LECTURE NOTES II

Outline

Derivatives Rules
Second derivatives
partial derivatives
Young's theorem
Hessian matrix

limits: Substitution, Factors method
Conjucates, l'Hopital Rule
limits to infinity

Inequalities

Absolute values

logarithmic and exponential functions
properties
economic meaning of e
Growth rate and discount factor

Total differential

Total derivative

Implicit function derivative

Derivative Rules:

Rule 1: $f(x) = c$; $f'(x) = 0$

Example: $f(x) = 3$; $f'(x) = 0$

Rule 2: $f(x) = x^a$; $f'(x) = ax^{a-1}$

Example 1: $f(x) = x^2$; $f'(x) = 2x^{2-1} = 2x$

Example 2: $f(x) = x$; $f'(x) = 1x^{1-1} = 1x^0 = 1$

Rule 3: $f(x) = c\,g(x)$; $f'(x) = c\,g'(x)$

Example: $f(x) = 3x^2$; $f'(x) = 3(2x) = 6x$

Rule 4: $f(x) = g(x) \pm h(x)$; $f'(x) = g'(x) \pm h'(x)$

Example: $f(x) = 2x + x^2$; $f'(x) = 2 + 2x$

Rule 5: $f(x) = g(x)\,h(x)$; $\cancel{f'(x) = g'(x)h'(x)}$

$$f'(x) = g'(x)h(x) + g(x)h'(x)$$

Example: $f(x) = x\sqrt{x} = x\,x^{1/2}$

$$f'(x) = (1)(x^{1/2}) + x\left(\tfrac{1}{2}\right)x^{-1/2}$$

$$= x^{1/2} + \frac{1}{2}x^{1/2} = \frac{3}{2}x^{1/2}$$

Rule 6: $f(x) = \dfrac{g(x)}{h(x)}$; $f'(x) = \dfrac{g'(x)h(x) - h'(x)g(x)}{h(x)^2}$

Example: $f(x) = \dfrac{1}{x}$; $f'(x) = \dfrac{0(x) - 1(1)}{x^2} = \dfrac{-1}{x^2}$

Rule 7: $f(x) = a^x$; $f'(x) = \ln(a)\, a^x$

Example: $f(x) = 2^x$; $f'(x) = \ln(2) 2^x$

~~Proof~~

Rule 8: generalization of Rule 7

$$f(x) = g(x)^{h(x)}$$

$$f'(x) = \left[h'(x)\log(x) + h(x)\frac{g'(x)}{g(x)}\right] g(x)^{h(x)}$$

proof will be provided ~~later~~ after covering logarithmic functions

Example $f(x) = 2^x$, here $g(x) = 2$

$h(x) = x$; $\Rightarrow h'(x) = 1$, and

$g'(x) = 0$.

Then $f'(x) = \left[(1)\ln(2) + x\left(\frac{0}{2}\right)\right] 2^x$

$= \ln(2)\, 2^x$.

Rule 9: $f(x) = e^{g(x)}$; $f'(x) = g'(x)e^{g(x)}$

Example: $f(x) = e^{2x}$, $f'(x) = 2e^{2x}$

Rule 10: $f(x) = \ln(g(x))$; $f'(x) = \dfrac{g'(x)}{g(x)}$

Example: $f(x) = \ln(x^2)$; $f'(x) = \dfrac{2x}{x^2} = \dfrac{2}{x}$

Rule 11: (Chain Rule)

$$f(x) = h(g(x))$$

$$f'(x) = h'(g(x)) \cdot g'(x)$$

Example: $f(x) = \underbrace{(\underbrace{x^2+2x}_{g(x)})^3}_{h(g(x))}$

$$f'(x) = \underbrace{3(x^2+2x)^2}_{h'(g(x))} \cdot \underbrace{(2x+2)}_{g'(x)}$$

Notation: I used $f'(x)$ as a notation for the derivative. Other popular notation include

$\dfrac{\partial f(x)}{\partial x}$, f_x ,

or if we define $y = f(x)$, then we can use y' or y_x as a notation for derivative

<u>Second derivative:</u> it's the derivative of the derivative.

$y = f(x)$

First derivative: $f'(x) = \frac{\partial f(x)}{\partial x}$

Second derivative $f''(x) = \frac{\partial^2 f(x)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f(x)}{\partial x}\right)$

$f''(x)$, f'', f_{xx}, y_{xx}, y'' are all ~~acceptable nota~~ valid notations for the second derivative

<u>Example:</u> $f(x) = x \ln(x)$.

$$f'(x) = \ln(x) + x\left(\frac{1}{x}\right) = \ln(x) + 1$$

$$f''(x) = \frac{\partial(f'(x))}{\partial x} = \frac{\partial}{\partial x}\left(\ln(x)+1\right)$$

$$= \frac{1}{x}.$$

Partial derivative: it applies when the underlying function has several input variables.

$$z = f(x, y)$$

* partial derivative of z with respect to y: $z_y = \frac{\partial z}{\partial y} = \frac{\partial f(x,y)}{\partial y}$

* partial derivative of z with respect to x

$$z_x = \frac{\partial z}{\partial x} = \frac{\partial f(x,y)}{\partial x}$$

Example $z = x^2 + y^2 + 2xy^2$

$$\frac{\partial z}{\partial x} = 2x + 2y^2$$

$$\frac{\partial z}{\partial y} = 2y + 4xy$$

Second partial derivative. $z = f(x,y)$

$$z_{xx} = \frac{\partial^2 f(x,y)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial x}\right)$$

$$z_{yy} = \frac{\partial^2 f(x,y)}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial y}\right)$$

also we can find the cross partial derivative

$$Z_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f(x,y)}{\partial x}\right)$$

$$Z_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f(x,y)}{\partial y}\right)$$

Consider the same function as in the previous example

$$Z = x^2 + y^2 + 2xy^2$$

$$Z_x = 2x + 2y^2$$

$$Z_y = 2y + 4xy$$

$$Z_{xx} = \frac{\partial}{\partial x}\left(2x + 2y^2\right) = 2$$

$$Z_{yy} = \frac{\partial}{\partial y}\left(2y + 4xy\right) = 2 + 4x$$

$$Z_{xy} = \frac{\partial}{\partial y}\left(2x + 2y^2\right) = 4y$$

$$Z_{yx} = \frac{\partial}{\partial x}\left(2y + 4xy\right) = 4y.$$

Young's theorm: states that cross partial derivatives are equal.

This ~~exactly~~ actually clear from the previous example where we found that $Z_{xy} = Z_{yx}$.

Hessian Matrize: matrix of second derivative

$$H = \begin{bmatrix} Z_{xx} & Z_{xy} \\ Z_{yx} & Z_{yy} \end{bmatrix}$$

we will use this matrix later in the optimization ~~so~~ part of the course.

Limits: it's the value of y (or $f(x)$) as x approaches a specific value.

$$y = f(x)$$

$\lim\limits_{x \to N} y$ reads limit of y as x approaches N.

* Left-hand limit $\lim\limits_{x \to N^-} y$

limit of y as x approaches N from below (x slightly smaller than N)

* Righ-hand limit $\lim\limits_{x \to N^+} y$

limit of y as x approaches N from above. (x slightly larger than N).

Recall the main indeterminate forms include $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, 1^∞, $\infty - \infty$, 0^0, ∞^0

also $\frac{a}{0}$ is undefined

Methods to evaluate limits:

1. Substitution: plug in the value of x ...

$y = x^2 + 1 - 2x$

$$\lim_{x \to 1} y = 1^2 + 1 - 2(1) = 0.$$

2. Factors Method: if you get an indeterminate form, try factoring and see if you have common factors that cancel out

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \frac{1^2 - 1}{1 - 1} = \frac{0}{0} \text{ indeterminate.}$$

Limit by factoring: $x^2 - 1 = (x-1)(x+1)$

$$\lim_{x \to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{(x-1)}} = \lim_{x \to 1} (x+1)$$

$$= 2.$$

3) Conjugates:

conjugate of $(x+y)$ is $(x-y)$

$$\lim_{x \to 4} \frac{2-\sqrt{x}}{4-x} = \frac{2-\sqrt{4}}{4-4} = \frac{0}{0}$$

indeterminate

$$\lim_{x \to 4} \frac{2-\sqrt{x}}{4-x} = \lim_{x \to 4} \frac{2-\sqrt{x}}{4-x} \frac{(2+\sqrt{x})}{(2+\sqrt{x})}$$

$$= \lim_{x \to 4} \frac{\cancel{4-x}}{\cancel{(4-x)}(2+\sqrt{x})} = \lim_{x \to 4} \frac{1}{2+\sqrt{x}}$$

$$= \frac{1}{2+\sqrt{4}} = \frac{1}{4}$$

$$\boxed{\begin{aligned} &\text{Recall } (x+y)(x-y) = x^2-y^2 \\ &\text{thus } (2-\sqrt{x})(2+\sqrt{x}) = 2^2-\sqrt{x}^2 \\ &\qquad = 4-x. \end{aligned}}$$

practice: find the limit of

$$\lim_{x \to 5} \frac{\sqrt{x+11} - 4}{x-5}$$

4) l'Hopital Rule:

if $\lim_{x \to N} \frac{f(x)}{g(x)}$ is indeterminate

then use l'Hopital Rule (if you have a ratio)

$$\lim_{x \to N} \frac{f(x)}{g(x)} = \lim_{x \to N} \frac{f'(x)}{g'(x)}$$

if you still get an indeterminate form

apply l'Hopital rule ~~one~~ again

$$\lim_{x \to N} \frac{f(x)}{g(x)} = \underbrace{\lim_{x \to N} \frac{f'(x)}{g'(x)}}_{\text{first time}} = \underbrace{\lim_{x \to N} \frac{f''(x)}{g''(x)}}_{\text{Second time}}$$

<u>Example 1</u>: $\lim_{x \to 0} \frac{e^x - 1}{x} = \frac{e^0 - 1}{0} = \frac{1-1}{0} = \frac{0}{0}$

indeterminate

l'Hopital Rule:

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = \frac{1}{1} = 1$$

<u>Example 2</u>: $\lim_{x \to 0} \frac{e^{xt} - 1 - xt}{x^2} = \frac{e^0 - 1 - 0}{0}$

$= \frac{0}{0}$ ~~indeterm~~

(indeterminate)

apply l'Hopital Rule $\lim_{x \to 0} \frac{t e^{xt} - t}{2x} = \frac{t e^0 - t}{2(0)}$

$= \frac{0}{0}$ (indeterminate)

apply l'Hopital Rule again.

$$\lim_{x \to 0} \frac{t^2 e^{xt}}{2} = \frac{t^2 e^0}{2} = \frac{t^2}{2}$$ ∎

5 limits to infinity.

$$\lim_{x\to\infty} \frac{10-x-9x^3}{5x^4-4x^2-1} = \lim_{x\to\infty} \frac{-9x^3}{5x^4}$$

$$= \lim_{x\to\infty} \frac{-9}{5x}$$

$$= \frac{-9}{\infty} = 0.$$

what I have done, ~~it~~ is that I focused only on the terms with highest powers as they dominate other terms.

You can only do this, when x approaches infinity.

Inequalities

* if $a \geq b$ and $b \geq c$, then $a \geq c$.
* if $a > b$ then $a \pm k > b \pm k$
* if $a > b$ $ka > kb$ if $k > 0$

Example: $13 > 5$ assume $k = 2$

then $13 \times 2 > 5 \times 2$

* if $a > b$ $ka < kb$ if $k < 0$

Example $13 > 5$ if $k = -2$

then $13(-2) < 5(-2)$

$-26 < -10$.

* if $a > b$ then $a^2 > b^2$ if $(b \geq 0)$
* if $a > b$ then $a^2 < b^2$ if $a < 0$.
* if $a > b$ the $-a < -b$
* if $a > b$ then $\frac{1}{a} < \frac{1}{b}$

if a and b have the same sign.

* if $a > b$ then $\frac{1}{a} > \frac{1}{b}$

if the signs of

a and b are different.

Example 1: Solving inequalities

$$3x + 5 > x + 2$$

then $3x - x > 2 - 5$

$$2x > -3$$

$$\boxed{x > -3/2}$$

Example 2: $-x - 1 > -2x + 3$

then $-(-x-1) < -(-2x+3)$

$$x + 1 < 2x - 3$$

$$1 + 3 < 2x - x$$

$\Rightarrow 4 < x$; or $\boxed{x > 4}$

Absolute value

- $|x| = x$ if $x > 0$
- $|x| = -x$ if $x < 0$
- $|x| = 0$ if $x = 0$.

Example: $|3|$: since $3 > 0$, $|3| = 3$.

$|-3|$: since $-3 < 0$, $|-3| = -(-3) = 3$

~~$|x| + |y| \geq |x+y|$~~

- $|x| \cdot |y| = |xy|$

$$\frac{|x|}{|y|} = \left|\frac{x}{y}\right|$$

Example: Solve the following inequality for x

$$|x| < 10$$

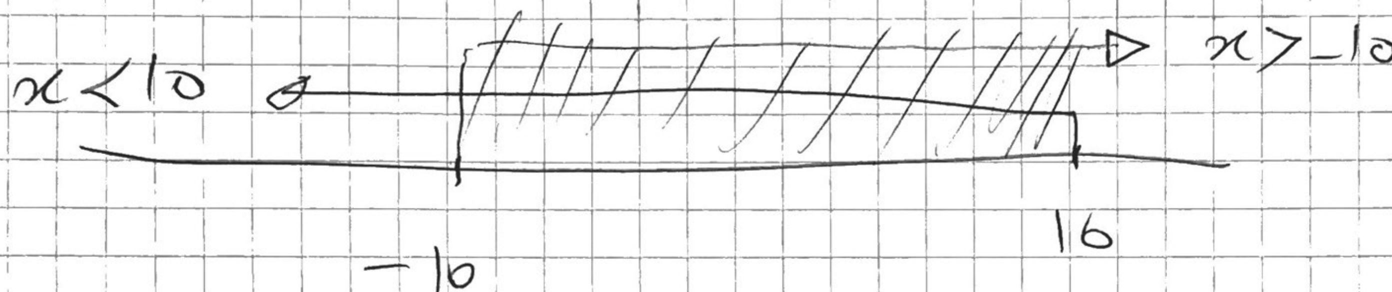
- Solution: $\hookrightarrow$ if $x > 0$ $|x| = x$

the $\boxed{x < 10}$

if $x < 0$ $|x| = -x$

then $-x < 10$

$$\boxed{x > -10}$$



Thus $\boxed{-10 < x < 10}$ final Solution

This also satisfies the trivial Case where $x = 0$

then $|x| = 0$ $\boxed{0 < 10}$

Example 2: $|1-x| \leq 3$

* if $1-x \geq 0$ then $|1-x| = 1-x$

then $1-x \leq 3$

$-x \leq 3-1$

$x \geq -2$

* if $1-x < 0$ then $|1-x| = -(1-x)$

then $-(1-x) \leq 3$

$-1+x \leq 3$

$x \leq 3+1 \Rightarrow x \leq 4$

~~Th~~

Combine the two conditions

$x \geq -2$

and $x \leq 4$ to get

$$-2 \leq x \leq 4$$

Properties of logarithmic, ~~and~~ exponential functions and exponents.

$$\ln(e) = 1$$

$$e^{\ln(x)} = x$$

$$\ln e^{x} = x$$

$$\ln(xy) = \ln(x) + \ln(y)$$

$$\ln\left(\frac{x}{y}\right) = \ln(x) - \ln(y)$$

$$\ln(1) = 0$$

$$\ln(x^{a}) = a\ln(x)$$

$$a\ln(x) = \ln(x^{a}).$$

$$a^{x} = e^{\ln a^{x}} = e^{x\ln(a)}$$

$$e^{x}e^{y} = e^{x+y}$$

$$x^{a}x^{b} = x^{a+b}$$

$$\frac{x^a}{x^b} = x^{a-b} = x^a x^{-b}$$

$$(x^a)^b = x^{ab}$$

$$\left(\frac{x}{y}\right)^a = \frac{x^a}{y^a} = x^a y^{-a} \qquad (y \neq 0)$$

$$x^{-a} = \frac{1}{x^a} \qquad (x \neq 0)$$

$$x^{\frac{1}{a}} = \sqrt[a]{x}$$

$$x^0 = 1 \quad (e^0 = 1).$$

$$x^{a/b} = \sqrt[b]{x^a}$$

$$\ln(1) = 0$$

$$e^{(\ln x)^y} = e^{y \ln(x)}$$

$$\lim_{x \to 0^+} \ln(x) = -\infty.$$

$$\lim_{x \to \infty} \ln(x) = \infty$$

The economic meaning of the exponential function e.

Assume you deposited one dollar at a bank ~~and go~~ where you were offered 100% interest rate, which will be compounded annually.

~~after one years the future dollars becomes~~

The future value of your deposit (the original capital and interest accrued) will equal to:

future value $V(1) = \left(1 + \frac{1}{1}\right)^1 = 2\ \$$.

frequency of compounding

* if alternatively, assume that the

interest is compounded semi-annually, ~~the~~ i.e, the compounding frequency $= 2$.

The future value ..

$$V(2) = \left(1 + \frac{1}{2}\right)^2 = 2.25 \ \$$$

$\times$ if the ~~compon~~ interest is compounded continuously (i.e, conpounding frequency is equal to infinity).

Future value

$$V(\infty) = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$$

proof let $y = \left(1 + \frac{1}{x}\right)^x$

$$\ln(y) = x \ln\left(1 + \frac{1}{x}\right) = \frac{1}{\frac{1}{x}} \ln\left(1 + \frac{1}{x}\right)$$

$$\lim_{x \to \infty} \ln(y) = \lim_{x \to \infty} \frac{1}{\frac{1}{x}} \left(\ln\left(1 + \frac{1}{x}\right)\right).$$

$$= \frac{0}{0}$$, thus apply

l'Hopital Rule.

l'Hopital Rule:

$$\lim_{x \to \infty} \frac{1}{\frac{1}{x}} \left(\ln \left(1 + \frac{1}{x} \right) \right)$$

$$= \lim_{x \to \infty} \frac{\frac{\cancel{\left(-\frac{1}{x^2}\right)}}{1 + \frac{1}{x}}}{\cancel{\left(-\frac{1}{x^2}\right)}} = \lim_{x \to \infty} \frac{1}{1 + \frac{1}{x}} = 1$$

Thus $\lim_{x \to \infty} \ln(y) = 1$

but we are interested in $\lim_{x \to \infty} y$,

$$e^{\lim_{x \to \infty} \ln(y)} = \cancel{\lim_{x \to \infty} e^{\ln(y)}}\ e^1$$

$$\lim_{x \to \infty} e^{\ln(y)} = e$$

$$\lim_{x \to \infty} y = e \quad \left(\text{Since } e^{\ln(y)} = y \right)$$

Thus a dollar deposited today, at $r = 100\%$, compounded continuously is worth e \$ after 1 year. Recall $e = 2.71828$

Growth rate and discount factor.

Relax the previous assumptions, so that we allow for any capital (A), any interest rate (r), any period (t), and any compounding frequency (m).

The future value of the capital A, is then obtained using this formula,

$$V(m) = A\left(1 + \frac{r}{m}\right)^{mt}.$$

we will show below that ~~this~~ as $m \to \infty$, $V(m) \to Ae^{rt}$, where e^{rt} represents the growth rate.

~~Proof: Let $w = \frac{m}{r}$~~

$$\cancel{\lim_{m\to\infty} V(m) = \lim_{m\to\infty} V(m)}$$

Proof: let $w = \frac{m}{r}$

Then $\lim\limits_{m \to \infty} V(m) = \cancel{\lim\limits_{w \to \infty} V(w)\left(1 + \frac{1}{w}\right)^{w}}$

$= \lim\limits_{w \to \infty} V(w)$ where $V(w) = A\left[\left(1 + \frac{1}{w}\right)^{w}\right]^{rt}$

Note $\frac{r}{m} = \frac{1}{\frac{m}{r}}$, but $\frac{m}{r} = w$

$\Rightarrow \frac{r}{m} = \frac{1}{w}$,

$mt = mt\frac{r}{r}$, since $\frac{m}{r} = w$

the $mt = mt\frac{r}{r} = wrt$.

using the above two results, we obtain $V(w) = A\left[\left(1 + \frac{1}{w}\right)^{w}\right]^{rt}$

~~since w~~

as $m \to \infty$ $w \to \infty$.

$$\lim_{w \to \infty} V(w) = A\Bigg[\underbrace{\left(1 + \frac{1}{w}\right)^{w}}_{e}\Bigg]^{rt} = Ae^{rt}$$

The future value of the Capital A, when Continuous compounding is applied, is equal to Ae^{rt}. e^{rt} is thus the growth rate.

Discount factor: we use it to find how much ~~the~~ a future value is worth today.

$$V = Ae^{rt}.$$

~~pre~~ multiply both Sides by e^{-rt},

$$Ve^{-rt} = Ae^{rt}e^{-rt}$$
$$= Ae^{(rt-rt)} = Ae^{0} = A$$

Thus the present value of V is equal to V times the discount factor (e^{-rt})

$$A = Ve^{-rt}$$

present value

future value

discount factor.

Total differential

$z = f(x, y)$

total differential of z is given by

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

dz: total change in z.

$\frac{\partial z}{\partial x}$: rate of change of z w.r.t x

dx: change in x

$\frac{\partial z}{\partial y}$: rat of change of z w.r.t y

dy: change in y

Example 1: $z = 5x_1^2 + 3y$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$dz = 10x \, dx + 3 \, dy$$

Example 2: $y = 3x_1^2 + x_1 x_2^2$

$$dy = (6x_1 + x_2^2) dx_1 + (2x_1 x_2) dx_2$$

Example 3 $\quad y = \dfrac{x_1 + x_2}{2x_1^2}$

$$dy = \frac{\partial y}{\partial x_1} dx_1 + \frac{\partial y}{\partial x_2} dx_2$$

$$= \left[\frac{2x_1^2 - 4x_1(x_1 + x_2)}{4x_1^4}\right] dx_1$$

$$+ \left(\frac{1}{2x_1^2}\right) dx_2 .$$

Total derivative:

$y = f(x, w)$ (y depends on x and w)

$x = g(w)$ (x depends on w)

$\frac{\partial y}{\partial w}$ ~~g~~

→ one: direct impact of w on y

→ two: indirect impact of w on y through x, (since x ~~do~~ changes as w changes, ~~g~~ and when x changes, y changes)

$$\begin{tikzcd} w \arrow[r, "\text{effects}"] \arrow[rr, bend right, "\text{effects directly}"'] & x \arrow[r, "\text{which in turns effects}"] & y \end{tikzcd}$$

$\overline{TD}$ deriving the formula:

Start with total differential

$$dy = \frac{\partial y}{\partial x} dx + \frac{\partial y}{\partial w} dw$$

divide both sides by dw

$$\frac{dy}{dw} = \frac{\partial y}{\partial x} \frac{dx}{dw} + \frac{\partial y}{\partial w} \cancel{\frac{dw}{dw}}$$

Thus total derivative $\frac{\partial y}{\partial w} \approx \frac{dy}{dw}$ (when dw is small).

$$\frac{\partial y}{\partial w} = \frac{\partial y}{\partial x} \frac{\partial x}{\partial w} + \frac{\partial y}{\partial w}$$

↳ total derivative

↳ partial derivative (holding x constant)

Example: ~~x~~ consider $y = 3x - w^2$
and $x = 2w^2 + w + 4$

$$\frac{\partial y}{\partial w} = \frac{\partial y}{\partial x}\frac{\partial x}{\partial w} + \frac{\partial y}{\partial w}$$

$$= 3(4w+1) + (-2w) = 10w + 3.$$

Example: Consider the production function

$$Q = f(K, L, t)$$

where K, L, t are capital, Labour and t respectively

Thus three factors affects production in our example: Capital, Labor and time.

However $K = f(t)$
$L = f(t)$, meaning that also capital and Labor ~~x~~ changes over time.

Thus the total impact of a change of time on production s comes from 3 sources

- → time directly affects production (Q)
- → time affects capital, which in turn affects Q.
- → ~~Labo~~ time affects Labour, which in turn affects Q.

$$\frac{\partial Q}{\partial t} = \frac{\partial Q}{\partial K}\frac{\partial K}{\partial t} + \frac{\partial Q}{\partial L}\cdot\frac{\partial L}{\partial t} + \frac{\partial Q}{\partial t}$$

<u>assume</u>

$$\begin{cases} Q = K^{\alpha} L^{\beta} t^{\gamma} \\ K = \sqrt{t} \\ L = 2\sqrt{t} \end{cases}$$

$$\frac{\partial Q}{\partial t} = \underbrace{\alpha K^{\alpha-1} L^{\beta} t^{\gamma}}_{\frac{\partial Q}{\partial K}} \underbrace{\left(\frac{1}{2} t^{-\frac{1}{2}}\right)}_{\frac{\partial K}{\partial t}}$$

$$+ \underbrace{\beta K^{\alpha} L^{\beta-1} t^{\gamma}}_{\frac{\partial Q}{\partial L}} \underbrace{\left(\frac{2}{2} t^{-\frac{1}{2}}\right)}_{\frac{\partial L}{\partial t}}$$

$$+ \underbrace{\gamma K^{\alpha} L^{\beta} t^{\gamma-1}}_{\frac{\partial Q}{\partial t}}$$

partial derivative holding L, and K constant.

Implicit function derivative,

The usual function that we use are **explicit** functions, where we isolate output (y) in terms of input(s) $(x_1, x_2, \ldots)$

$$y = f(x_1, x_2)$$

$$y = f(x)$$

Implicit functions on the other hands combines output and inputs together on the left hand side (without the requirement to isolate the output (y)).

$$F(x) = y - f(x) = 0$$

we use $F(x)$ to denote Implicit functions and we use $f(x)$ to denote explicit functions.

~~Example~~ $\cancel{y = \frac{\sqrt{x}}{2}}$

Example: $y = \frac{\sqrt{x}}{2}$ explicit.

$\underbrace{2y - \sqrt{x}}_{F(x)} = 0$ implicit.

Implicit function derivative

$$\frac{\partial y}{\partial x} = - \frac{\partial F(x)/\partial x}{\partial F(x)/\partial y}$$

or using a more compact notation,

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} \qquad F_x = \partial F(x)/\partial x \qquad F_y = \partial F(x)/\partial y$$

Using the implicit function $2y - \sqrt{x} = 0$ we can obtain $\frac{\partial y}{\partial x}$:

$$\frac{\partial y}{\partial x} = -\frac{\partial(2y-\sqrt{x})/\partial x}{\partial(2y-\sqrt{x})/\partial y} = -\frac{\left(-\frac{1}{2}x^{-1/2}\right)}{2}$$

$$= \frac{1}{4x^{1/2}}.$$

using the explicit function: $\frac{\partial y}{\partial x} = \frac{\partial\left(\frac{\sqrt{x}}{2}\right)}{\partial x} = \frac{1}{4}x^{-1/2} = \frac{1}{4x^{1/2}}$