x-1) - \log(x-3) = 2$
32. $\log(x-3) + \log(x-5) = 1$
33. $\log_2(5x+1) = 4 - \log_2(3x-2)$
34. $\log(x+2)^2 = 2$, where $x > 0$
35. $\log_2\left(\frac{2}{x}\right) = 3 + \log_2 x$
36. $\ln(x-2) = \ln(2x-1) + 3$

37. **Rooted Plants** In a study of rooted plants in a certain geographic region,[11] it was determined that on plots of size A (in square meters), the average number of species that occurred was S. When $\log S$ was graphed as a function of $\log A$, the result was a straight line given by

$$\log S = \log 12.4 + 0.26 \log A$$

Solve for S.

38. **Gross National Product** In an article, Taagepera and Hayes refer to an equation of the form

$$\log T = 1.7 + 0.2068 \log P - 0.1334(\log P)^2$$

Here T is the percentage of a country's gross national product (GNP) that corresponds to foreign trade (exports plus imports), and P is the country's population (in units of 100,000).[12] Verify the claim that

$$T = 50P^{(0.2068 - 0.1334 \log P)}$$

You may assume that $\log 50 = 1.7$. Also verify that, for any base b, $(\log_b x)^2 = \log_b(x^{\log_b x})$.

39. **Radioactivity** The number of milligrams of a radioactive substance present after t years is given by

$$Q = 100e^{-0.035t}$$

(a) How many milligrams are present after 0 years?
(b) After how many years will there be 20 milligrams present? Give your answer to the nearest year.

40. **Blood Sample** On the surface of a glass slide is a grid that divides the surface into 225 equal squares. Suppose a blood sample containing N red cells is spread on the slide and the cells are randomly distributed. Then the number of squares containing no cells is (approximately) given by ... If 100 of the squares contain no cells, estimate the number of cells the blood sample contained.

41. **Population** In one city, the population P grows at the rate of 2% per year. The equation $P = 1{,}000{,}000(\ldots)$ gives the population t years after 1998. Find the value of t for which the population is 1,500,000. Give your answer to the nearest tenth.

42. **Market Penetration** In a discussion of market penetration by new products, Hurter and Rubenstein[13] refer to the function

$$F(t) = \frac{q - pe^{-(t+C)(p+q)}}{q[1 + e^{-(t+C)(p+q)}]}$$

where p, q, and C are constants. They claim that if $F(0) = 0$, then

$$C = -\frac{1}{p+q}\ln\frac{q}{p}$$

Show that their claim is true.

43. **Demand Equation** The demand equation for a consumer product is $q = 80 - 2^p$. Solve for p and express your answer in terms of common logarithms, as in Example 4. Evaluate p to two decimal places when $q = 60$.

44. **Investment** The equation $A = P(1.105)^t$ gives the value A at the end of t years of an investment of P dollars compounded annually at an annual interest rate of 10.5%. How many years will it take for an investment to double? Give your answer to the nearest year.

45. **Sales** After t years the number of units of a product sold per year is given by $q = 1000\left(\frac{1}{2}\right)^{0.8^t}$. Such an equation is called a *Gompertz equation* and describes natural growth in many areas of study. Solve this equation for t in the same manner as in Example 4, and show that

$$t = \frac{\log\left(\dfrac{3 - \log q}{\log 2}\right)}{\log 0.8}$$

Also, for any A and suitable b and a, solve $y = Ab^{a^x}$ for x and explain how the previous solution is a special case.

46. **Learning Equation** Suppose that the daily output of units of a new product on the tth day of a production run is given by

$$q = 500(1 - e^{-0.2t})$$

Such an equation is called a *learning equation* and indicates that as time progresses, output per day will increase. This may be due to a gain in a worker's proficiency at his or her job. Determine, to the nearest complete unit, the output on (a) the first day and (b) the tenth day after the start of a production run. (c) After how many days will a daily production run of 400 units be reached? Give your answer to the nearest day.

47. Verify that 4 is the only solution to the logarithmic equation in Example 6 by graphing the function

$$y = 5 - \log_2(x+4) - \log_2 x$$

and observing when $y = 0$.

48. Solve $2^{3x+0.5} = 17$. Round your answer to two decimal places.

49. Solve $\ln(x+2) = 5 - x$. Round your answer to two decimal places.

50. Graph the equation $(3)2^y - 4x = 5$. (*Hint:* Solve for y as a function of x.)

[11] ...tion to Quantitative Ecology (New York: ...

[12] R. Taagepera and J. P. Hayes, "How Trade/GNP Ratio Decreases with Country Size," *Social Science Research*, 6 (1977), 108–...

[13] A. P. Hurter, Jr., A. H. Rubenstein, et al., "Market Penetration by New Innovations: The Technological Literature," *Technological Forecasting and Social Change*, 11 (1978), 197–221.

4.5 Review

Important Terms and Symbols

		Examples
Section 4.1	**Exponential Functions**	
	exponential function, b^x, for $b > 1$ and for $0 < b < 1$	Ex. 2,3, p. 164,165
	compound interest principal compound amount	Ex. 6, p. 167
	interest period periodic rate nominal rate	
	e natural exponential function, e^x	Ex. 8, p. 170
	exponential law of decay initial amount decay constant half-life	Ex. 11, p. 172
Section 4.2	**Logarithmic Functions**	
	logarithmic function, $\log_b x$ common logarithm, $\log x$	Ex. 5, p. 178
	natural logarithm, $\ln x$	Ex. 5, p. 178
Section 4.3	**Properties of Logarithms**	
	change-of-base formula	Ex. 8, p. 185
Section 4.4	**Logarithmic and Exponential Equations**	
	logarithmic equation exponential equation	Ex. 1, p. 187

Summary

An exponential function has the form $f(x) = b^x$. The graph of $f(x) = b^x$ has one of two general shapes, depending on the value of the base b. (See Figure 4.3.) An exponential function is involved in the compound interest formula

$$S = P(1+r)^n$$

where S is the compound amount of a principal of P at the end of n interest periods at the periodic rate r.

A frequently used base in an exponential function is the irrational number $e \approx 2.71828$. This base occurs in economic analysis and many situations involving growth or decay, such as population studies and radioactive decay. Radioactive elements follow the exponential law of decay,

$$N = N_0 e^{-\lambda t}$$