Lecture Notes III

Outline

1) Unconstrained optimization: one choice variable

   [illegible]

   [illegible]

   F.O.C

   first derivative test

   S.O.C

   N^{th} derivative test

   ~~Examples~~

2) Unconstrained optimization: $\underbrace{\text{Multiple}}_{\text{two or more}}$ choice variables

2 variables: F.O.C

   S.O.C: first ~~a~~ method

   S.O.C Secon method (using the Hessian)

3 variables F.O.C

   S.O.C

~~Comparative Statics~~:

3) Examples

4) Comparative Statics

Unconstrained optimization: One Choice Variable

$$\underset{x}{\text{optimize}} \quad y = f(x)$$

- choose x that optimizes y
- optimization refers to either maximization or minimization
  - the optimal (most desirable) outcome
    - might be maximizing profits for instance, or
    - minimizing costs
- another example would be to

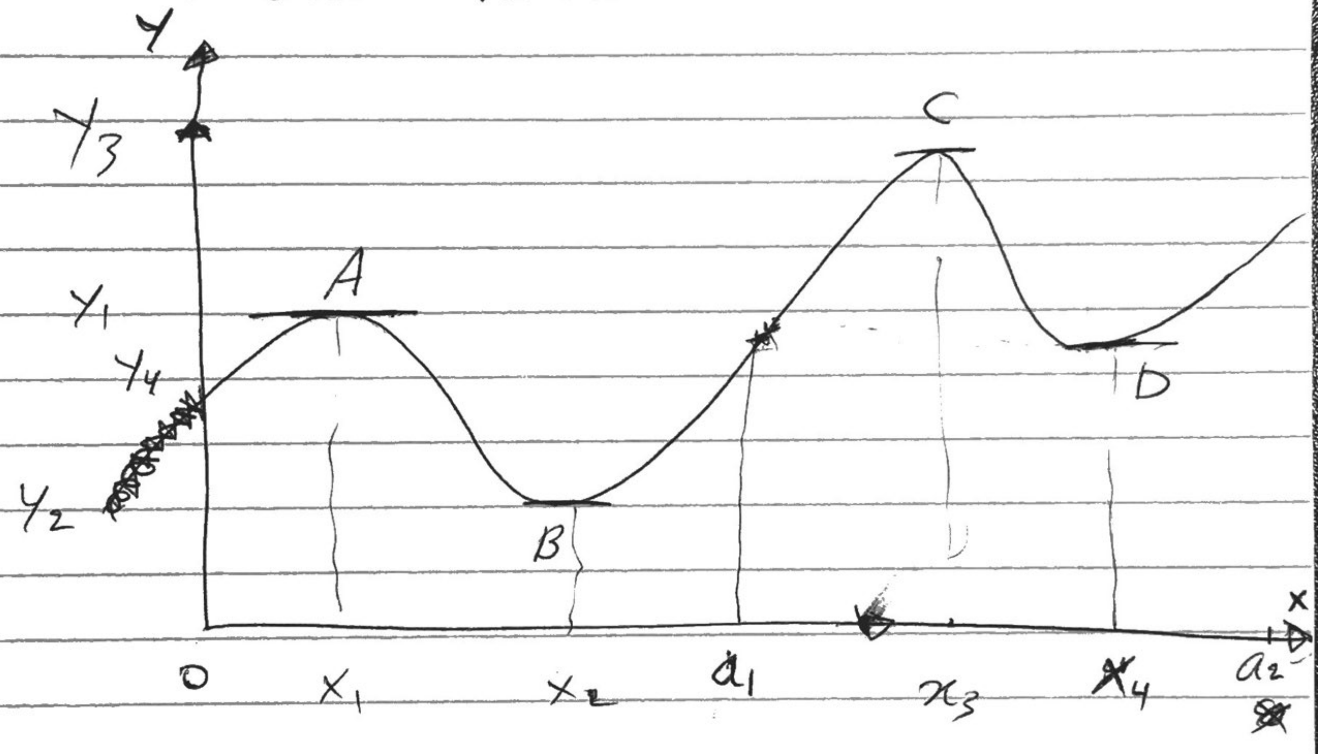
  maximize returns on investment

  or to minimize risks
- ~~I will explain the optimization problem in the context of maximization, but the method applies equally to minimization problems.~~

Notation

optimization (maximization) problem $\left\{ \max\limits_{x} \; \underbrace{y = f(x)}_{\text{objective function}} \right.$

x → choice variable



- point A represents a maximum over the domain $[0, a_1]$
- point C represents a maximum over the whole domain $[0, \infty[$
- point B represent a minimum over $[0, \infty[$
- point D represents a minimum over $[a_1, \infty[$

~~a maximum or~~

* we call point A a relative or local maximum, we call C an absolute or global maximum.

* Similarly D represents a relative or local minimum, while B represents an absolute or global minimum.

~~any global maximum (minimum) is also relative~~

* In this course we won't focus on specifying whether a maximum or a minimum is relative or absolute (local or global)

* Note that global optimums are also local

* Terminology: optimum ~~refers~~ is the general term for maximum / minimum

Find the optimum point(s) : F.O.C

we use what is known as the F.O.C to find the optimum point(s).

F.O.C : set the first total differential equal to zero and solve for x.

$$y = f(x)$$

$$dy = \frac{\partial f(x)}{\partial x} dx = 0$$

Since $dx \neq 0$ then this condition only holds if $\boxed{\frac{\partial f(x)}{\partial x} = 0}$

Thus in practice we use:

$$\boxed{\text{F.O.C :} \quad \frac{\partial f(x)}{\partial x} = 0}$$

Example 1: $y = f(x) = x^2 - 4x + 4$.

the optimal value of y is associated with $\boxed{x^* = 2}$ which is obtained as follows:

F.O.C $\dfrac{\partial f(x)}{\partial x} = 0$.

$$\Rightarrow \quad 2x - 4 = 0 \quad \Rightarrow \boxed{x^* = 4/2 = 2}$$

$$y^* = x^{*2} - 4x^* + 4 = (2)^2 - 4(2) + 4 = 0$$

Example 2: $y = f(x) = x^3 - 12x^2 + 36x + 8$

F.O.C $\quad 3x^2 - 24x + 36 = 0$

$$3(x^2 - 8x + 12) = 0$$

$$3(x - 6)(x - 2) = 0$$

Thus $x_1^* = 6$ and $x_2^* = 2$

In this example we have two stationary points $(x_1^* = 6)$ and $(x_2^* = 2)$.

First Derivative test and S.O.C

The F.O.C condition ~~f~~ helps us find the ~~of~~ stationary point(s) (x^*) and the ~~optin~~ corresponding optimal value(s) (y^*). However, it doesn't tell us whether this optimal is a maximum or a minimum. To find that, we use ~~the~~ either the First Derivative Test, or the Second derivative test. ~~w~~ The Latter is known as the Second order Condition (S.O.C).

First Derivative test

* if ~~f~~ $\frac{\partial f(x)}{\partial x}$ changes its sign from positive to negative from the immediate left of x^* to its immediate right, then x^* represents a maximum

$$\left[\text{Max if } \lim_{x \to x^{*-}} f'(x) \text{ is +ve and } \lim_{x \to x^{*+}} f'(x) \text{ is -ve} \right.$$

* Similarly $\left[\text{Min if } \lim_{x \to x^{*-}} f'(x) \text{ is -ve and } \lim_{x \to x^{*+}} f'(x) \text{ is +ve} \right.$

* Neither maximum nor minimum if there is no change in the sign

Example: Consider again

$$y = x^3 - 12x^2 + 36x + 8$$

F.O.C gave us two stationary points

$$x_1^* = 6 \quad \text{and} \quad x_2^* = 2$$

we have to check if $x_1^* = 6$ pertains to a maximum or minimum

and whether $x_2^* = 2$ pertains to a maximum or minimum.

Case of $x_1^* = 6$

$$\lim_{x \to 6^-} f'(x)$$

$$= \lim_{x \to 6^-} 3(x-6)(x-2)$$

$$= 3\underbrace{(6^- - 6)}_{-ve}\underbrace{(6^- - 2)}_{+ve} < 0$$

$$\lim_{x \to 6^+} f'(x) = 3(\underset{+ve}{6^+ - 6})(\underset{+ve}{6^+ - 2}) > 0$$

Since the left limit ~~chan~~ is negative and the right limit is positive ∴ $x_1^* = 6$ pertains to a minimum.

~~Case of $x_1^* = 2$~~

$$\cancel{\lim_{x \to 2^-} 3(2^- - 6)(2^- - 2) > 0}$$

~~$\lim_{x \to 2^+} 3($~~

Case of $x_2^* = 2$

$$\lim_{x \to 2^-} 3(x-6)(x-2) = 3(\underset{-ve}{2^- - 6})(\underset{-ve}{2^- - 2}) > 0$$

$$\lim_{x \to 2^+} 3(x-6)(x-2) = 3(\underset{-ve}{2^+ - 6})(\underset{+ve}{2^+ - 2}) < 0$$

Thus, maximum.

Second Order Condition

$$\begin{cases} \text{if } d^2y < 0 \Rightarrow \text{Max} \\ \text{if } d^2y > 0 \Rightarrow \text{Min} \\ \text{if } d^2y = 0 \text{ Neither.} \end{cases}$$

<u>Recall</u>: $d^2y = d(dy) = d(f'(x)dx)$

Since dx is a non-zero constant

then $d(f'(x)dx) = f''(x)dx$

where $f''(x) = \dfrac{\partial^2 f(x)}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f(x)}{\partial x}\right)$

~~Thus, in practice it is sufficient to:~~

Thus we can reformulate the above conditions

$$\begin{cases} \text{if } f''(x) < 0 \Rightarrow \text{max} \\ \text{if } f''(x) > 0 \Rightarrow \text{min} \\ \text{if } f''(x) = 0 \Rightarrow \text{Neither.} \end{cases}$$

Example:

Let's apply S.O.C to the same example

$$y = f(x) = x^3 - 12x^2 + 36x + 8$$

$$f'(x) = 3x^2 - 24x + 36 = 3(x-6)(x-2)$$

F.O.C $f'(x) = 0 \Rightarrow x_1^* = 6, \; x_2^* = 2$

S.O.C $f''(x) = \dfrac{\partial f'(x)}{\partial x}$

$$= 6x - 24$$

$x_1^* = 6$

$f''(6) = 6(6) - 24 > 0$

$\Rightarrow$ minimum

$x_2^* = 2$

$f''(2) = 6(2) - 24 < 0$

$\Rightarrow$ maximum

↳ Same conclusions as those obtained from first derivative test.

Example: Revenue function $R(Q) = 1200Q - 2Q^2$

Cost function:

$$C(Q) = Q^3 - 61.25Q^2 + 1528.5Q + 2000$$

a) Write down the profit function

$$\pi = R(Q) - C(Q)$$
$$= 1200Q - 2Q^2 - (Q^3 - 61.25Q^2 + 1528.5Q + 2000)$$

$$= -Q^3 + 59.25Q^2 - 328.5Q - 2000$$

b) Find the Stationary point(s).

F.O.C $\frac{\partial \pi}{\partial Q} = 0$

$$-3Q^2 + 118.5Q - 328.5 = 0$$

Using the quadratic formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$Q_{A_1}^{*} = 3$

$Q_{A_2}^{*} = 36.5$

c) Characterize these stationary point(s) or check whether the stationary point(s) pertain to a maximum or to a minimum.

S.O.C $\pi''(Q) = \frac{\partial^2 \pi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \pi}{\partial x}\right)$

$= -6Q + 118.5$

$Q_{A_1}^{*} = 3$

$\pi''(3) = -18 + 118.5 > 0$

$\Rightarrow$ Minimum

$Q_{A_2}^{*} = 36.5$

$\pi''(36.5)$

$= -219 + 118.5 < 0$

$\Rightarrow$ Maximum.

6pt

d) find the optimal profit

$$\underbrace{\pi^*(36.5) = 16318.44}$$

plug $Q = 36.5$ in the main profit function obtained in part a).

N^{th} derivative test

if the first nonzero derivative value at x^* encountered is that of N^{th} derivative, then the ~~[illegible]~~ optimal value of $f(x^*)$ will be

* Max if N is even and $f^N(x^*) < 0$
* Min if N is even and $f^N(x^*) > 0$
* inflection point if N is odd

Notation: $f'(x^*)$: first derivative evaluated at x^*

$f''(x^*)$ second derivative evaluated at x^*

⋮

$f^N(x^*)$ N^{th} derivative evaluated at x^*

Example: $f(x) = (7-x)^4$

F.O.C $\quad f'(x) = 0 \Rightarrow -4(7-x)^3 = 0$

$$\Rightarrow x^* = 7.$$

S.O.C $\quad f''(x^*) = +12(7-x)^2 \Big|_{x^*=7}$

$$= 12(7-7)^2 = 0$$

Since $f''(x^*) = 0$, we have to find the third derivative

$$f'''(x^*) = -24(7-x)\Big|_{x^*=7}$$

$$= 0$$

$$f''''(x^*) = +24\Big|_{x^*=7} = 24 > 0$$

$N = 4$

N is even

$f''''(x^*) > 0$

Thus $x^* = 7$ pertains to a Minimum

Unconstrained optimization with more than one choice variable

The goal is to optimize (minimize or maximize) an objective function with more than one choice variable

$$\max (\min) \quad z = f(x,y)$$

(Choose the values of x and y that leads to highest value of z (if you are maximizing) or the lowest value of z (if you are minimizing).)

As before, this involves two steps

① Finding the stationary points (F.O.C)

② Checking whether the stationary point(s) pertain to a minimum or a maximum.

F.O.C

$$\begin{cases} Z_x = 0 \\ Z_y = 0 \end{cases}$$

System of two equations and two unknowns (x and y).

where $Z_x = \dfrac{\partial Z}{\partial x} = \dfrac{\partial f(x,y)}{\partial x}$

$Z_y = \dfrac{\partial Z}{\partial y} = \dfrac{\partial f(x,y)}{\partial y}$

or we can use f_x instead of Z_x and f_y instead of Z_y

any notation of the above is valid.

you can solve this system by substitution or matrix inverse method or Cramer's Rule.

S.O.C:

- Find the Hessia matrix, which is the matrix of the second derivatives

$$H = \begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix}$$

- Compute the leading principal minors

$$H_1 = [Z_{xx}]$$

$$H_2 = \begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix}$$

- $$\begin{cases} \text{if } |H_1| > 0 \text{ and } |H_2| > 0 \Rightarrow \text{Min} \\ \text{if } |H_1| < 0 \text{ and } |H_2| > 0 \Rightarrow \text{Max} \\ \text{if } |H_2| < 0 \Rightarrow \text{Saddle point.} \end{cases}$$

Find the Stationary points and Classify them.

<u>Example</u>: $z = f(x,y) = x^2 - 6x + 2xy + 2y^2 + 5$

F.O.C $\begin{cases} Z_x = 2x - 6 + 2y = 0 \\ Z_y = 2x + 4y = 0 \end{cases}$

→ You Can Solve this System using matrix inverse method

$$\begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 0 \end{bmatrix}$$

→ alternatively you can use cramer's Rule.

-) OR you can use substitution method (which I will use here)

$$\begin{cases} 2x - 6 + 2y = 0 & \text{eq (1)} \\ 2x + 4y = 0 & \text{eq (2)} \end{cases}$$

$2x = -4y \Rightarrow x = -2y$

plug in eq (1)

$2(-2y) - 6 + 2y = 0$

$\Rightarrow -2y = 6 \Rightarrow \boxed{y^* = -3}$

plug $y^* = -3$ in $x = -2y$

$x^* = -2(-3) = 6$

Stationary point $(x^*, y^*) = (6, -3)$

S.O.C

$Z_x = 2x - 6 + 2y$
$Z_y = 2x + 4y$

$Z_{xx} = 2$
$Z_{xy} = Z_{yx} = 2$
$Z_{yy} = 4$

$$\Rightarrow H = \begin{bmatrix} 2 & 2 \\ 2 & 4 \end{bmatrix}$$

$|H_1| = |2| = 2 > 0$

$$|H_2| = \begin{vmatrix} 2 & 2 \\ 2 & 4 \end{vmatrix} = 8 - 4 = 4 > 0$$

Since $|H_1| > 0$ & $|H_2| > 0 \Rightarrow$ Min

Example:

$$z = f(x, y) = 8x^3 + 2xy - 3x^2 + y^2 + 1$$

F.O.C $\begin{cases} z_x = 24x^2 + 2y - 6x = 0 & eq(1) \\ z_y = 2x + 2y = 0 & eq(2) \end{cases}$

from eq(2) $\boxed{x = -y}$

$$24(-y)^2 + 2y - 6(-y) = 0$$

$$= 24y^2 + 8y = 0$$

$\cancel{8(y}$ $8y(3y+1) = 0$

$\Rightarrow$ two points

$\hookrightarrow y_1^* = 0 \quad (8y = 0)$

$\hookrightarrow y_2^* = -\frac{1}{3} \quad (3y + 1 = 0)$

Now $\boxed{x = -y}$ $\Rightarrow \begin{cases} x_1^* = -0 = 0 \\ x_2^* = -\left(-\frac{1}{3}\right) = \frac{1}{3} \end{cases}$

Thus two Stationary Points

$$(x_1^*, y_1^*) = (0, 0)$$

$$(x_2^*, y_2^*) = (1/3, -1/3).$$

S.O.C

$$Z_{xx} = 48x - 6$$

$$Z_{yy} = 2$$

$$Z_{xy} = 2$$

$$H = \begin{bmatrix} 48x-6 & 2 \\ 2 & 2 \end{bmatrix}$$

At $(x_1^*, y_1^*) = (0, 0)$

$$H = \begin{bmatrix} -6 & 2 \\ 2 & 2 \end{bmatrix} \quad |H_1| = |-6| = -6 < 0$$

$$|H_2| = \begin{vmatrix} -6 & 2 \\ 2 & 2 \end{vmatrix} = -16 < 0$$

$\Rightarrow$ Saddlepoint.

At $(x_2^*, y_2^*) = (1/3, -1/3)$

$$H = \begin{bmatrix} 48(1/3) - 6 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 2 & 2 \end{bmatrix}$$

$|H_1| = |10| = 10 > 0$

$$|H_2| = \begin{vmatrix} 10 & 2 \\ 2 & 2 \end{vmatrix} = 20 - 4 = 16 > 0$$

$\Rightarrow$ this point $[(1/3, -1/3)]$ pertains to a minimum.

Unconstrained optimization

Three choice variables

$$\max(\min) \quad z = f(x, y, w)$$

F.O.C
$$\begin{cases} Z_x = 0 \\ Z_y = 0 \\ Z_w = 0 \end{cases}$$

System of three equations three unknowns.

Solve for x^*, y^*, w^*

S.O.C:
$$H = \begin{bmatrix} Z_{xx} & Z_{xy} & Z_{xw} \\ Z_{yx} & Z_{yy} & Z_{yw} \\ Z_{wx} & Z_{wy} & Z_{ww} \end{bmatrix}$$

Leading principal Minors:

if

$$H_1 = [Z_{xx}]$$

$$H_2 = \begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix}$$

$$H_3 = \begin{bmatrix} Z_{xx} & Z_{xy} & Z_{xw} \\ Z_{yx} & Z_{yy} & Z_{yw} \\ Z_{wx} & Z_{wy} & Z_{wy} \end{bmatrix}$$

$$\begin{cases} \text{if } |H_1| > 0,\ |H_2| > 0,\ |H_3| > 0 \Rightarrow \text{Min} \\ \text{if } |H_1| < 0,\ |H_2| > 0,\ |H_3| < 0 \Rightarrow \text{Max} \end{cases}$$

More Examples:

(1) The height of a plant after t months is given by $h(t) = \sqrt{t} - \frac{t}{2}$, $t \in [0, 3]$.

At what time is the plant at it's highest? Test whether t^* is a maximizing solution.

Answer: $\frac{\partial h(t)}{\partial t} = 0$

$$\frac{1}{2} t^{-1/2} - \frac{1}{2} = 0$$

$$t^{-1/2} = 1$$

$$(t^{-1/2})^{-2} = (1)^{-2}$$

$$\Rightarrow \boxed{t^* = 1}$$

The ~~tree~~ height of the tree at $t^* = 1$

$$h(1) = \sqrt{1} - \frac{1}{2} = \frac{1}{2}.$$

S.O.C $\frac{\partial^2 h(t)}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial h(t)}{\partial t}\right)$

$= \frac{\partial}{\partial t}\left(\frac{1}{2}t^{-\frac{1}{2}} - \frac{1}{2}\right)$

$= \underbrace{-\frac{1}{4}}_{-ve} \underbrace{t^{-\frac{3}{2}}}_{+ve} < 0 \Rightarrow$ Max.

(2) A sports club plans to charter a plane. The charge for 60 passengers is 800 \$ each. For each additional person above 60, all travelers get a discount of 10 \$. The plane ~~p.~~ can take at most 80 passengers.

a) if $60 + x$ ~~[illegible]~~ passengers fly, what is the total Cost?

b) find the number of passengers that maximizes the total airfare paid by the club members.

Answer: a) $\underline{\underline{\text{cost}}}$ $(60+x)(800-10x)$

explanation: * if 61 passengers fly $x=1$ and all passengers get a 10 \$ discount. thus

$(60+1)(800-10(1))$

$(61)(790)$

passengers — fare after 10 \$ discount.

* if 62 passengers fly, $x=2$ and all passengers get 20 \$ discount (10 \$ / additional

passenger).

$$(60+2)(800-10(2))$$

~~(62~~

$$(62)\ (780)$$

passenger

↳ fare with 20 $ discount

b) F.O.C $\frac{\partial C}{\partial x}=0$

$$\frac{\partial\left[(60+x)(800-10x)\right]}{\partial x}=0$$

$$(800-10x)+(-10)(60+x)=0$$

$$200-20x=0$$

$$\Rightarrow x^{*}=10.$$

3 Show that a profit maximizing monopolist's output is unaffected by a proportional profit tax, but is affected by a tax of t \$ per unit of output.

Answer: A No tax Case.

$$\Pi = \text{Revenue} - \text{Cost}$$
$$= R(x) - C(x)$$

F.O.C $\frac{\partial \Pi}{\partial x} = 0 \Rightarrow R'(x) - C'(x) = 0$

or $\boxed{MR = MC}$

↳ profit maximizing condition with no tax.

B Proportional tax (~~rate~~

~~use the~~

$\Pi = (1-\tau)(R(x) - C(x))$

Net profit

$\frac{\partial \Pi}{\partial x} = (1-\tau)(R'(x) - C'(x)) = 0$

Since $0 < \tau < 1$, the above equation holds if and only if

$R'(x) - C'(x) = 0$

$MR - MC = 0$

This again, produces the same profit maximizing condition as in case A. $(MR = MC)$

Thus, a proportional tax don't affect the level of output since in Both cases it is determined by the Same Condition.

c) t \$ per unit.

$$\Pi = R(x) - C(x) - tx.$$

$$\frac{\partial \Pi}{\partial x} = 0$$

$$R'(x) - C'(x) - t = 0$$

Thus $\boxed{MR = MC + t}$

as obviously, this condition differs from that implied by the case (A) and thus the resulting output level (x^*) will be different as well.

4) Consider the function

$$C(x,y) = \frac{x^2}{100} - 10x + \frac{y^3}{300} - 9y + 20600$$

: assume $x \geq 0$ and $y \geq 0$

a) find the stationary point(s).

b) Classify the stationary point(s) obtained in part (a).

a) F.O.C

C_x: $\frac{2x}{100} - 10 = 0 \Rightarrow \frac{x}{50} = 10 \Rightarrow \boxed{x^* = 500}$

C_y: $\frac{3y^2}{300} - 9 = 0 \Rightarrow \frac{y^2}{100} = 9 \Rightarrow \boxed{y^* = \pm 30}$

however $y \geq 0$, thus we can exclude the $y^* = -30$, thus

$$\boxed{y^* = 30}$$

4 Consider the function

$$C(x,y) = \frac{x^2}{100} - 10x + \frac{y^3}{300} - 9y + 20600$$

assume $x \geq 0$ and $y \geq 0$

a) find the stationary point(s).

b) Classify the stationary point(s) obtained in part (a).

a) F.O.C

$C_x = 0$: $\frac{2x}{100} - 10 = 0 \Rightarrow \frac{x}{50} = 10 \Rightarrow \boxed{x^* = 500}$

$C_y = 0$: $\frac{3y^2}{300} - 9 = 0 \Rightarrow \frac{y^2}{100} = 9 \Rightarrow \boxed{y^* = \pm 30}$

however $y \geq 0$, thus we can exclude the $y^* = -30$, thus

$$\boxed{y^* = 30}$$

b) $C_{xx} = \frac{\partial C_x}{\partial x} = 50$

$C_{xy} = \frac{\partial}{\partial y}(C_x) = 0$

$C_{yx} = C_{xy}$ by Young's theorem

$C_{yy} = \frac{\partial (C_y)}{\partial y} = y/50$

$$H = \begin{bmatrix} 50 & 0 \\ 0 & y/50 \end{bmatrix}$$

$$y^* = 30 \Rightarrow H\Big|_{\substack{x=x^* \\ y=y^*}} = \begin{bmatrix} 50 & 0 \\ 0 & 30/50 \end{bmatrix}$$

$$\left.\begin{array}{l} |H_1| = |50| = 50 > 0 \\ |H_2| = 50 \times \frac{30}{50} - 0 = 30 > 0 \end{array}\right\} \Rightarrow \text{Minimum}$$

5 ~~Example~~ Optimal timing: Timber Cutting Problem

Suppose that the value of timber is defined ~~by~~ according to the following increasing function of time:

$$V = 2^{\sqrt{t}}$$

r is the interest rate and A is the present value of timber. Assume that Both r an A are greater than zero.

a) what is the optimal time to cut the timber for sale?

b) Test whether t^* obtained in a) is a maximizing solution.

a) $$\underbrace{A}_{\text{present value}} = \underbrace{V}_{\text{future value}} \underbrace{e^{-rt}}_{\text{discount factor}} = 2^{\sqrt{t}} e^{-rt}$$

F.O.C $\quad \frac{\partial A}{\partial t} = 0$

$$\frac{\partial A}{\partial t} = \frac{\partial}{\partial t}\left(2^{\sqrt{t}} e^{-rt}\right) = 0$$

$$= \left(\frac{\partial 2^{\sqrt{t}}}{\partial t}\right) \times e^{-rt} + 2^{\sqrt{t}} \frac{\partial e^{-rt}}{\partial t} = 0$$

$$= \underbrace{\frac{1}{2} t^{-\frac{1}{2}} \ln(2) \overbrace{2^{\sqrt{t}} e^{-rt}}^{A}} + \overbrace{2^{\sqrt{t}} (-r) e^{-rt}}^{A} = 0$$

$$= \frac{1}{2} t^{-\frac{1}{2}} \ln(2)\, A - rA = 0$$

$$= A\left(\frac{1}{2} t^{-\frac{1}{2}} \ln(2) - r\right) = 0$$

Since $A > 0$

then this relation holds if and only if $\frac{1}{2} t^{-\frac{1}{2}} \ln(2) - r = 0$

solving for t, we get

$$t^* = \left(\frac{\ln(2)}{2r}\right)^2$$

b) S.O.C $\quad \dfrac{\partial^2 A}{\partial t^2} = \dfrac{\partial}{\partial t}\left(\dfrac{\partial A}{\partial t}\right)$

$$= \frac{\partial \left[A\left(\frac{1}{2}t^{-1/2}\ln(2) - r\right)\right]}{\partial t}$$

Important: $A = 2^{\sqrt{t}} e^{-rt}$, thus it is a function of t, so we have to use the product rule to compute the derivative above:

$$= \frac{\partial A}{\partial t}\left[\frac{1}{2}t^{-1/2}\ln(2) - r\right] + A\,\frac{\partial}{\partial t}\left(\frac{1}{2}t^{-1/2}\ln(2) - r\right)$$

$$= \frac{\partial A}{\partial t}\left(\frac{1}{2}t^{-1/2}\ln(2) - r\right) + A\left(-\frac{1}{4}t^{-3/2}\ln(2)\right)$$

To test whether t^* is a maximizing solution we have to evaluate the second derivative at t^* and check its sign.

$$\left.\frac{\partial A}{\partial t}\right|_{t=t^*} = 0$$ This is exactly the F.O.C.

Then $$\frac{\partial^2 A}{\partial t^2} = 0 + A\left(-\frac{1}{4}t^{-3/2}\ Ln(2)\right)$$

<0 >0 >0

+ve -ve +ve +ve

thus $$\frac{\partial^2 A}{\partial t^2} < 0$$

$\Rightarrow$ Maximum.

Comparative Statics

Comparative statics is concerned with the comparison of different equilibrium states that are associated with different sets of values of parameters and exogeneous variables.

Example
$$\begin{cases} Q^d = a - bP & \text{demand} \\ Q^s = -c + dP & \text{Supply} \end{cases}$$

Demand = Supply

$$a - bP = -c + dP \Rightarrow \boxed{P^* = \frac{a+c}{b+d}}$$

plug P^* in either Q^d or Q^s:

$$Q^* = \frac{ad - bc}{b+d}$$

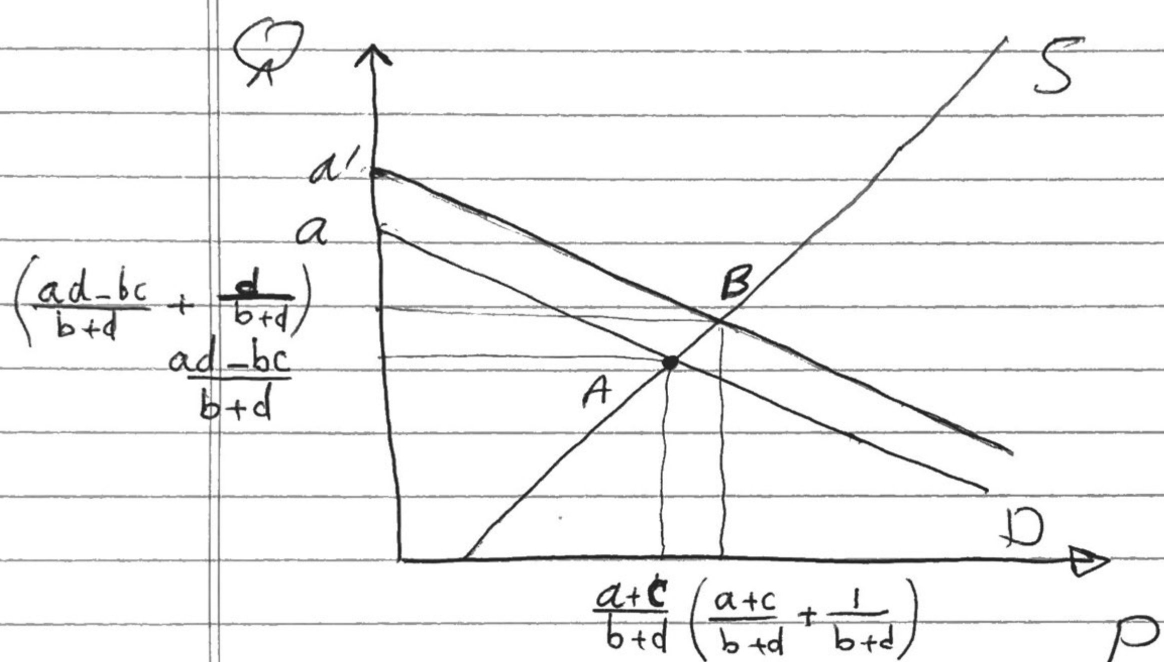
question: Study the compartive statics with respect to a.

The question is asking us to find the rate of change of the equilibruim values of P and Q as a changes:

$$\begin{cases} \dfrac{\partial P^*}{\partial a} = \dfrac{1}{b+d} \\ \dfrac{\partial Q^*}{\partial a} = \dfrac{d}{b+d} \end{cases}$$

interpretation: if a↑ by one unit, the equilibruim value of P^* changes by $\dfrac{1}{b+d}$. we can interpret $\dfrac{\partial Q^*}{\partial a}$ in the same manner

Graphical interpretation



if $a \uparrow$ to a', $P^* \uparrow$ by $\frac{1}{b+d}$ and $Q^* \uparrow$ by $\frac{d}{b+d}$.

So initial equilibrium $P_1^* = \frac{a+c}{b+d}$

New equilibrium $P_2^* = P_1^* + \frac{\partial P^*}{\partial a}$

$$= \frac{a+c}{b+d} + \frac{1}{b+d}$$

New equilibrium quantity : $Q_2^* = Q_1^* + \frac{\partial Q^*}{\partial a}$

$$Q_2^* = \frac{ad-bc}{b+d} + \frac{d}{b+d}.$$

Comparative Statics without the need to solve for the equilibrium values.

Example Consider the Same example: Study comparative statics w.r.t to a.

$$\begin{cases} Q = a - bP \\ Q = -c + dP \end{cases}$$

① rewrite the system of equations in the implicit function form

F^1: $Q - a + bP = 0$

F^2: $Q + c - dP = 0$

② use the following relationship:

$$\underbrace{\begin{bmatrix} \dfrac{\partial F^1}{\partial Q} & \dfrac{\partial F^1}{\partial P} \\ \dfrac{\partial F^2}{\partial Q} & \dfrac{\partial F^2}{\partial P} \end{bmatrix}}_{\text{Jacobian}} \begin{bmatrix} \dfrac{\partial Q}{\partial a} \\ \dfrac{\partial P}{\partial a} \end{bmatrix} = \begin{bmatrix} -\dfrac{\partial F^1}{\partial a} \\ -\dfrac{\partial F^2}{\partial a} \end{bmatrix}$$

$$\begin{bmatrix} 1 & b \\ 1 & -d \end{bmatrix} \begin{bmatrix} \dfrac{\partial Q}{\partial a} \\ \dfrac{\partial P}{\partial a} \end{bmatrix} = \begin{bmatrix} -(-1) \\ -0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & b \\ 1 & -d \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \partial Q/\partial a \\ \partial P/\partial a \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{d}$$

$$\boxed{Ax = d}$$

we can solve this system of equations either using the matrix inverse method or Cramer's Rule:

$$\partial Q/\partial a = \frac{|A_1|}{|A|} = \frac{\begin{vmatrix} 1 & b \\ 0 & -d \end{vmatrix}}{\begin{vmatrix} 1 & b \\ 1 & -d \end{vmatrix}} = \frac{-d}{-d-b}$$

$$= \frac{d}{b+d}$$

$$\frac{\partial P}{\partial a} = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & b \\ 1 & -d \end{vmatrix}} = \frac{-1}{-d-b} = \frac{1}{b+d}$$

Example 2 Study the comparative statics w.r.t to G_0.

$$Y = C + I_0 + G_0$$
$$C = \alpha + \beta(Y - T)$$
$$T = \gamma + \delta Y$$

F^1: $Y - C + I_0 + G_0 = 0$
F^2: $C - \alpha - \beta Y + \beta T = 0$

F^3: $T - \gamma - \delta Y = 0$

$$\begin{bmatrix} \partial F^1/\partial Y & \partial F^1/\partial C & \partial F^1/\partial T \\ \partial F^2/\partial Y & \partial F^2/\partial C & \partial F^2/\partial T \\ \partial F^3/\partial Y & \partial F^3/\partial C & \partial F^3/\partial T \end{bmatrix} \begin{bmatrix} \partial Y/\partial G_0 \\ \partial C/\partial G_0 \\ \partial T/\partial G_0 \end{bmatrix} = \begin{bmatrix} -\partial F^1/\partial G_0 \\ -\partial F^2/\partial G_0 \\ -\partial F^3/\partial G_0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \partial Y/\partial G_0 \\ \partial C/\partial G_0 \\ \partial T/\partial G_0 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}}_{d}$$

$$\frac{\partial Y}{\partial G_0} = \frac{|A_1|}{|A|}$$

$$= \frac{\begin{vmatrix} -1 & -1 & 0 \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{vmatrix}} = \frac{1}{1-\beta+\beta\delta}$$

$$\frac{\partial C}{\partial G_0} = \frac{|A_2|}{|A|} = \frac{\begin{vmatrix} 1 & -1 & 0 \\ -\beta & 0 & \beta \\ -\delta & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -\beta & 1 & \beta \\ -\delta & 0 & 1 \end{vmatrix}} = \frac{\quad}{1-\beta+\beta\delta}$$

Similarly find

$$\frac{\partial T}{\partial G_0} = \frac{|A_3|}{|A|} = \frac{\begin{vmatrix} 1 & -1 & -1 \\ -\beta & 1 & 0 \\ -\delta & 0 & 0 \end{vmatrix}}{1-\beta+\beta\delta} = \frac{\quad}{1-\beta+\beta\delta}$$