Problem 1 (20 pts)

(a) Which of the following matrices are in the reduced row echelon form?

$$\text{(i)} \quad \begin{pmatrix} 1 & 3 & 1 & 4 & -3 \\ 0 & 0 & 3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

$$\text{(ii)} \quad \begin{pmatrix} 0 & & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\text{(iii)} \quad \begin{pmatrix} 1 & 3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) In each of the following, the augmented matrix is in row echelon form. For each case, indicate whether the corresponding system of linear equations is consistent. If the system is consistent, find the solution set.

$$\text{(i)} \quad \left(\begin{array}{cc|c} 4 & -2 & 1 \\ 0 & 5 & -3 \\ 0 & 0 & 2 \end{array}\right).$$

$$\text{(ii)} \quad \left(\begin{array}{ccc|c} 1 & -2 & 4 & 2 \\ 0 & -1 & 2 & -7 \\ 0 & 0 & 2 & -6 \end{array}\right).$$

Problem 2 (20 pts)

(a) Describe the solution set of the following linear system:

$$\begin{cases} x_1 + 3x_2 + 5x_3 - x_4 = -4 \\ 2x_1 + x_3 + x_4 = 4 \\ -x_1 + x_2 + 2x_3 + 3x_4 = 4 \end{cases}$$

(b) Describe the solution set of the associated homogeneous system.

(c) Is the corresponding linear transformation one-to-one?

(d) Is the corresponding linear transformation onto?
(Recall that the corresponding linear transformation is the function $T(\mathbf{x}) = A\mathbf{x}$ where A is the coefficient matrix of the system.)

Problem 3 (20 pts)

(a) Determine whether the following vectors are linearly independent. Justify each answer.

(i) $\begin{pmatrix} -3 \\ -2 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$

(ii) $\begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix}.$

(iii) $\begin{pmatrix} 1 \\ 5 \\ -2 \\ 3 \end{pmatrix}, \begin{pmatrix} -7 \\ 3 \\ 7 \\ 6 \end{pmatrix}.$

(b) Consider $\mathbf{u} = \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 4 \\ -3 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 1 \\ a \end{pmatrix}$. For what value of a will $\mathbf{w}$ lie in $\text{Span}(\mathbf{u}, \mathbf{v})$?

Problem 4 (20 pts)

Let $A = \begin{pmatrix} 2 & -1 \\ -3 & 2 \\ 5 & -1 \end{pmatrix}$ and let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(\mathbf{x}) = A\mathbf{x}$$

for every $\mathbf{x} \in \mathbb{R}^3$.

(a) Compute $T\begin{pmatrix} 2 \\ -3 \end{pmatrix}$.

(b) Determine whether T is one-to-one and whether T is onto. Justify your answer.

Problem 5 (20 pts)

(a) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the transformation defined by

$$T\begin{pmatrix} x_1 \\ x_2+3 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 \\ 3x_3 + 2x_2 \end{pmatrix}.$$

Determine whether T is linear. Justify your answer.

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2x_1 - x_2 \\ x_2 + 4x_3 \\ 2x_1 + x_3 \end{pmatrix}.$$

Find the 3×3 matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^3$.

Bonus Problem (9 pts)

(a) Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^m$ be linearly independent. Prove that, for a nonzero $c \in \mathbb{R}$, $\mathbf{u}+\mathbf{v}$ and $c\mathbf{u}$ are linearly independent.

(b) Let $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$ and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T(\mathbf{u}) = \begin{pmatrix} 5 \\ 1 \end{pmatrix} \text{ and } T(\mathbf{v}) = \begin{pmatrix} 5 \\ -2 \end{pmatrix}.$$

Find $T(\mathbf{w})$. Determine the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

(c) Let A be a 3×4 matrix, $\mathbf{b}$ be a vector in $\mathbb{R}^3$. Suppose that $\begin{pmatrix} 1 \\ -2 \\ 4 \\ 7 \end{pmatrix}$ is a solution to the equation $A\mathbf{x} = \mathbf{b}$ and $\begin{pmatrix} 1 \\ 0 \\ 2 \\ -3 \end{pmatrix}$ is a solution to $A\mathbf{x} = \mathbf{0}$. Find another solution to $A\mathbf{x} = \mathbf{b}$.