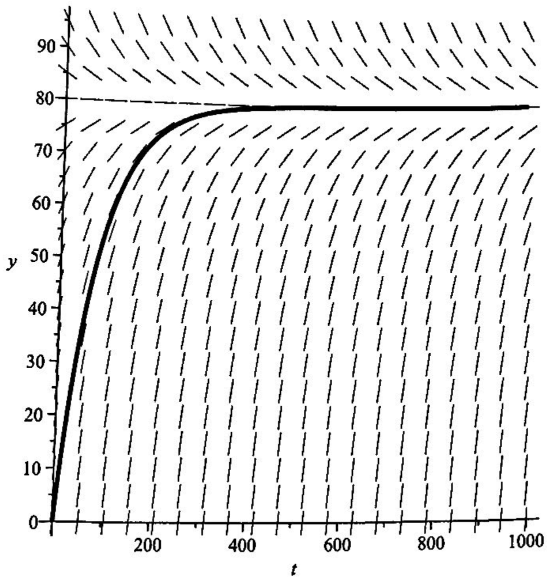


Figure 1.13: Slope field for $y' = 1 - y/80$

The problem in this example is called a *mixing problem.* Generalizing this example, we see that the DE describing $y(t)$ has the form

$$\frac{dy}{dt} = \begin{pmatrix} \text{Rate of sol} \\ \text{flowing in} \end{pmatrix} \cdot \begin{pmatrix} \text{Conc of sol} \\ \text{flowing in} \end{pmatrix} - \begin{pmatrix} \text{Rate of sol} \\ \text{flowing out} \end{pmatrix} \cdot \left(\frac{y}{\text{Vol of sol in tub}} \right).$$

We will use this model in later sections.

Exercises

1.3.1 A decaying material, such as a radioactive material, is said to be in *exponential decay* if the rate at which the amount of material decreases is proportional to the amount of remaining material.

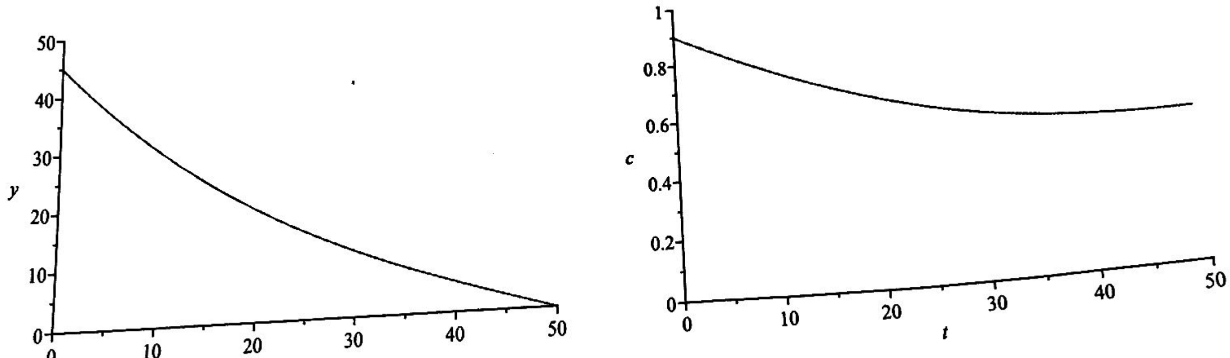
a. Let $y(t)$ represent the amount of material remaining at time t. Derive a DE to model y. Use $k > 0$ as the constant of proportionality.

b. Verify that a general solution to the DE in part a. is $y = Ce^{-kt}$ and show that $C = y(0)$.

c. Suppose y is measured in grams (g) and t is measured in seconds (sec). Use the balance of units principle to find the units of k.

d. The *half-life*, $t_{\text{half-life}}$, of a material in exponential decay is the time required for y to be reduced to half of the original quantity. That is $t_{\text{half-life}}$ is the time at which $y(t) = 0.5y(0)$. Use the

Graphs of $y(t)$ and $c(t)$ are shown in Figure 2.3. Note that as $t \to 50$, $y(t) \to 0$ and $c(t) \to 0.5$. This should be expected because the total volume of solution in the tank is approaching 0, so the volume of alcohol in the tank is also approaching 0. Also, the remaining solution in the tank is being replaced by the 50% solution being added.



(a) Volume

(b) Concentration

Figure 2.3: Volume and Concentration of Alcohol

□

Exercises

2.1.1 Find a general solution to the basic population model $y' = ky$.

For each equation below state whether it is linear or not.

1. $x' = t\sin(x)$
2. $x' = t + \sin(x)$
3. $x' = x\sin(t)$
4. $x' = x + \sin(t)$
5. $y' + ty^2 = t$
6. $y' + x^2y = x^2$

Solve each of the following linear equations.

7. $x' + 2x = e^{-2t}\cos(t)$
8. $x' = x + 1$
9. $tx' + 2x = 1 + t$

Exercises

Determine whether or not each equation is separable.

2.3.1 $x' + 2x = e^{-t}$

2.3.2 $x' + 2x = 1$

2.3.3 $x' = \dfrac{x+1}{t+1}$

2.3.4 $x' = \frac{\sin t}{\cos x}$

Solve each below by the method of separation of variables:

2.3.5 $x' = \dfrac{x}{t}$

2.3.6 $x' = \dfrac{t}{x}$

2.3.7 $x' = x + 5$

2.3.8 $x' = 3x - 2$

2.3.9 $x' = x\cos(t)$

2.3.10 $x' = (1+t)(2+x)$

Solve each of the following initial-value problems:

2.3.11 $y' = y + 1$, $y(0) = 2$

2.3.12 $y' = ty$, $y(0) = 3$

2.3.13 $x' = x\cos(t)$, $x(0) = 1$

2.3.14 $x' = (1+t)(2+x)$, $x(0) = -1$

2.3.15 $P' = 2P(1-P)$, $P(0) = 1/2$

2.3.16 (Population growth) A population P is growing according to the growth law $\frac{dP}{dt} = rP$. Time t is measured in years. If the population is initially 100, and after 1 year the population is 150, how many will there be after 2 years? Hint: solve the differential equation for P as a function of t. Then use the two conditions $P(0) = 100$ and $P(1) = 150$ to determine r and the integration constant. What happens to the population in the long term?

2.3.17 (Population growth) Repeat the previous problem, but this time assume the growth law $\frac{dP}{dt} = rP(1 - P/300)$. What happens to the population in the long term?

calculate using separation of variables, or computer algebra) and give the amount of error for each method.

2.4.2 For the IVP $x' = -x + e^t$, $x(0) = 2$, estimate the value of $x(1)$ by making a table of values using RK2 with 4 steps and step size $\Delta t = 0.25$ (do the work without using built-in computer or calculator methods, as in example 2.4.2). Repeat with RK4. Compare to the exact solution (which you can calculate using the integrating factor method, or computer algebra) and give the amount of error for each method.

For each I.V.P. 3 - 6, estimate the value of the dependent variable accurate to three significant digits at the point where the independent variable is equal to 5. First use Euler's method, then repeat using either fourth-order Runga-Kutta or an adaptive step size RK method (as on the TI-89 calculator). Use a computer or calculator built-in method, or alter the first-order applet at .

2.4.3 $y' = -y + \sin(t)$, $y(0) = 1$.

2.4.4 $x' = x^2 - t$, $x(0) = 0$.

2.4.5 $y' = -0.1xy$, $y(0) = 4$.

2.4.6 $p' = p(1-p) + 0.5\cos(t)$, $p(0) = 0$.

Create accurate phase portraits of each differential equation 7 - 10, and include in the phase portrait the solution curve corresponding to the initial condition given. They are the same as the previous four problems, and you can use the same technology you used there.

2.4.7 $y' = -y + \sin(t)$, $y(0) = 1$.

2.4.8 $x' = x^2 - t$, $x(0) = 0$.

2.4.9 $y' = -0.1xy$, $y(0) = 4$.

2.4.10 $p' = p(1-p) + 0.5\cos(t)$, $p(0) = 0$.

2.5 Autonomous first-order equations and bifurcations

Differential equations always involve a dependent and at least one independent variable. In the DE