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1. SYSTEM THEORY

1.1 Definitions and Fundamentals

Signal

A signal is any function that contains time t as an independent variable. Typical examples of signals are the position and velocity of a moving body, e.g. a ship. The symbols used for signals are commonly lower-case letters of the Greek or Latin alphabet. In classical control theory signals are only functions of time. However, there has been recently a growing interest for distributed signals, i.e. signals that contain more than one independent variables. For example, a beam supported at one end is described by signals that depend on both time and the distance, x , from the support point.

System

A confined space that accepts input signals (or excitations) and generates output signals (or responses) is a system. Typical examples of engineering systems include electric networks, internal combustion engines, turbines etc. Commonly, a system can be subdivided to subsystems with respect to its complexity. A marine engine, for example, includes its turbocharger, which in turn includes the turbine and the air compressor, the cylinders, the crankshaft etc. All these subsystems may be regarded as systems with their own inputs and outputs. However, the engine is also a subsystem of a higher complexity system i.e. the propulsion plant or the vessel.

Recently the concept of system has been generalized in order to include, except technological or physical systems, biological systems, economical systems, administrative systems etc. In correspondence the concept of signals has been expanded in order to include inputs and outputs to such systems.

Automatic control and feedback

The concept of control is directly related to that of *feedback*. Specifically, Wiener defines automatic control as the method of controlling a system by applying as inputs the system's past responses. A similar description is given by the IEEE (Institute of Electrical and

Electronics Engineers): An automatic control system, such as the cruise control system in automobiles, implements in the controller a decision process, also called the control law, that dictates the appropriate control actions to be taken for the speed to be maintained within acceptable tolerances. These decisions are taken based on how different the actual speed is from the desired, called the error, and on the knowledge of the car's response to fuel increases and decreases. This knowledge is typically captured in a mathematical model. Information about the actual speed is fed back to the controller by sensors, and the control decisions are implemented via a device, the actuator, that increases or decreases the fuel flow to the engine. This idea is also depicted in the following block diagram.

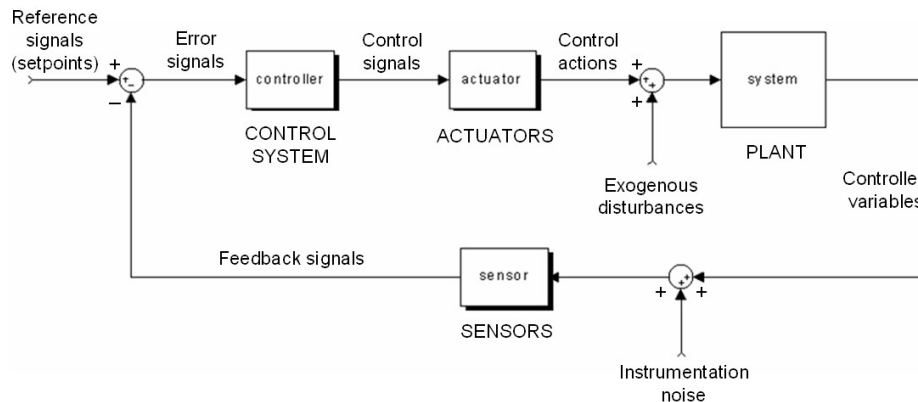


Figure 1-1: Structure of system with automatic feedback control.

The use of block diagrams is wide in the control community and is not limited to demonstration of functionality as in the figure above. In the contrary, it is connected with mathematical properties and operations. However, coming back to Figure 1-1, it can be seen that an automatic feedback control system includes a plant (controlled system) and a control system. The control system is composed of a controller, sensor(s) and actuator(s). When the feedback path (i.e. the line supplying information on the plant status through the sensors) is broken intentionally or unintentionally the system is called open-loop; otherwise, it is a closed-loop system.

An example of a closed-loop system is the automatic course-keeping system of a surface ocean vessel. This system includes all the components shown in Figure 1-1. The plant in this case is the vessel. The control system consists of the autopilot unit (controller), the steering gear (actuator) and the compass and GPS receiver (sensors). The controlled variable is the vessel's course and the control action is the rudder deflection from the centerline. Exogenous disturbances are present in the form of wind, wave and current generated forces and torques. The effect of the disturbances is demonstrated as a deviation of the actual heading of the vessel from the desired one, entered by the human operator as a waypoint (i.e. setpoint). In the case that either the waypoint is modified or excessive deviation is introduced to the controlled variable, as an effect of disturbance presence, the autopilot unit (controller) issues in response rudder position command signals (control signals) that drive the steering gear to appropriately adjust the rudder. The feedback signals

are the positioning information provided by the GPS receiver and the heading indication provided by the compass. Both have additive noise which leads to more or less inaccurate readings; also, especially in the case of the GPS receiver, significant and in general variable delay between the actual occurrence of a change and its indication is experienced.

Before embarking on the methods to develop a control system, a clear understanding of the behavior and analysis of the response of the plant is needed. This is commonly achieved by developing a mathematical model of the system. The type and the complexity of such a model, though, depend on various salient features of the plant in hand. One of the most important is whether the plant is static or dynamic. A static system is one for which the output signal values at any given time instant can be determined by employing exclusively the input signal values at the exact same time instant. In effect, static systems are commonly described by algebraic equations of the following general form.

$$\bar{y} = f(\bar{u}) \quad (1.1)$$

In the above, $\bar{u}$ and $\bar{y}$ are the input and output signals, respectively, conveniently packed in vectors.

Another class of systems, though, which is usually the case in feedback control theory and engineering, are dynamic systems. A dynamic system is described by a number of *dynamic* equations, i.e. integro-differential equations including at least time as an independent variable. Specifically, the description of a dynamic system must encompass all the constraint equations as well as the full set of initial, or in some cases, final conditions required for the determination of a special solution of the integro-differential equations. A deeper insight in this definition as well as the fundamental description methods for linear and time-invariant dynamic systems is presented in the remainder of this chapter.

1.2 The Laplace Transform

The Laplace transform is a linear integral transform that is defined as follows for a signal $x(t)$.

$$X(s) = \mathcal{L}\{x(t)\} = \int_0^{\infty} x(t) \cdot e^{-st} \cdot dt \quad (1.2)$$

The (one-sided) Laplace transform is used extensively for analyses of either signals or systems in the complex frequency domain, $s = a + j\omega$. The complex frequency comes as generalization of the (natural) frequency as defined especially for periodic signals, e.g. sinusoidal signals or sums of such. This is manifested also by the fact that the Fourier transformation of any signal can be derived by substituting $s = j\omega$ in its Laplace transformation or equivalently setting $a = 0$. These remarks are investigated in the following example.

Consider the signal:

$$x(t) = \begin{cases} \exp(a_0 t) \cdot \sin(\omega_0 t), & t \geq 0 \\ 0, & t < 0 \end{cases}$$

The Laplace transform is calculated by the definition as follows:

$$X(s) = \int_0^{\infty} \exp(a_0 t) \cdot \sin(\omega_0 t) \cdot e^{-st} \cdot dt = \int_0^{\infty} e^{a_0 t} \cdot \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \cdot e^{-st} \cdot dt$$

In the above, use of a well known identity for the calculation of the sine signals has been made. By substituting $s_0 = a_0 + j\omega_0$ and taking into account that $s_0^* = a_0 - j\omega_0$, the following is obtained.

$$X(s) = \frac{1}{2j} \cdot \int_0^{\infty} e^{(s_0^* - s)t} dt - \frac{1}{2j} \cdot \int_0^{\infty} e^{(s_0 - s)t} dt = \frac{1}{2j} \cdot \left[-\frac{1}{s_0 - s} + \frac{1}{s_0^* - s} \right] = \frac{\omega_0}{(s - a_0)^2 + \omega_0^2}$$

It is evident that signal $x(t)$ is a sine wave of angular frequency ω_0 which is either attenuating or growing, depending on the value of constant a_0 . This fact is reflected to the Laplace transformation of $x(t)$, i.e. $X(s)$. Indeed, the denominator polynomial of $X(s)$, $p_c(s) = (s - a_0)^2 + \omega_0^2$ obtains the following roots.

$$(s - a_0)^2 = -\omega_0^2 \Rightarrow s - a_0 = \pm j\omega_0 \Rightarrow s = s_0 \text{ or } s = s_0^*$$

In the case that the sine wave is growing it holds that $\text{Re}(s_0) = \text{Re}(s_0^*) = a_0 < 0$, whilst in the case that it is growing $a_0 > 0$. Finally, in the case that $a_0 = 0$, evidently it holds that $\text{Re}(s_0) = \text{Re}(s_0^*) = 0$.
 $x(t) = \sin(\omega_0 t), t \geq 0$.

Some of the most important properties of the Laplace transform are given below.

1) Linearity:

$$\left. \begin{array}{l} X_1(s) = \mathcal{L}\{x_1(t)\} \\ X_2(s) = \mathcal{L}\{x_2(t)\} \\ k_1, k_2 \in \mathbb{R} \end{array} \right\} \Rightarrow k_1 \cdot X_1(s) + k_2 \cdot X_2(s) = \mathcal{L}\{k_1 \cdot x_1(t) + k_2 \cdot x_2(t)\} \quad (1.3)$$

2) Differentiation wrt time:

$$\mathcal{L}\{\dot{x}(t)\} = sX(s) - x(t=0) \quad (1.4)$$

3) Integration wrt time:

$$\left. \begin{aligned} \mathcal{L}\{x(t)\} &= X(s) \\ y(t) &= \int x(\xi) d\xi \end{aligned} \right\} \Rightarrow \mathcal{L}\left\{\int_0^t x(\xi) d\xi\right\} = \frac{X(s)}{s} \text{ and } \mathcal{L}\left\{\int_{-\infty}^t x(\xi) d\xi\right\} = \frac{X(s)}{s} + \frac{x(t=0)}{s} \quad (1.5)$$

4) Initial and Final value theorems:

$$\mathcal{L}\{x(t)\} = X(s) \Rightarrow \begin{cases} \lim_{t \rightarrow 0} (x(t)) = \lim_{s \rightarrow \infty} (sX(s)) \\ \lim_{t \rightarrow \infty} (x(t)) = \lim_{s \rightarrow 0} (sX(s)) \end{cases} \quad (1.6)$$

5) Time and Complex Frequency Shift:

$$\left. \begin{aligned} X(s) &= \mathcal{L}\{x(t)\}, x(\forall t < 0) = 0 \\ t_0 &\in \mathbb{R}_+, s_0 \in \mathbb{C} \end{aligned} \right\} \Rightarrow \begin{cases} \mathcal{L}\{x(t-t_0)\} = \exp(-st_0) \cdot X(s) \\ \mathcal{L}\{e^{s_0 t} \cdot x(t)\} = X(s-s_0) \end{cases} \quad (1.7)$$

6) Theorem of Convolution:

$$\left. \begin{aligned} X_1(s) &= \mathcal{L}\{x_1(t)\} \\ X_2(s) &= \mathcal{L}\{x_2(t)\} \\ x(t) &= x_1(t) * x_2(t) \triangleq \int_{-\infty}^{+\infty} x_1(v) \cdot x_2(t-v) \cdot dv \end{aligned} \right\} \Rightarrow X(s) = \mathcal{L}\{x(t)\} = X_1(s) \cdot X_2(s) \quad (1.8)$$

NOTE: The signal $x(t)$ resulting from a convolution operation must be **causal**; i.e. it may not depend on the values of signals $x_1(t_1)$ and $x_2(t_2)$ that will occur at time instants t_1 and t_2 , respectively, later than t . In effect, the limits of the convolutional integral in (1.8) must be bounded as follows.

$$\left. \begin{aligned} t_1 &= v \leq t \\ t_2 &= t-v \leq t \Rightarrow v \geq 0 \end{aligned} \right\} \Rightarrow x_1(t) * x_2(t) = \int_0^t x_1(v) \cdot x_2(t-v) \cdot dv$$

Furthermore, the convolution operation, for which symbol $\otimes$ is also used in some texts, obtains the *commutative*, *associative* and *distributive* with respect to addition properties. The *identity element* of convolution is the Dirac delta signal defined in the next section.

The proof of the above Laplace transform properties as well as various other lemmas of the transform can be sought after in either ordinary differential equations or introductory control systems literature.

1.3 Linear Time Invariant Systems

Definition of LTI systems

A category of dynamic systems of high importance in feedback control theory is *Linear Time Invariant* (LTI) systems. Such systems are described by linear differential equations with constant coefficients. The general form of such differential equations is given below for a Single-Input, Single-Output (SISO) system.

$$\begin{aligned} \left(a_n \cdot \frac{d^n}{dt^n} + a_{n-1} \cdot \frac{d^{n-1}}{dt^{n-1}} + \dots + a_1 \cdot \frac{d}{dt} + a_0 \right) \cdot y(t) = \\ = \left(b_m \cdot \frac{d^m}{dt^m} + b_{m-1} \cdot \frac{d^{m-1}}{dt^{m-1}} + \dots + b_1 \cdot \frac{d}{dt} + b_0 \right) \cdot u(t) \end{aligned} \quad (1.9)$$

In the above, $y(t)$ and $u(t)$ are the input and output signal, respectively, of the system. The initial conditions required are those for the up to the $(n-1)$ -th derivative wrt time of the output signal $y(t)$ at some initial time instant t_0 (commonly t_0 is set to zero by an appropriate translation of the time axis).

$$\left. \frac{d^v}{dt^v} y(t) \right|_{t=t_0} = y_0^{(v)}, \quad v = 0, 1, \dots, (n-1) \quad (1.10)$$

In practically realizable systems, i.e. systems that can be implemented by interconnecting electrical (e.g. induction coils, capacitors, resistors etc.) or mechanical (e.g. masses, springs, dampers etc.) elements, it holds that $m \leq n$; this assumption will be observed in the following of this text as well.

Order of an LTI system

The *order* of any LTI system is integer n , i.e. the order of its differential equation.

Transfer function of an LTI system

Considering a SISO-LTI system as in equation (1.9) with **zero initial conditions** in (1.10), by consecutively applying Laplace transform property (1.3) and then property (1.4), the following **algebraic** relationship between the Laplace transformations of the output and input signal is obtained.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_m \cdot s^m + b_{m-1} \cdot s^{m-1} + \dots + b_1 \cdot s + b_0}{a_n \cdot s^n + a_{n-1} \cdot s^{n-1} + \dots + a_1 \cdot s + a_0}, \quad s \in \mathbb{C}, a_n \neq 0 \quad (1.11)$$

The complex polynomial ratio $H(s)$ is the system's transfer function. Transfer functions are used extensively in control theory. Commonly, the polynomial coefficients are normalized with respect to $a_n \neq 0$.

Characteristic polynomial of an LTI system

The polynomial $p_c(s) = a_n \cdot s^n + a_{n-1} \cdot s^{n-1} + \dots + a_1 \cdot s + a_0$ appearing at the denominator of the transfer function of a SISO-LTI system, $H(s)$, is the characteristic polynomial of the system. If the transfer function is normalized with respect to $a_n \neq 0$, it follows that $a_n = 1$. In the sequel, normalization will be assumed.

System poles

The n distinct or multiple complex roots of the characteristic polynomial $p_c(s)$ of an LTI system are the system poles. In effect, an LTI system's poles are calculated as roots of the equation:

$$p_c(s) = s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0 = 0 \quad (1.12)$$

In the above, the polynomial coefficients are assumed normalized in order that the highest order coefficient is equal to one.

Otherwise it is noted that if all coefficients of a polynomial are *purely real* then the roots are either also purely *real* or form pairs of *complex conjugate* numbers.

System zeros

The m distinct or multiple complex roots of the polynomial appearing at the numerator of the transfer function, $H(s)$, of an LTI system are the system zeros.

System impulse response

The response of an LTI system when the input is the Dirac delta signal and the initial conditions are set to zero is the impulse response. In effect, the impulse response, $h(t)$, can be calculated as the solution of the following differential equation.

$$\begin{aligned} \left(a_n \cdot \frac{d^n}{dt^n} + a_{n-1} \cdot \frac{d^{n-1}}{dt^{n-1}} + \dots + a_1 \cdot \frac{d}{dt} + a_0 \right) \cdot h(t) = \\ = \left(b_m \cdot \frac{d^m}{dt^m} + b_{m-1} \cdot \frac{d^{m-1}}{dt^{m-1}} + \dots + b_1 \cdot \frac{d}{dt} + b_0 \right) \cdot \delta(t) \end{aligned}$$

In the above, $\delta(t)$ stands for the Dirac delta signal, referred to also as impulse function or signal. The Dirac signal is defined on the basis of the following properties:

$$1) \int_{-\infty}^{+\infty} \delta(t) dt = 1 \quad (1.13)$$

$$2) \quad \delta(t) = \begin{cases} 0, & t \neq 0 \\ \text{undefined}, & t = 0 \end{cases} \quad (1.14)$$

By using the above definition properties the following properties can be derived.

$$3) \quad \int_{-\infty}^{+\infty} \delta(t) f(t) dt = f(0) \quad (1.15)$$

$$4) \quad \Delta(s) = \mathcal{L}\{\delta(t)\} = \int_0^{+\infty} \delta(t) e^{-st} dt = 1 \quad (1.16),$$

By using (1.16) it can be shown that the transfer function and the impulse response are connected through the Laplace transform.

$$\mathcal{L}\{h(t)\} = H(s) \cdot \mathcal{L}\{\delta(t)\} = H(s) \cdot \Delta(s) = H(s) \quad (1.17)$$

In effect, by employing the Theorem of Convolution (1.8), the time-domain input-output relationship equivalent to the one given in (1.11) in the complex frequency domain is derived.

$$\begin{aligned} Y(s) &= H(s) \cdot U(s) \\ &\Updownarrow \\ y(t) &= h(t) * u(t) = \int_{-\infty}^{+\infty} h(v) \cdot u(t-v) \cdot dv = \int_0^t h(v) \cdot u(t-v) \cdot dv \end{aligned} \quad (1.18)$$

The interval of the convolutional integral is also reduced from $(-\infty, +\infty)$ to $(0, t)$ on the basis of *causality*. Therefore, the output signal value $y(t)$ at time t depends on input signal values $u(v)$ occurring at time instants not later than t .

Conclusively, the determination of an LTI system's transfer function can be done on the basis of equation (1.17), i.e. by Laplace transformation of the system's impulse response. This is why it is interesting to determine the impulse response, $h(t)$, of the system. The direct method to determine $h(t)$ is to excite the system by the Dirac signal, $\delta(t)$, and record the response. However, in many practical cases this is not feasible, especially because implementation of the Dirac signal $\delta(t)$ is not without problems or approximations.

Alternatively, the impulse response $h(t)$ and, consequently, the transfer function of a system, $H(s)$, can be calculated using its *step response*, $y_{step}(t)$. The step response is the output of the system when the input signal is the *unit step signal* (also referred to as Heaviside signal or function), $u_{step}(t)$, defined as follows.

$$u_{step}(t) = \begin{cases} 1, & t > 0 \\ 0, & t \leq 0 \end{cases} \Rightarrow u_{step}(t) = \int_{-\infty}^t \delta(v) dv \Leftrightarrow U_{step}(s) = \mathcal{L}\{u_{step}(t)\} = \frac{\Delta(s)}{s} = \frac{1}{s} \quad (1.19)$$

In effect, on the basis of property (1.4) it can be shown that:

$$h(t) = \frac{d}{dt} y_{step}(t) \Leftrightarrow H(s) = s \cdot Y_{step}(s) = s \cdot \mathcal{L}\{y_{step}(t)\} \quad (1.20)$$

Furthermore, in the case that the transfer function of an LTI system is known and the impulse response is sought after, the technique of *partial fraction expansion* is employed. In that end, the following fact is needed:

$$\frac{1}{s - \lambda} = \mathcal{L}\{e^{\lambda t}\}, \quad \lambda \in \mathbb{C}, \quad t \geq 0 \quad (1.21)$$

At first, assume that we are presented with an n -th order system obtaining n distinct, purely real poles; this means that the characteristic polynomial has no multiple roots and no complex roots. Then, the characteristic polynomial can be rewritten in the following *factorized* form.

$$p_c(s) = \prod_{v=1}^n (s - \lambda_v) \quad (1.22)$$

In the above λ_v , $v=1, \dots, n$ are the poles of the system and as explained previously it has been assumed that $a_n = 1$. On the basis of the factorization of $p_c(s)$ the transfer function $H(s)$ may be expanded to a sum of fractions of the form shown in (1.21).

$$H(s) = \sum_{v=1}^n \frac{k_v}{s - \lambda_v} \quad (1.23)$$

In the above constants (coefficients) k_v , $v=1, \dots, n$ can be calculated by using the following relationship.

$$k_v = \lim_{s \rightarrow \lambda_v} [(s - \lambda_v) \cdot H(s)], \quad v=1, \dots, n \quad (1.24)$$

Based on the fact (1.21) and the expansion (1.23) the following expression is derived for the system's impulse response.

$$h(t) = \sum_{v=1}^n k_v \cdot \exp(\lambda_v t), \quad t \geq 0 \quad (1.25)$$

The partial fraction expansion can be employed also in cases for which the characteristic polynomial obtains multiple real or complex roots. Firstly, the case of a purely real pole, λ_0 , with multiplicity $l > 1$, is examined. Then, the factorised form of the characteristic polynomial of the system includes factor $(s - \lambda_0)^l$. Consequently, in the system's transfer function this factor appears in the denominator. Therefore, the partial function expansion of the following general form is needed.

$$p_c(s) = (s - \lambda_0)^l \prod_{v=1}^n (s - \lambda_v) \Rightarrow H(s) = \sum_{v=1}^n \frac{k_v}{s - \lambda_v} + \sum_{\xi=1}^l \frac{k_{0\xi}}{(s - \lambda_0)^\xi} \quad (1.26)$$

In the above, additional terms augment the expansion of $H(s)$ in (1.23) in order to encompass factor $(s - \lambda_0)^l$ of the characteristic polynomial. To obtain the values of constants $k_{0\xi}$ of the expansion, the following formula is used.

$$k_{0\xi} = \lim_{s \rightarrow \lambda_0} \left\{ \frac{1}{(l - \xi)!} \cdot \frac{d^{(l - \xi)}}{ds^{(l - \xi)}} \left[(s - \lambda_0)^l \cdot H(s) \right] \right\}, \quad \xi = 1, \dots, l \quad (1.27)$$

The inverse Laplace transform of each additional term in the expansion (1.26) is given by the following.

$$\mathcal{L}^{-1} \left\{ \frac{1}{(s - \lambda_0)^\xi} \right\} = \frac{1}{(\xi - 1)!} t^{(\xi - 1)} e^{\lambda_0 t} \quad (1.28)$$

The above comes as a generalization of (1.21). In conclusion, each additional term (1.26) corresponds to a term of the form (1.28) in the impulse response expression (1.25).

In the case that the characteristic polynomial obtains a pair of complex conjugate roots, λ_c and λ_c^* , the factorized form includes the factor:

$$(s - \lambda_c) \cdot (s - \lambda_c^*) = s^2 - 2 \cdot \text{Re}(\lambda_c) \cdot s + |\lambda_c|^2$$

In this case, formulas (1.21), (1.23) and (1.24) are still valid for each one of the poles λ_c and λ_c^* , separately. Furthermore, the following fact for the combined partial fraction expansion of this factor can be shown.

$$\frac{1}{s^2 - 2 \cdot \text{Re}(\lambda_c) \cdot s + |\lambda_c|^2} = \frac{k_c}{s - \lambda_c} + \frac{k_{c*}}{s - \lambda_c^*} \Rightarrow k_{c*} = (k_c)^* \quad (1.29)$$

Therefore, for calculating the constant k_{c*} corresponding to factor $(s - \lambda_c^*)$ it suffices to calculate constant k_c corresponding to factor $(s - \lambda_c)$.

1.4 Multivariable Systems and State-Space

State equations

State equations constitute a description in the time domain encompassing, besides LTI systems, other categories of systems such as time varying or nonlinear systems. A system described by state equations may in general accept more than one scalar input signals and

produce more than one scalar output signals, also referred to as measurements. If the number of scalar input signals is m and that of scalar output signals is p , the following column vectors are introduced for convenience in notation.

$$\bar{\mathbf{u}} = \mathbf{u}(t) = [u_1 \quad u_2 \quad \dots \quad u_m]^T \text{ and } \bar{\mathbf{y}} = \mathbf{y}(t) = [y_1 \quad y_2 \quad \dots \quad y_p]^T \quad (1.30)$$

In the disciplines of dynamic systems and control, the term *state* or *state vector* is used for a set of $n \geq 1$ variables (*state variables*) conveniently packed in a column vector as follows.

$$\bar{\mathbf{x}} = \mathbf{x}(t) = [x_1 \quad x_2 \quad \dots \quad x_n]^T \quad (1.31)$$

If the value of $\bar{\mathbf{x}}$ is known at some initial time instant t_0 , then it must be possible to determine the values of both $\bar{\mathbf{x}}$ and $\bar{\mathbf{y}}$ for times $t \geq t_0$ provided that: a) the values of $\bar{\mathbf{u}}$ are given and b) the mathematical law connecting the input, output and state vectors is known.

Commonly, the afore mentioned mathematical law is expressed as a set of n first-order differential equations with respect to time, i.e. dynamic equations. The dynamic equations connect the input vector with the state vector.

$$\dot{\mathbf{x}}(t) \triangleq \frac{d}{dt} \mathbf{x}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t) \quad (1.32)$$

The description of the system in state-space requires also a set of p **algebraic** output equations connecting the input and state vectors with the output one.

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \quad (1.33)$$

By employing (1.32) it is directly concluded that if the initial state of a system is given at a time instant t_0 then the state at any future time instant can be calculated as follows.

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^t \mathbf{f}(\mathbf{x}(\xi), \mathbf{u}(\xi), \xi) d\xi \quad (1.34)$$

Multivariable LTI systems

In the case of LTI systems equations (1.32) and (1.33) assume the following form.

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{u} \\ \mathbf{y} &= \mathbf{C} \cdot \mathbf{x} + \mathbf{D} \cdot \mathbf{u} \end{aligned} \quad (1.35)$$

In the above $\mathbf{A}$ is an $n \times n$ matrix, $\mathbf{B}$ an $n \times m$ matrix, $\mathbf{C}$ a $p \times n$ matrix and $\mathbf{D}$ a $p \times m$ matrix.

An input-output relationship can be deduced for an LTI system described in state-space by the state and output equations of the form (1.35). This relationship comes as a natural extension of the transfer function concept already introduced for Single-Input, Single-Output (SISO) LTI systems to the case of MIMO (Multi-Input, Multi-Output) LTI systems. To obtain it the Laplace transform is applied to the time-domain expression (1.35) by employing property (1.4).

$$\begin{aligned} s \cdot X(s) - x(0) &= A \cdot X(s) + B \cdot U(s) \\ Y(s) &= C \cdot X(s) + D \cdot U(s) \end{aligned} \quad (1.36)$$

In the above, it has been assumed that $t_0=0$ for convenience and without any loss of generality. The complex frequency domain algebraic equations (1.36) can be solved to yield the following.

$$\begin{aligned} X(s) &= (s \cdot I_n - A)^{-1} \cdot B \cdot U(s) + (s \cdot I_n - A)^{-1} \cdot x(0) \\ Y(s) &= (C \cdot (s \cdot I_n - A)^{-1} \cdot B + D) \cdot U(s) + C \cdot (s \cdot I_n - A)^{-1} \cdot x(0) \end{aligned} \quad (1.37)$$

In the above, I_n stands for the $n \times n$ identity matrix. Formulas (1.37) allow the analytic calculation of any LTI system's response and are referred to as full response equations in literature. The following special cases are investigated.

$\alpha)$ Zero initial state response, i.e. $x(0)=0$:

$$Y(s) = (C \cdot (sI - A)^{-1} \cdot B + D) \cdot U(s) \quad (1.38)$$

$\beta)$ Zero input response, i.e. $u=0$:

$$Y(s) = C \cdot (sI - A)^{-1} \cdot x(0) \quad (1.39)$$

In order to obtain the response in the time domain, the following formula of the Laplace transform theory is needed.

$$(s \cdot I_n - A)^{-1} = \mathcal{L}\{e^{At}\}, \quad t > 0 \quad (1.40)$$

The above is the matrix generalization of formula (1.21). Expression e^{At} , also referred to by the term system transient matrix, is defined as an $n \times n$ matrix with the following properties.

$$e^{At} = I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \dots \Leftrightarrow \frac{d}{dt} e^{At} = A \cdot e^{At} \quad (1.41)$$

By using (1.41), the full response equation in the time domain is obtained.

$$x(t) = \int_0^t e^{A_v} \cdot B u(t-v) \cdot dv + e^{A_t} \cdot x(0), t > 0 \quad (1.42)$$

The following matrix version of the Theorem of Convolution has been employed in the above.

$$(sI - A)^{-1} \cdot B \cdot U(s) = \mathcal{L}\{e^{At} * Bu(t)\} = \mathcal{L}\left\{\int_0^t e^{Av} \cdot Bu(t-v) \cdot dv\right\} \quad (1.43)$$

Transfer function matrix

In the input-output relationship (1.38), the $p \times m$ matrix $H(s) = C \cdot (sI - A)^{-1} \cdot B + D$ is the system's transfer matrix.

It is straightforward to show that

$$(sI - A)^{-1} = |sI - A|^{-1} \cdot \text{adj}(sI - A)$$

In the above $|\bullet| = \det(\bullet)$ stands for the determinant of a matrix and $\text{adj}(\bullet)$ is the conjugate transpose of the matrix cofactors. In effect, the following conclusions are made.

- * The characteristic polynomial of an LTI system with description in state-space as in (1.35) is the determinant of matrix $(sI - A)$, i.e. $p_c(s) = |sI - A|$.
- * The poles of a system with the general form (1.35), i.e. the roots of $p_c(s) = |sI - A|$, are the eigenvalues of matrix A .

The above general remarks are now demonstrated in the special case of an LTI-SISO system, with a transfer function of the form below.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1} \cdot s^{n-1} + \dots + a_1 \cdot s + a_0} \quad (1.44)$$

It is noted here that although the system is SISO the state vector is of n dimension, i.e. equal to the order of the system. However, there are more than one choices of the state vector's components. The problem of choosing the components of the state vector in order to formulate a state-space description of an LTI system which is otherwise known only in the form of either a scalar transfer function or a transfer function matrix, is referred to as the *realization* problem. In the general case of MIMO-LTI systems is not solvable. However, in the case of LTI-SISO systems with transfer function of the form (1.44), a *minimal* description, i.e. a description that includes no redundant components can be deduced as follows.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1} \cdot s^{n-1} + \dots + a_1 \cdot s + a_0} = \frac{b_0}{s^n + a_{n-1} \cdot s^{n-1} + \dots + a_1 \cdot s + a_0} \cdot \frac{X_1(s)}{X_1(s)}$$

After introducing the auxiliary signal as above, the following differential equation is obtained in the time domain. Appropriate initial conditions of the form (1.10) for solving the differential equation must also be provided.

$$\left(\frac{d^n}{dt^n} + a_{n-1} \cdot \frac{d^{n-1}}{dt^{n-1}} + \dots + a_1 \cdot \frac{d}{dt} + a_0 \right) \cdot x_1(t) = u(t) \Leftrightarrow x_1^{(n)}(t) = u(t) - \sum_{v=0}^{n-1} a_v \cdot x_1^{(v)}(t) \quad (1.45)$$

$$\&$$

$$y(t) = b_0 \cdot x_1(t)$$

In the above $x_1^{(v)}(t)$ stands for the v -th order time derivative of the auxiliary signal $x_1(t)$. An appropriate state vector, x , is as follows.

$$x = \begin{bmatrix} x_1(t) & x_1^{(1)}(t) & \dots & x_1^{(n-1)}(t) \end{bmatrix}^T \quad (1.46)$$

For the components of x the following differential equations hold.

$$\dot{x}_v = x_{v+1}, \quad v = 1, \dots, (n-1) \quad \text{and} \quad \dot{x}_n = -\sum_{v=1}^n a_{(v-1)} \cdot x_v + u \quad (1.47)$$

If the above are arranged in matrix format the following equations are obtained.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \cdot u$$

$$y = \begin{bmatrix} b_0 & 0 & 0 & \dots & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} \quad (1.48)$$

This form of the state equations and especially of matrix A is also known as the *companion canonical form*.

1.5 Nonlinear Systems and Linearization

Consider a nonlinear system with m inputs, p outputs and n state variables with the following description in state space.

$$\begin{aligned}\dot{x} &= f(x, u) \\ y &= g(x, u)\end{aligned}\tag{1.49}$$

Significant progress has been achieved in recent years in the analysis and controller synthesis for a wide class of nonlinear systems, especially through the studies of Isidori and Kokotovic. However, the technique of linearization, i.e. the reduction to or approximation by a linear system is still extensively used.

The first step towards the linearization of a nonlinear system is to determine the equilibrium points. These points are defined by the following vector equation.

$$\dot{x} = \vec{0} \Leftrightarrow f(x_0, u_0) = \vec{0}\tag{1.50}$$

Linearization is then applied around some equilibrium point (x_0, u_0) by using the Taylor expansions for functions f and g :

$$\begin{aligned}f(x, u) &\approx f(x_0, u_0) + \tilde{f}'_x(x_0, u_0) \cdot (x - x_0) + \tilde{f}'_u(x_0, u_0) \cdot (u - u_0) \\ g(x, u) &\approx g(x_0, u_0) + \tilde{g}'_x(x_0, u_0) \cdot (x - x_0) + \tilde{g}'_u(x_0, u_0) \cdot (u - u_0)\end{aligned}\tag{1.51}$$

In the above, matrices $\tilde{f}'_x(x_0, u_0)$, $\tilde{f}'_u(x_0, u_0)$, $\tilde{g}'_x(x_0, u_0)$, $\tilde{g}'_u(x_0, u_0)$ with dimensions $n \times n$, $n \times m$, $p \times n$ and $p \times m$, respectively are defined as follows.

$$\tilde{f}'_x(x_0, u_0) = \left[\left. \frac{\partial}{\partial x_j} f_i(x, u) \right|_{\substack{x=x_0 \\ u=u_0}}, i, j = 1, \dots, n \right], \tilde{f}'_u(x_0, u_0) = \left[\left. \frac{\partial}{\partial u_j} f_i(x, u) \right|_{\substack{x=x_0 \\ u=u_0}}, \begin{matrix} i = 1, \dots, n \\ j = 1, \dots, m \end{matrix} \right]\tag{1.52}$$

$$\tilde{g}'_x(x_0, u_0) = \left[\left. \frac{\partial}{\partial x_j} g_i(x, u) \right|_{\substack{x=x_0 \\ u=u_0}}, \begin{matrix} i = 1, \dots, p \\ j = 1, \dots, n \end{matrix} \right], \tilde{g}'_u(x_0, u_0) = \left[\left. \frac{\partial}{\partial u_j} g_i(x, u) \right|_{\substack{x=x_0 \\ u=u_0}}, \begin{matrix} i = 1, \dots, p \\ j = 1, \dots, m \end{matrix} \right]\tag{1.53}$$

The following change of variables is then adopted.

$$\delta x = x - x_0, \delta u = u - u_0, \delta y = y - y_0 = y - g(x_0, u_0)\tag{1.54}$$

In effect, the following linearized equations are deduced for the nonlinear system (1.49) which is valid exclusively in the vicinity of the equilibrium point (x_0, u_0) .

$$\begin{aligned}
\delta \dot{x} &= \tilde{f}'_x(x_0, u_0) \cdot \delta x + \tilde{f}'_u(x_0, u_0) \cdot \delta u \\
\delta y &= \tilde{g}'_x(x_0, u_0) \cdot \delta x + \tilde{g}'_u(x_0, u_0) \cdot \delta u
\end{aligned} \tag{1.55}$$

Despite its virtues, the technique of linearization has also a major drawback. This drawback originates from the fact that the elements of matrices (1.52) and (1.53) depend on the equilibrium point in the vicinity of which they are calculated. Therefore, a *family* of linear systems is obtained in order to describe a single nonlinear system. In effect, the form of the results of the analysis and control scheme synthesis must be parameterized with respect to the equilibrium point. Furthermore, linearization cannot be used for nonlinear systems that do not demonstrate equilibrium points as well as systems whose behavior is of interest when operating away from any equilibrium points.

2. ANALYSIS OF LTI SYSTEMS

2.1 Block Diagrams

The block diagram of a system serves as a depiction of the interconnection structure of the various subsystems as well as the signals circulation and traffic. Each sub-block of the overall block diagram is defined by an input-output (I/O) relationship, which for LTI systems is commonly expressed in the complex frequency domain. For example consider the following transfer function.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{1}{s+1} \quad (2.1)$$

The corresponding block is shown below.

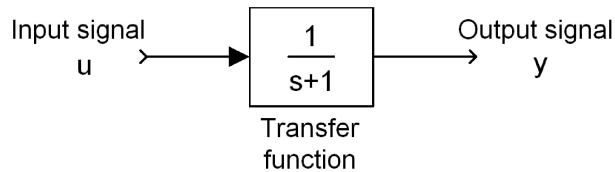


Figure 2-1: Block representing a scalar (SISO) transfer function.

Using transfer functions for the representation of the I/O relationships of the various blocks allows to conveniently obtain the overall transfer function of a system that comprises of more than blocks interconnected in cascade. For example, consider the system depicted in the following block diagram.

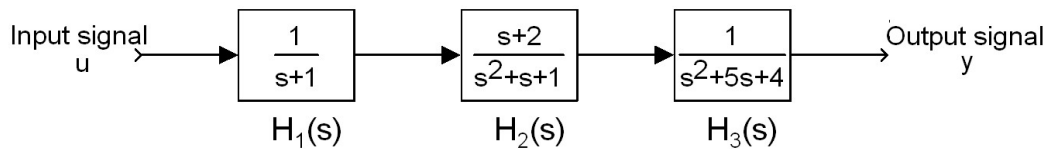


Figure 2-2: Cascade interconnection of scalar transfer functions.

The overall transfer function of the full, interconnected system is as follows.

$$H(s) = \frac{Y(s)}{U(s)} = H_1(s) \cdot H_2(s) \cdot H_3(s) = \frac{1}{s+1} \cdot \frac{s+2}{s^2+s+1} \cdot \frac{1}{s^2+5s+4} \quad (2.2)$$

Except the individual transfer function blocks, another common element encountered frequently in block diagrams is the adder (or summing junction). An adder produces as output the algebraic sum of all the input signals.

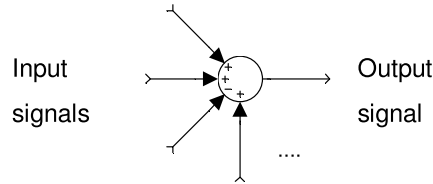


Figure 2-3: Summing junction used in block diagrams.

Furthermore, a system may be defined as either open-loop or closed-loop on the basis of its block diagram. Indeed, if at least one output signal of a system is fed as input to at least one of its own blocks, the system is called closed-loop; otherwise, it is called open-loop. A closed-loop system is shown in the block diagram below.

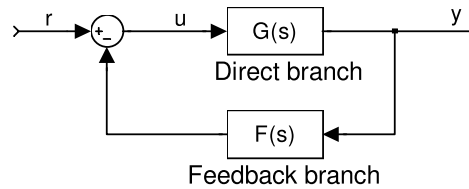


Figure 2-4: Block diagram of a typical closed-loop system.

The system transfer function of the closed-loop system shown above, i.e. the I/O relationship connecting the reference signal r to the output signal y is given below.

$$H(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s) \cdot F(s)} \quad (2.3)$$

By inspection of equation (2.3), it is evident that any closed-loop can be reduced to a single-block system, i.e. an equivalent open-loop system. The inverse is also possible; an open-loop system can also be converted to a closed-loop one with unity feedback branch (i.e. $F(s) \equiv 1$). Indeed, if $F(s) = 1$, by solving equation (2.3) with respect to $G(s)$, it is easily concluded that the direct (feedforward) branch must obtain the following transfer function.

$$G(s) = \frac{Y(s)}{U(s)} = \frac{H(s)}{1 - H(s)} \quad (2.4)$$

The case that the closed-loop system in Figure 2-4 is multivariable will be now examined, too. The dimensions of the input and reference vectors, u and r respectively, is assumed to be m and that for the output vector y , p . Evidently, transfer function matrices $G(s)$ and $F(s)$ must be of dimensions $p \times m$ and $m \times p$, respectively. Following the same method used for deriving the scalar case expression (2.4), it can be shown that the $p \times m$ closed-loop, transfer function matrix, $H(s)$, is given by the following.

$$Y(s) = H(s) \cdot U(s) \text{ where } H(s) = [I_p + G(s) \cdot F(s)]^{-1} \cdot G(s) \quad (2.5)$$

In the above, I_p stands for the identity $p \times p$ matrix. Equation (2.5) comes as a generalization of (2.3); it is reminded here though that commutivity in matrix multiplication does not hold.

2.2 Stability

Stability is a substantial structural characteristic of any dynamic system. Stability is not optional but compulsory and, therefore, must be guaranteed after the application of control to a system; otherwise, control must be employed to stabilize an unstable, open-loop system.

A working definition of stability is the following:

A system is stable if and only if it generates a bounded output when driven by any bounded input.

The above definition is generic and applicable to both linear as well as nonlinear systems. In the case of linear, time-invariant systems (LTI) it is possible to determine stability without taking into account the class of the input signals applied; actually, the characterization can be done if either the transfer function or the state-space description is available.

Consider an LTI system with description in state space as follows.

$$\begin{aligned} \dot{x} &= A \cdot x + B \cdot u \\ y &= C \cdot x + D \cdot u \end{aligned} \quad (2.6)$$

The system is said to be (*asymptotically*) stable iff under zero input conditions (i.e. $u(t) \equiv 0$), the following criteria are met.

$$\begin{aligned} \exists M (x(t=0)) \geq 0: \forall t \geq 0, \|x(t)\| < M \\ \lim_{t \rightarrow \infty} \|x(t)\| &= 0 \end{aligned} \quad (2.7)$$

It is evident that the value of the bound M for the (Euclidean) norm of the state vector depends on the actual values for the initial condition. In the case that only the first of the criteria in equation (2.7) is met the system is said to be *critically stable*.

Some further insight is gained in these definitions of asymptotically and critically stable systems by applying them to two specific cases of LTI systems. Consider two LTI systems, (S1) and (S2), with one inputs and two outputs each. For their state-space representation, as in equation (2.6), assume that matrices B and C are the same for both (S1) and (S2).

$$B^T = [0 \ 1] \quad \text{and} \quad C = I_2$$

Matrix A for both (S1) and (S2) is given below.

$$(S1): A = \begin{bmatrix} 0 & 1 \\ -1.04 & -0.4 \end{bmatrix} \quad \& \quad (S2): A = \begin{bmatrix} 0 & 1 \\ -0.09 & 0 \end{bmatrix} \quad (2.8)$$

In the figure below, the trajectories for both systems in state space are shown when the initial conditions are set to $x^T(t=0) = [10.0 \ 10.0]$ for both. It can be seen that the trajectories will be similar for any other value of the initial conditions. Both systems are stable as their trajectory remains in the interior of a disc as time elapses. For example, for the specific initial conditions value the radius of the disc can be set to $M = 40.0$. Furthermore, (S1) is asymptotically stable as the state vector's norm is converging to zero whilst (S2) is only critically stable as the state vector's norm, although remaining inside the disc with radius $M = 40.0$, it never decreases to zero (not even after infinite time).

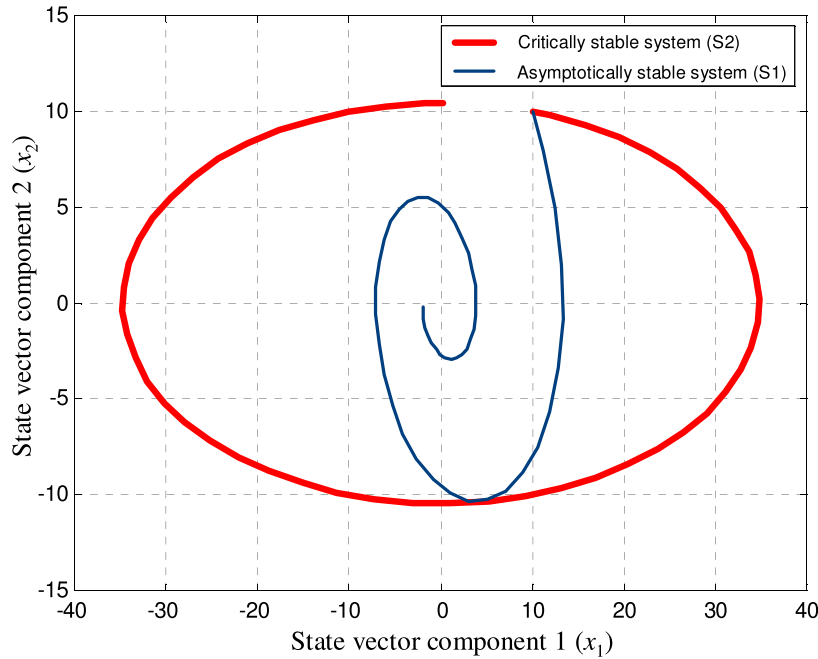


Figure 2-5: State-space trajectories of critically and asymptotically stable systems.

Equation (2.7) is associated with the definition of stability in the time domain. The lemma given below, allows stability characterization of any LTI system (SISO or MIMO), with transfer function matrix given below, in the complex frequency domain.

$$H(s) = C \cdot (sI - A)^{-1} \cdot B + D = C \cdot \frac{\text{adj}(sI - A)}{|sI - A|} \cdot B + D \quad (2.9)$$

The system above is asymptotically stable iff ***each and every one of its poles***, i.e. the roots of the system's characteristic polynomial $p_c(s) = |sI - A|$, has ***negative real parts***. In the case that at least one pair of conjugate complex poles have zero real parts (i.e. they are purely imaginary) the system is critically stable. It is noted here that the poles should be calculated as the *eigenvalues* of the system's matrix A; this enables to avoid issues raised by a potential loss of *controllability* or *observability* (examined in the next sections) and are of concern if the poles are calculated as roots of the denominator polynomial of the transfer function.

Returning to example (2.8), one may observe that matrix A for both systems (S1) and (S2) are in companion canonical form. In effect, their characteristic polynomial is calculated as follows.

$$(S1): p_c(s) = s^2 + 0.4s + 1.04 = (s + 0.2 + j) \cdot (s + 0.2 - j)$$

$$(S2): p_c(s) = s^2 + 0.0s + 0.09 = (s + 0.3j) \cdot (s - 0.3j)$$

In conclusion (S1) is asymptotically stable because both its poles obtain negative real part: $-0.2 < 0$. On the other hand, system (S2) is critically stable because it obtains poles with zero real part. Finally, consider the SISO system with the following transfer function.

$$H(s) = \frac{1}{s^3 + 0.5s^2 - s + 3} = \frac{1}{(s + 1.8803)(s - 0.6902 + 1.0579j)(s - 0.6902 - 1.0579j)} \quad (2.10)$$

This system is unstable because it obtains a pair of conjugate poles with positive real part: $0.6902 > 0$. The step response of the system is shown in the figure below. As can be seen the response cannot be bounded.

There is a criterion allowing the characterization of an ***unstable*** system by simple inspection of its characteristic polynomial in reduced form, with no need to determine its roots. The criterion states that any polynomial, the coefficients of which are *real* and exhibit at least one ***sign change***, is guaranteed to obtain at least one root in the *right-hand complex plane*. In effect, if the coefficients of the characteristic polynomial of an LTI system exhibit at least one sign change, the system is unstable. The converse, however, is not true in general; it is true in the case of 1st and 2nd order polynomials and systems, only.

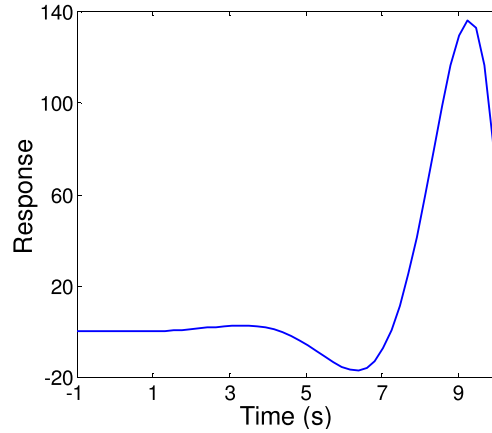


Figure 2-6: Step response of an unstable system.

Consider, for example, the characteristic polynomial of the transfer function in equation (2.10). This one exhibits two sign changes and therefore obtains roots in the right-hand complex plane. On the other hand, the following 3rd-order polynomial also obtains roots in the right-hand complex plane, i.e. with positive real part, although no coefficient sign changes occur.

$$p(s) = s^3 + 0.5s^2 + s + 3 = (s + 1.3694)(s - 0.4347 + 1.4148j)(s - 0.4347 - 1.4148j) \quad (2.11)$$

2.3 Controllability and Observability

The concepts of controllability and observability are fundamental in modern control system theory. For an LTI system with m inputs, p outputs and n states, controllability and observability may be investigated by using its state-space description.

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A} \cdot \mathbf{x} + \mathbf{B} \cdot \mathbf{u} \\ \mathbf{y} &= \mathbf{C} \cdot \mathbf{x} + \mathbf{D} \cdot \mathbf{u} \end{aligned} \quad (2.12)$$

The dimensions for matrix $\mathbf{A}$ are $n \times n$, for $\mathbf{B}$ $n \times m$, for $\mathbf{C}$ $p \times n$ and for $\mathbf{D}$ $p \times m$.

State controllability

State vector controllability is the situation where there exists a control action vector that can drive the state vector to any finite value in a finite time interval. The formal definition is provided below.

The state vector $\mathbf{x}$ of an LTI system as in equation (2.12) is controllable if there exists a piecewise continuous control signal $\mathbf{u}$ that can drive the state vector from any initial condition $\mathbf{x}_0$ to any final value $\mathbf{x}_f$ in a finite time interval $(t_f - t_0)$.

In the case that system's matrix A in equation (2.12) is diagonal, one can easily see that, by the definition, matrix B must contain no rows that are identically zero in order for the LTI system to be controllable.

In the general case that A is not diagonal, the criterion to check controllability of the state vector for an LTI system, as in equation (2.12), is based on the rank of the **controllability matrix**, $\mathbf{C}^\dagger$, defined below.

$$\mathbf{C}^\dagger = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \quad \dots \quad \mathbf{A}^{(n-1)}\mathbf{B}] \quad (2.13)$$

If the rank of $\mathbf{C}^\dagger$ is no less than n then the system is controllable, otherwise not. It is reminded here that the *rank* of a matrix is defined as the dimension of the vector subspace spanned by either its row or its column vectors. In effect, for an LTI system as in equation (2.12) to be controllable, its controllability matrix $\mathbf{C}^\dagger$ must contain n linearly independent row vectors or n linearly independent column vectors.

State observability

State vector observability is the situation where the initial value, at any time instant, of the state vector may be determined solely on the basis of the recorded inputs (controls) and outputs (measurements). The formal definition is provided below.

The state vector $\mathbf{x}$ of an LTI system as in equation (2.12) is observable in the time interval $[t_0, t_f]$ if it is possible to determine the initial value of the state vector, $\mathbf{x}_0 = \mathbf{x}(t_0)$, when presented with the values of the control signal $\mathbf{u}$ and output signal $\mathbf{y}$ in the finite time interval $[t_0, t_f]$.

In the case that system's matrix A in equation (2.12) is diagonal, one can easily see that, by the definition, matrix C must contain no columns that are identically zero in order for the LTI system to be observable.

In the general case that A is not diagonal, the criterion to check observability of the state vector for an LTI system, as in equation (2.12), is based on the rank of the **observability matrix**, $\mathbf{O}^\dagger$, defined below.

$$\mathbf{O}^\dagger = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{(n-1)} \end{bmatrix} \quad (2.14)$$

If the rank of $\mathbf{O}^\dagger$ is no less than n then the system is observable, otherwise not. In effect, for an LTI system as in equation (2.12) to be observable, its observability matrix $\mathbf{O}^\dagger$ must contain n linearly independent row vectors or n linearly independent column vectors.

Zero-pole cancellation

An important lemma connecting the transfer function (or the transfer function matrix) with the concepts of controllability and observability is given in this section.

If the *same* system pole (at least one) is cancelled by a zero in *each and every one* of the transfer function matrix elements, then the system in hand is either uncontrollable or unobservable or both.

The lemma above is now investigated in depth for the case of SISO systems. Consider an n -th order, SISO system with description in state space as follows.

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A} \cdot \mathbf{x} + \mathbf{b} \cdot u \\ y &= \mathbf{c}^T \cdot \mathbf{x}\end{aligned}\tag{2.15}$$

In the above, $\mathbf{b}$ and $\mathbf{c}$ are n -dimensioned column vectors. For simplicity, assume that $\mathbf{A}$ in equation (2.15) is diagonal.

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}\tag{2.16}$$

The (scalar) transfer function of system is, then, given by the following.

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{c}^T \cdot (s\mathbf{I}_n - \mathbf{A})^{-1} \cdot \mathbf{b}\tag{2.17}$$

Based on the vector dynamic equation (2.15) and the diagonal form of matrix $\mathbf{A}$ it can be concluded that

$$X_i(s) = \frac{b_i}{s - \lambda_i} \cdot U(s)\tag{2.18}$$

Furthermore, using the algebraic output equation in (2.15), the following is obtained.

$$Y(s) = \sum_{i=1}^n c_i \cdot X_i(s) = \sum_{i=1}^n \frac{c_i \cdot b_i}{s - \lambda_i} \cdot U(s) \Rightarrow H(s) = \sum_{i=1}^n \frac{c_i \cdot b_i}{s - \lambda_i}\tag{2.19}$$

If the expansion of the transfer function in equation (2.19) is compared against the one on in equation (1.23), that has been obtained through partial fraction expansion, it can be

concluded that they are identical, provided that $k_i \equiv c_i \cdot b_i$. Therefore, if one of the poles, λ_i , of the system, which are determined as the eigenvalues of matrix A, is absent from (1.23), this means that $k_i \equiv c_i \cdot b_i = 0$. In effect, this is due to either $b_i = 0$, or $c_i = 0$ or both. Taking into account that matrix A is diagonal, it is concluded that the system has to be either uncontrollable or unobservable or both.

In effect, the transfer function matrix of any LTI system encompasses information only for that part of the system which is both controllable and observable. As a result, it is a well-defined description of the system only if controllability and observability are guaranteed.

The importance of the above is demonstrated in the following example. Consider a 2-inputs, 2-outputs and 3-states system with matrices A, B, C, D as follows.

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -0.4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, B = C^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, D = \mathbf{0}_{2 \times 2}$$

This system is evidently both uncontrollable and unobservable. Due to the diagonal form of matrix A, poles are easily determined to be -2 , -0.4 and $+1$. Therefore, the system is unstable, too. However, the 2X2 transfer function matrix is as follows.

$$H(s) = \frac{1}{s^2 + 2.4s + 0.8} \cdot \begin{bmatrix} s + 0.4 & 0 \\ 0 & s + 2 \end{bmatrix}$$

Based solely on the transfer function, one might erroneously characterize the system to be a 2nd order and stable one, as the apparent characteristic polynomial is determined to be the following.

$$p(s) = s^2 + 2.4s + 0.8 = (s + 0.4) \cdot (s + 2)$$

However, the correct characteristic polynomial is the following.

$$p_c(s) = |sI - A| = s^3 + 1.4s^2 - 1.6s - 0.8 = (s + 0.4) \cdot (s + 2) \cdot (s - 1)$$

Therefore, the system is actually 3rd order and unstable. This simple example shows, in effect, that in order to arrive to safe conclusions for the system, especially concerning stability, by using solely the transfer function controllability and observability must be guaranteed in advance.

2.4 Sinusoidal Steady-State Response and Bode Plots

In Section 1.3 the impulse and the step response of any LTI system have been introduced. Equally important is the response of an LTI system when sinusoids as applied as inputs.

When presented with the transfer function of an LTI-SISO system, the sinusoidal steady-state response, i.e. its response when driven with a pure sine wave of arbitrary frequency ω and unity amplitude, can be calculated by replacing $s = j\omega$ in the transfer function. Referring to the I/O relationship in equation (1.11), it holds that

$$u(t) = \sin(\omega t) \Rightarrow y(t) = y_{ss}(t) = |H(j\omega)| \cdot \sin(\omega t + \angle H(j\omega)) \quad (2.20)$$

In the above, $|\bullet|$ stands for a complex number's magnitude (or modulus) and $\angle \bullet$ for a complex number's phase (or argument).

The graphs of the real functions $A(\omega) = 20 \log |H(j\omega)|$ and $\phi(\omega) = \angle H(j\omega)$ for real and positive ω ($\omega \geq 0$) on plots with the x-axis, where the angular frequency ω is depicted, in logarithmic scale are called magnitude and phase Bode plots respectively.

For example, in the figure below, the Bode plots of a 3rd-order system are shown. The transfer function of the system is as follows.

$$H(s) = \frac{s-1}{s^3 + 2.2s^2 + 5.4s + 1} = \frac{s-1}{(s+0.2)(s+1+2j)(s+1-2j)}$$

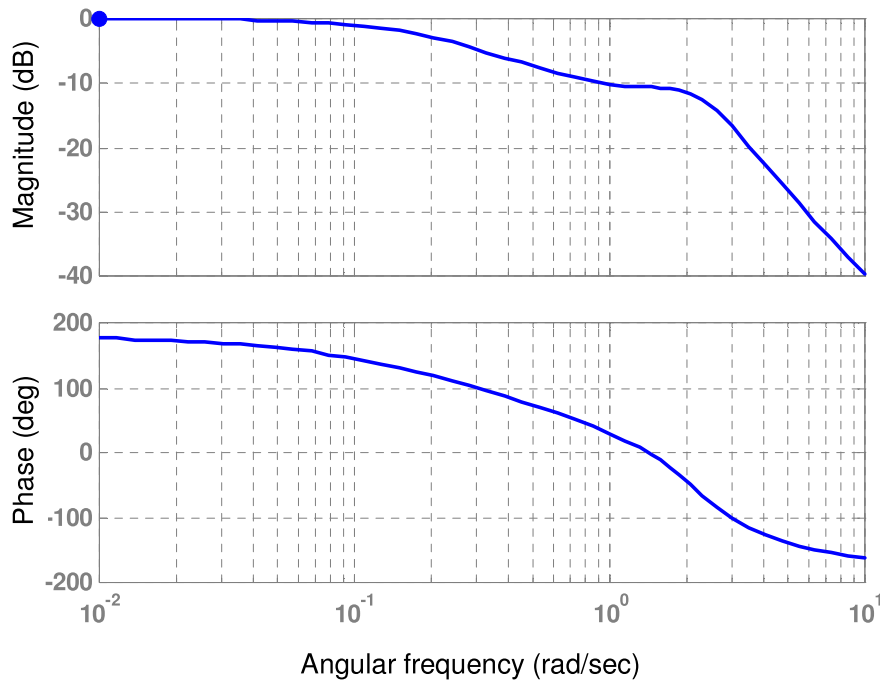


Figure 2-7: Typical magnitude and phase Bode plots for an LTI system.

Approximate (or asymptotic or corner) Bode plots

The Bode plots are widely used for frequency response characterization mainly because they can be drawn approximately on the basis of a limited set of rules of thumb. Such

approximate plots, also called asymptotic or corner plots, starts with bringing the transfer function of the system to the following factorized form.

$$H(s) = K \cdot \frac{(1 + \tau_1 s)(1 + \tau_2 s) \dots (1 + \tau_m s)}{s^q (1 + T_1 s)(1 + T_2 s) \dots (1 + T_n s)} \quad (2.21)$$

As a working example consider the following 3rd-order transfer function.

$$H(s) = K \omega_0^2 \cdot \frac{(1 + \tau s)}{s(1 + Ts)(s^2 + 2\zeta \omega_0 s + \omega_0^2)} \quad (2.22)$$

The above is augmented, with respect to the transfer function in (2.21), by the quadratic factor $(s^2 + 2\zeta \omega_0 s + \omega_0^2)$ for which it is assumed that $\omega_0 > 0$ and $|\zeta| < 1$. This factor introduces two complex conjugate poles, $p_{1,2} = \omega_0 \cdot (-\zeta \pm j \cdot \sqrt{1 - \zeta^2})$, and therefore its investigation must be done with caution.

Substituting $s = j\omega$ in equation (2.22) the following is obtained.

$$H(j\omega) = K \cdot \frac{(1 + \tau \omega j)}{(\omega j) \cdot (1 + T \omega j) \cdot \left[1 - \left(\frac{\omega}{\omega_0} \right)^2 + j 2\zeta \left(\frac{\omega}{\omega_0} \right) \right]} \quad (2.23)$$

To draw the magnitude and phase Bode plots the following manipulation is needed on the basis of equation (2.23).

$$\begin{aligned} A(\omega) = 20 \log |H(j\omega)| &= 20 \log |K| + 20 \log |1 + \tau \omega j| - 20 \log \omega \\ &\quad - 20 \log |1 + T \omega j| - 20 \log \left| 1 - \left(\frac{\omega}{\omega_0} \right)^2 + j 2\zeta \left(\frac{\omega}{\omega_0} \right) \right| \end{aligned} \quad (2.24)$$

$$\begin{aligned} \phi(\omega) = \angle H(j\omega) &= \angle K + \angle(1 + \tau \omega j) - \angle \omega \\ &\quad - \angle(1 + T \omega j) - \angle \left[1 - \left(\frac{\omega}{\omega_0} \right)^2 + j 2\zeta \left(\frac{\omega}{\omega_0} \right) \right] \end{aligned} \quad (2.25)$$

Therefore, it is concluded that in order to draw the Bode plots of a transfer function it suffices to graphically add up the various components appearing as terms in equations (2.24) and (2.25). The magnitude and phase contribution of each one of these components to the overall Bode plots is examined below.

- Constant gain K:

The contribution of a constant to both Bode plots of a transfer function is a straight line with zero slope as follows.

$$A(\omega) = 20 \log |K| \text{ and } \phi(\omega) = \begin{cases} 0^\circ, & K \geq 0 \\ 180^\circ, & K < 0 \end{cases} \quad (2.26)$$

- Factor $(j\omega)$:

For the magnitude Bode plot of a transfer function that contains this factor as part of its denominator, the contribution consists of a straight line with slope -20dB/dec (decibels per decade). For the phase Bode plot of a transfer function that contains this factor as part of its denominator, the contribution consists of a horizontal straight line as follows.

$$A(\omega) = -20 \log \omega \text{ and } \phi(\omega) = -90^\circ \quad (2.27)$$

In the case that the pole has multiplicity $q > 1$, as for example in equation (2.21), then equation (2.27) is modified accordingly.

$$A(\omega) = -20q \cdot \log \omega \text{ and } \phi(\omega) = -q \cdot 90^\circ \quad (2.28)$$

Finally, if factor $(j\omega)$ appears on the numerator instead the denominator of the transfer function the above still hold but the sign has to be switched to positive.

- Factor $(1 + j\omega T)$:

Calculation of the contribution of this factor, when appearing as part of the denominator of the transfer function, is done on the basis of the following approximations.

$$A(\omega) = -20 \log |1 + j\omega T| = -20 \log \sqrt{1 + \omega^2 T^2} \approx \begin{cases} 0, & \omega T \ll 1 \\ -3, & \omega T \simeq 1 \\ -20 \log(\omega T), & \omega T \gg 1 \end{cases} \quad (2.29)$$

$$\phi(\omega) = -\arctan(\omega T) \approx \begin{cases} 0^\circ, & \omega T < 0.1 \\ -90^\circ, & \omega T > 10.0 \end{cases} \quad (2.30)$$

As an example, consider the following 1st-order transfer function.

$$H(s) = \frac{1}{s+1} \Rightarrow H(j\omega) = \frac{1}{1+j\omega} \quad (2.31)$$

In the above, evidently $T = 1.0$.

The Bode plots of this transfer function are shown in the figure below. It is clear that these graphs can be approximated by straight lines. In this respect the magnitude plot consists of a horizontal line until the breakpoint (or corner point) corresponding to $\omega T = 1.0$; beyond that point it can be approximated by a straight line with a slope of -20dB/dec . The phase plot can be approximated by horizontal lines at the level of 0° for $\omega T < 0.1$ and -90° for $\omega T > 10.0$. These sections can then be connected with an appropriately negatively sloped straight line.

Finally, these remarks can be easily extended for the cases where factor $(1 + j\omega T)$ appears in the numerator of the transfer function or obtains multiplicity higher than one.

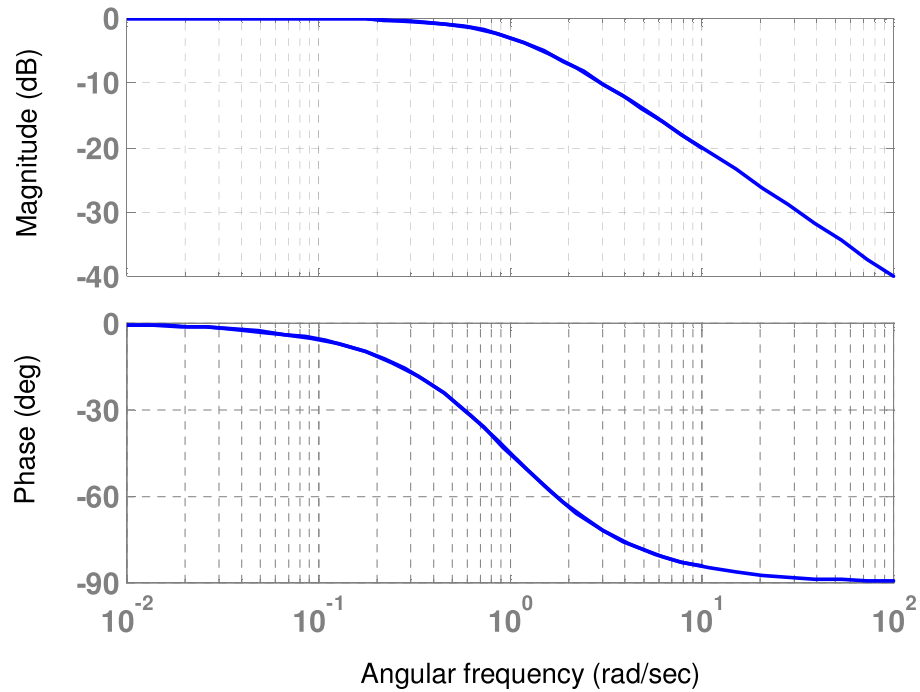


Figure 2-8: Magnitude and phase Bode plots of a 1st-order factor in the denominator.

- Factor $\left[1 - \left(\frac{\omega}{\omega_0} \right)^2 + j2\zeta \left(\frac{\omega}{\omega_0} \right) \right], \quad |\zeta| < 1$

Calculation of the contribution of this factor, when appearing as part of the denominator of the transfer function, is done on the basis of the following approximations.

$$A(\omega) = -20 \log \left| 1 - \left(\frac{\omega}{\omega_0} \right)^2 + j2\zeta \left(\frac{\omega}{\omega_0} \right) \right| = -20 \log \sqrt{\left[1 - \left(\frac{\omega}{\omega_0} \right)^2 \right]^2 + 4\zeta^2 \left(\frac{\omega}{\omega_0} \right)^2} \quad (2.32)$$

↓

$$A(\omega) = \begin{cases} 0, & \omega \ll \omega_0 \\ -20 \log(2|\zeta|), & \omega = \omega_0 \\ -40 \log\left(\frac{\omega}{\omega_0}\right), & \omega \gg \omega_0 \end{cases}$$

$$\phi(\omega) = -\arctan \left[2\zeta \left(\frac{\omega}{\omega_0} \right) / \left(1 - \left(\frac{\omega}{\omega_0} \right)^2 \right) \right] \Rightarrow \phi(\omega) = \begin{cases} \approx 0^\circ, & \omega \ll \omega_0 \\ -90^\circ, & \omega = \omega_0 \\ \approx -180^\circ, & \omega \gg \omega_0 \end{cases} \quad (2.33)$$

The dependence of the corresponding plots on parameter ζ (known as the *damping factor* or *coefficient*) is shown in the figure below. Evidently, the smaller the damping coefficient is, the higher the intensity of resonance. By resonance we mean the phenomenon according to which the system exhibits response of significantly amplified amplitude when driven by a sinusoidal of a specific frequency, known as the resonance frequency, $\omega = \omega_0$. In the extreme case that $\zeta = 0$, the output's amplitude is infinite. On the other hand, when $\zeta = 1$ the system obtains a double real pole located at $\omega = \omega_0$ and the analysis can be done as in the case of a 1st-order factor $(1 + j\omega T)$ with multiplicity equal to two.

The 2nd-order factor Bode plots can be approximated by straight lines as in the case of factor $(1 + j\omega T)$, especially for frequencies far from the resonance point $\omega = \omega_0$. For $\omega \ll \omega_0$ a horizontal straight line is a good approximation whilst for $\omega \gg \omega_0$ a straight line with slope -40dB/dec represents the frequency dependence of the magnitude Bode plot. For the vicinity of the resonance point $\omega = \omega_0$, the intensity of the resonance, depending on the value of ζ , needs to be estimated. For the phase plot similar considerations with the first order case hold; the final value, though, is now -180° instead of -90° . Furthermore, the smaller the absolute value of the damping coefficient, the steeper the phase plot of the factor.

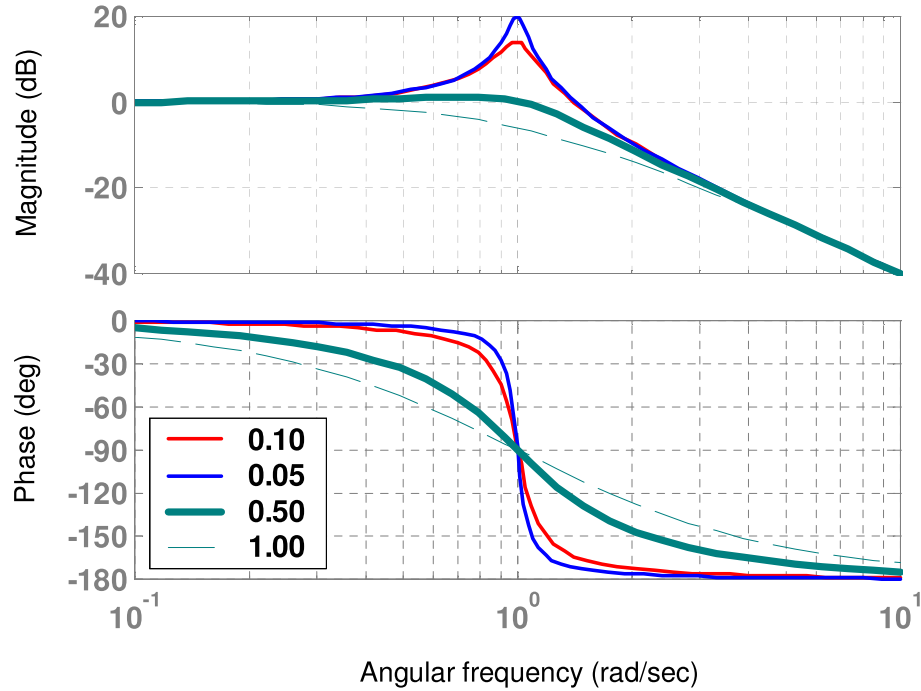


Figure 2-9: Bode plots of a 2nd-order factor in the denominator for various values of ζ .

2.5 Analysis of Second-Order Systems

A large class of practical systems encountered in control engineering is either of 1st or 2nd order. Furthermore, many other higher-order systems can be approximated or “reduced” to 1st or 2nd order ones, provided that certain conditions are valid. The importance of 1st and 2nd order LTI systems is additionally emphasized by the fact that their poles can be conveniently calculated as functions of the coefficients of their characteristic polynomial.

In this section, the analysis of a generic 2nd-order system without zeros, as in the transfer function below, is presented.

$$H(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2} \quad (2.34)$$

Parameter $\omega_0 \geq 0$ is referred to as the eigenfrequency of the system and parameter ζ as the coefficient (or damping) coefficient.

The poles of the system are given by the following expression.

$$p_{1,2} = \omega_0 \cdot \left(-\zeta \pm \sqrt{\zeta^2 - 1} \right) \quad (2.35)$$

In the case that $|\zeta| < 1$, parameter $\omega_n = \omega_0 \cdot \sqrt{1 - \zeta^2}$ is defined as the natural frequency of the system.

The step response of the generic 2nd-order system is given by the following relationship in the complex frequency domain, if equation (1.20) is employed.

$$Y_{step}(s) = H(s) \cdot U_{step}(s) = \frac{\omega_0^2}{s(s^2 + 2\zeta\omega_0 s + \omega_0^2)} \quad (2.36)$$

The analysis of the 2nd-order system in hand will proceed on the basis of the following cases for the value of the damping coefficient. As shown, this coefficient plays a crucial role in determining the location of the system poles on the s-plane and effectively its stability and transient behavior.

(a) $\zeta \leq -1$

Two positive purely real poles (or a single double one if $\zeta = -1$) – Unstable system

Step response:

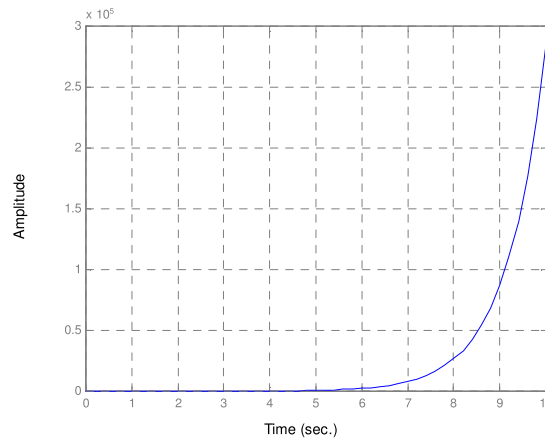


Figure 2-10: 2nd-order system step response when $\zeta \leq -1$.

(b) $-1 < \zeta < 0$

Two complex conjugate poles on the right-hand s-plane – Unstable system

Step response:

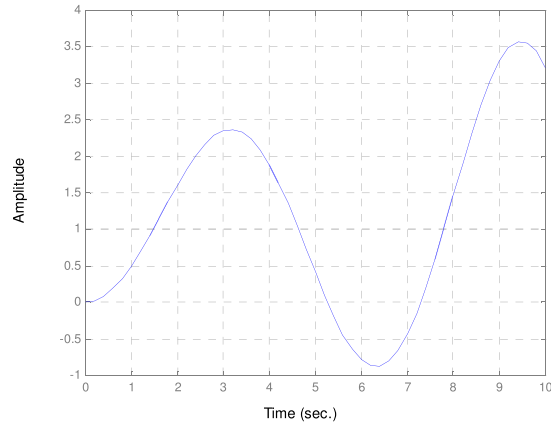


Figure 2-11: 2nd-order system step response when $-1 < \zeta < 0$.

(c) $\zeta = 0$

Two conjugate purely imaginary poles – Critically stable system

Step response:

$$y_{step}(t) = 1 - \cos(\omega_0 t) \quad (2.37)$$

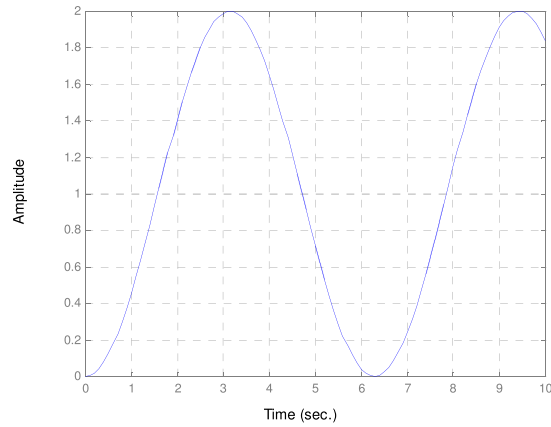


Figure 2-12: 2nd-order system step response when $\zeta = 0$.

(d) $0 < \zeta < 1$

Two complex conjugate poles on the left-hand s-plane – Stable system

Step response:

$$y_{step}(t) = 1 - e^{-\sigma t} \cos(\omega_n t) - \frac{\sigma}{\omega_n} \cdot e^{-\sigma t} \sin(\omega_n t) = 1 - \frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}} \cdot \sin(\omega_n t + \theta) \quad (2.38)$$

where $\sigma = \text{Re}\{p_{1,2}\} = -\zeta\omega_0$ and $\theta = \arctan\left(\sqrt{1-\zeta^2}/\zeta\right)$.

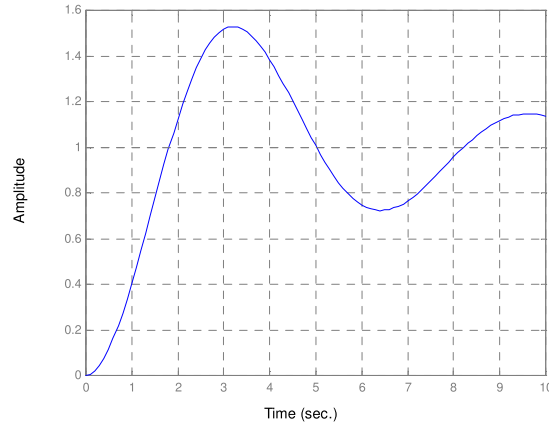


Figure 2-13: 2nd-order system step response when $0 < \zeta < 1$.

In Figure 2-13 it is of interest to calculate the time (at which occurs) and the value of maximum overshoot. This is achieved by differentiating the expression for the step response in equation (2.36). The derivative calculated is then equated to zero. The time of the first maximum's occurrence is the time instant of maximum overshoot, $t_{\max}$, given below.

$$t_{\max} = \frac{\pi}{\omega_n} \quad (2.39)$$

(e) $\zeta = 1$

One double, purely real pole in the right-hand s-plane – Stable system

Step response:

$$y_{\text{step}}(t) = 1 - e^{-\omega_0 t} - \omega_0 t e^{-\omega_0 t} \quad (2.40)$$

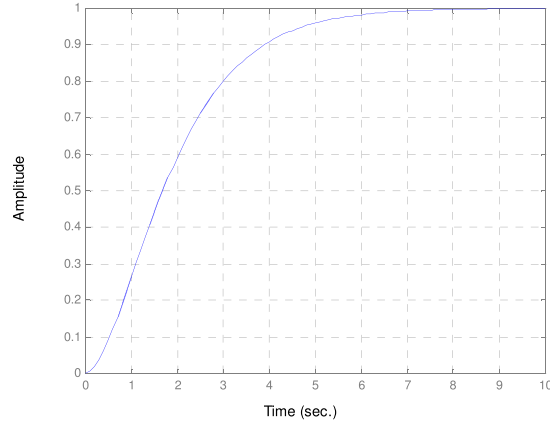


Figure 2-14: 2nd-order system step response when $\zeta = 1$.

(f) $\zeta > 1$

Two negative purely real poles – Stable system

Step response:

$$y_{step}(t) = 1 - e^{-\sigma t} \cdot \cosh(at) - \frac{\sigma}{a} e^{-\sigma t} \cdot \sinh(at) \quad (2.41)$$

where $\sigma = \text{Re}\{p_{1,2}\} = -\zeta\omega_0$ and $a = \omega_0 \cdot \sqrt{\zeta^2 - 1}$.

The Bode plots for various positive values of $\zeta > 0$ (therefore asymptotically stable cases), are given in Figure 2-9.

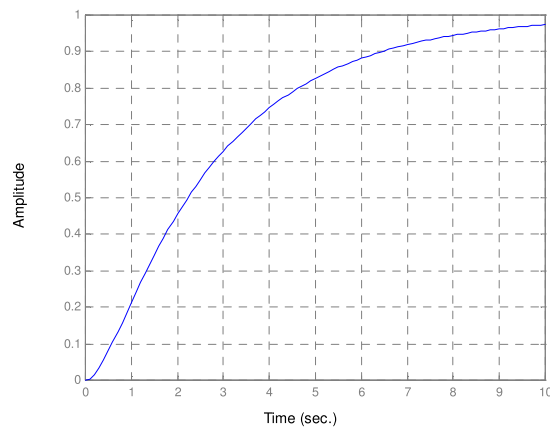


Figure 2-15: 2nd-order system step response when $\zeta > 1$.

3. SISO SYSTEM CONTROLS

3.1 Introduction

The subject of this chapter is to present some fundamental techniques for the control of LTI-SISO systems, with structure as shown in the following schematic.

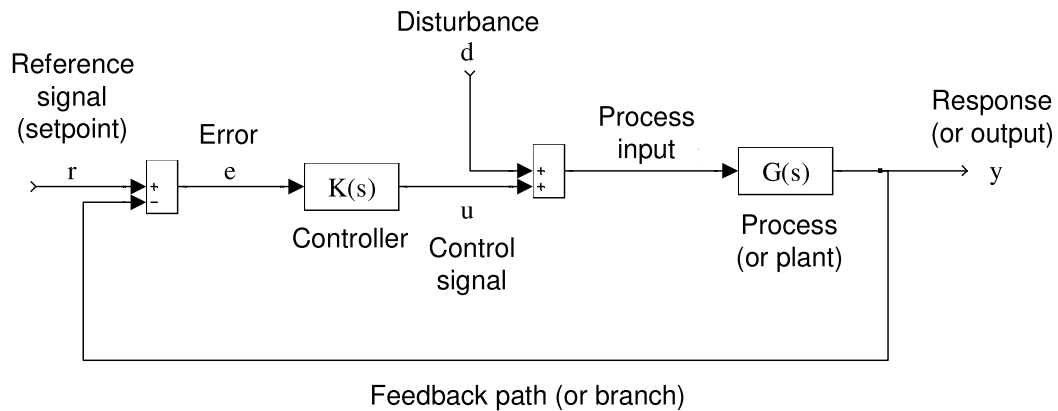


Figure 3-1: Typical block diagram of a closed-loop system.

The closed-loop control system given above is encountered in many practical, industrial plants and processes. The objective in introducing feedback control is to maintain the process response signal, y , as close as possible to the reference signal (setpoint), r . Deflection of the process output signal from the reference one is generated because either there is significant disturbance present, manifested by signal d , or the setpoint is modified or both. The first problem is referred to as the *regulation problem* whilst the second as the *tracking problem*.

In any of these cases, however, the controller (control system) is inserted in order to reduce the error signal, $e = r - y$, i.e. the deviation of the output from the reference signal, to the lowest possible value; this can be achieved by appropriately processing the error signal in order to generate the driving control signal, that will compensate the effect of the disturbance. The block diagram in Figure 3-1 is similar to the one shown Figure 1-1; for

the sake of simplicity, however, the sensor and actuator blocks have been omitted, or equivalently included in the controller and process block respectively, and the plant is assumed to be SISO.

In the complex frequency domain, the output signal is given by the following relationship.

$$\left. \begin{aligned} Y(s) &= G(s) \cdot [U(s) + D(s)] \\ U(s) &= K(s) \cdot E(s) = K(s) \cdot [R(s) - Y(s)] \end{aligned} \right\} \Rightarrow \quad (3.1)$$

$$\Rightarrow Y(s) = \frac{G(s)K(s)}{1 + G(s)K(s)} \cdot R(s) + \frac{G(s)}{1 + G(s)K(s)} \cdot D(s)$$

The above can be alternatively demonstrated by separately applying the general equation (2.3) for each one of the signals r and d and setting the other one identically equal to zero.

In the rest of this chapter, the following controller synthesis methods are examined: a) ON/OFF control; b) PID (Proportional-Integral-Differential) control, and c) Controller design by use of the *Root Locus*.

3.2 ON/OFF Control

ON/OFF control is the simplest control technique and widely used in various practical systems. Typical applications include thermostats; two-positions pressure regulators; pneumatic and hydraulic actuator valves like the ones used e.g. in marine steering gears etc.

The control law, i.e. the relationship connecting the error to the control signal as implemented in the controller, is as follows.

$$u = \begin{cases} A \cdot \text{sgn}(e), & |e| \geq Z_d \\ 0, & |e| < Z_d \end{cases} = \begin{cases} +A, & e \geq +Z_d \\ 0, & |e| < Z_d \\ -A, & e \leq -Z_d \end{cases} \quad (3.2)$$

In the above, A is a constant and $Z_d \geq 0$ a *threshold value* for the absolute error $|e|$ at which the control signal is activated. The region $|e| < Z_d$ for which the control signal is zero is called the *dead zone*; the dead zone need not be necessarily symmetrical around zero as shown in the figure below for an ON/OFF controller with $A=1$. Furthermore, this figure as well as equation (3.2) refer to controllers with *three switch positions*. Commonly, the controller switch may have only two positions, as indicated by the name ON/OFF, or four or more. However, the same features are shared by all controllers whose control law is based on switching as in equation (3.2); therefore, the analysis is general enough.

An important feature of all ON/OFF controllers is that the associated control law is nonlinear and, in effect, cannot be investigated with tools from linear control theory, like e.g. transfer functions. In this end, an example is presented.

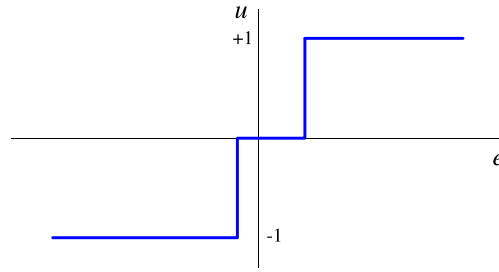


Figure 3-2: General form of an ON/OFF controller with three positions switch.

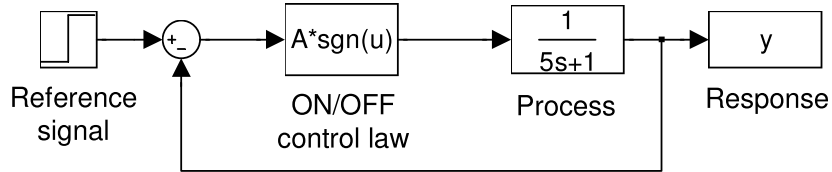


Figure 3-3: Example for the analysis of ON/OFF controllers.

The step response of the closed-loop system for various values of parameter A and no dead zone, i.e. $Z_d = 0$, is shown in Figure 3-4. Starting with $A = 1$, one can easily see that the response is the same with the one obtained when the switch is not intermitted between the summing junction and the process. For $A = 0.1$, on the other hand, the response exhibits large steady-state error (almost 0.9) from the setpoint value (equal to 1). For $A = 2$, one obtains a very good, from a practical viewpoint, step response as it approaches the setpoint in almost half the time needed in the case of $A = 1$. However, if one attempts to further reduce the *rise time*, by simply increasing A beyond 2, *limit cycle oscillations* of the control signal are induced. This is highly undesirable as it can cause various elements in the system (e.g. the actuator) to rapidly wear and fail.

In practical systems, dead zone cannot be exactly zero; this is mainly due to the fact that the control loop elements are not ideal and demonstrate finite sensitivity. Even a small dead zone may sever the limit cycle phenomenon. This is clearly seen in the leftmost plot of Figure 3-5, where $A = 2$ and the dead zone is just 3% of the absolute maximum error, e , observed (which is equal to 1.0 at start time). Furthermore, a relatively large dead zone is often deliberately introduced in order to reduce the frequency of the limit cycle oscillations. This is seen in the rightmost plot of Figure 3-5 where $Z_d = 0.3$ (i.e. 30% of the absolute maximum error observed). However, this comes at the price of increasing the error in the response; indeed, the final value of the response as can be seen in Figure 3-5 is around 0.7, i.e. 30% off the setpoint.

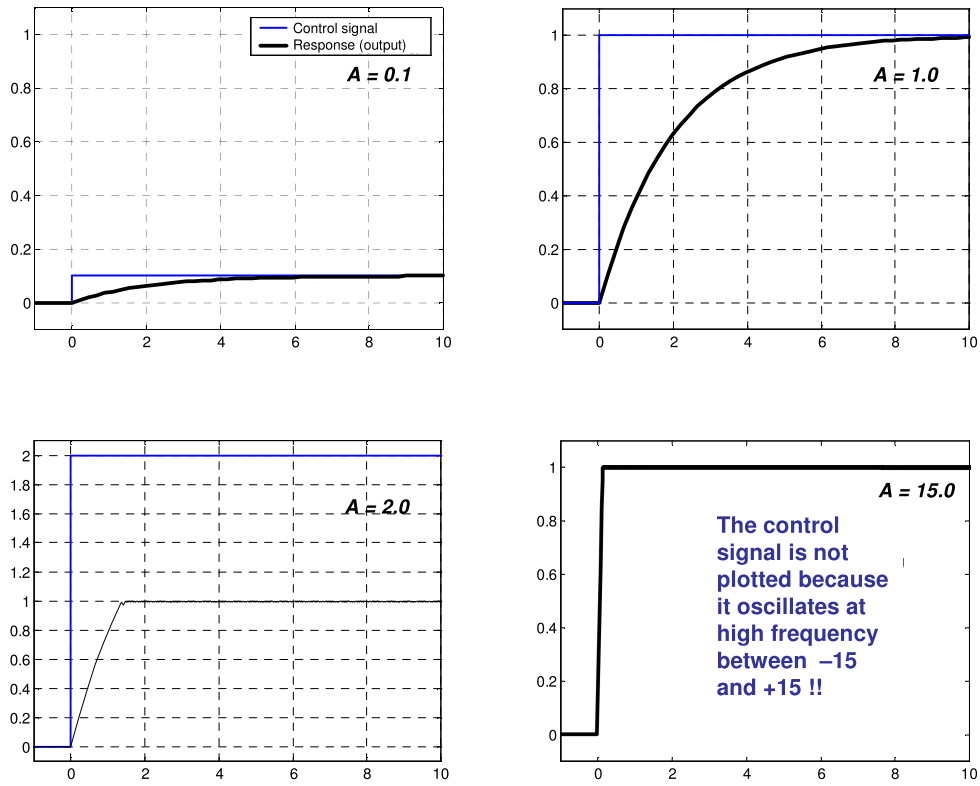


Figure 3-4: Response with ON/OFF control various values of $A = 2$ and $Z_d = 0$.

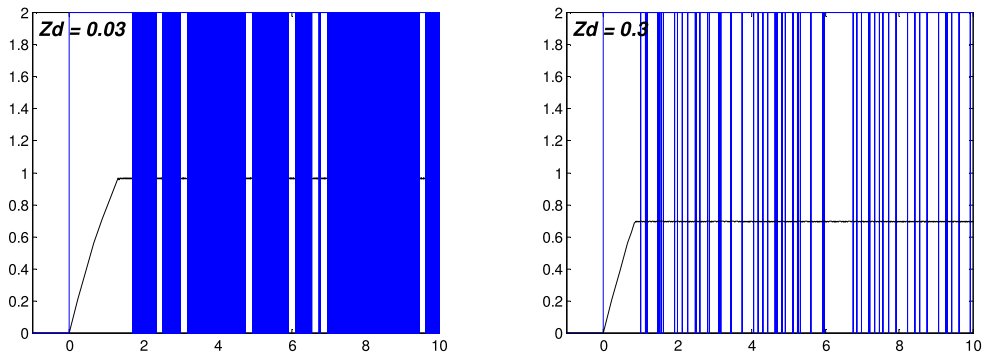


Figure 3-5: Response with ON/OFF control for $A = 2$ and $Z_d = 0.03$ and 0.3 .

3.3 P-I-D Control

Generalities

The Proportional-Integral-Differential (PID) control law was introduced as an alternative to ON/OFF control because it actually deals with practical control issues of SISO plants in a much improved manner.

The PID control law is commonly encountered in the following alternative forms (reference of symbols is Figure 3-6).

(i) The analytic form which is also used in the rest of this text. The expressions for the control law in both the time and complex frequency domain are given below.

$$\begin{aligned}
 u(t) &= K_p \cdot e(t) + K_i \cdot \int_0^t e(\xi) d\xi + K_d \cdot \frac{d}{dt} e(t) \\
 &\quad \Downarrow \\
 U(s) &= K(s) \cdot E(s) = \left(K_p + \frac{K_i}{s} + K_d \cdot s \right) \cdot E(s) = \frac{K_i + K_p \cdot s + K_d \cdot s^2}{s} \cdot E(s)
 \end{aligned} \tag{3.3}$$

In the analytic form the control signal depends exclusively on the error signal. Furthermore, note that a) The P-control law is a static system; b) The PID controller transfer function with non-zero I-term demonstrates a pole at $s = 0$; c) The numerator polynomial of the PID controller transfer function with non-zero D-term is of higher order than its denominator one.

The block diagram of the generic PID controller is given in the figure below.

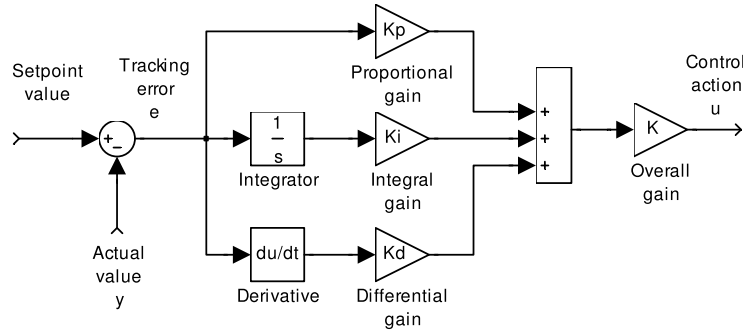


Figure 3-6: General form of the PID controller.

(II) The practical form which is adopted commonly in industrial practice.

$$u(t) = K_p \cdot \left(-y(t) + \frac{1}{T_i} \cdot \int_0^t e(\xi) d\xi - T_d \cdot \frac{d}{dt} y(t) \right) \tag{3.4}$$

In this form, the error signal in both the P and D-term is replaced by that of the process output signal (negated). In this way the closed-loop system response to abrupt changes of the setpoint, r , is significantly smoother

The problem of *tuning* a PID controller consists of determining the three P-I-D control law gains K_p , K_i and K_d so that certain transient and steady-state specifications for the closed-loop system are satisfied.

Proportional control

For proportional control (P-control) it holds that: $K_i = K_d = 0$. Therefore, the tuning problem is limited to the determination of coefficient K_p in the control law. In this respect, P-control exhibits significant similarity to the ON/OFF controller presented previously. Below, a widespread version of P-control is given. In this version, the control signal is generated as the superposition of the output of a P controller plus an open-loop control action manifested as offset u_0 .

$$u(t) = \begin{cases} u_0 + K_p \cdot e_0, & e(t) > e_0 \\ u_0 + K_p \cdot e(t), & -e_0 \leq e(t) \leq e_0 \\ u_0 - K_p \cdot e_0, & e(t) < -e_0 \end{cases} \quad (3.5)$$

In both the above relationship as well as in the figure below, one can easily assert the similarity of P-control to ON/OFF control (at least with a 3-positions switch, as examined in the text).

As in the case of ON/OFF control, the control signal demonstrates saturation when the absolute value of the error signal exceeds a threshold in order to prevent excessive control signal values.

The ideal (i.e. without saturation) P controller eliminates the problem of limit cycle oscillations. On the other hand, it reduces, but does not fully eliminate, steady-state error. The effect of P-control is now investigated in two typical practical cases. At first, zero disturbance is assumed and the process block in Figure 3-1 is assumed to be purely static as follows.

$$y = K_{ss} \cdot u = K_{ss} \cdot (u_0 + K_p \cdot e) \quad (3.6)$$

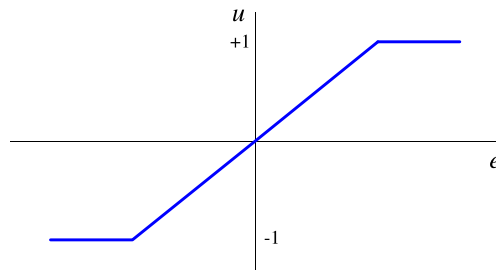


Figure 3-7: P-control law with saturation and zero offset.

In the above, K_{ss} stands for the steady-state gain of the process plant. After algebraic manipulation the output of the closed-loop system with P-control is obtained as follows.

$$y = \frac{K_{ss} \cdot u_0 + K_{ss} K_p \cdot r}{1 + K_{ss} K_p} \Rightarrow e = \frac{r - K_{ss} \cdot u_0}{1 + K_{ss} K_p} \quad (3.7)$$

In effect, for the steady-state error to be zero it is necessary for one of the following to hold.

- $u_0 = r / K_{ss}$
- $K_p \rightarrow \infty$

The second case to be examined is when the process exhibits 1st-order linear dynamics and non-zero disturbance is present. Assume that $G(s)$ in equation (3.1) is as follows.

$$G(s) = \frac{K_{ss}}{\tau s + 1} \quad (3.8)$$

Then equation (3.1) with P-control ($K(s) = K_p$) yields

$$Y(s) = \frac{K_{ss} K_p}{\tau s + 1 + K_{ss} K_p} \cdot R(s) + \frac{K_{ss}}{\tau s + 1 + K_{ss} K_p} \cdot D(s) \quad (3.9)$$

By assuming that $R(s) = 0$, without any loss of generality, the Final Value Theorem of the Laplace Transform (see equation (1.6)) may be employed in equation (3.1), to obtain the closed-loop response when there is a step change in disturbance (i.e. $d(t) = u_{step}(t) \Leftrightarrow D(s) = 1/s$).

$$\begin{aligned} \lim_{t \rightarrow \infty} (y(t)) &= \lim_{s \rightarrow 0} (sY(s)) = \lim_{s \rightarrow 0} \left(s \cdot \frac{K_{ss}}{\tau s + 1 + K_{ss} K_p} \cdot D(s) \right) \\ &\Downarrow \\ \lim_{t \rightarrow \infty} (y(t)) &= \lim_{s \rightarrow 0} \left(s \cdot \frac{K_{ss}}{\tau s + 1 + K_{ss} K_p} \cdot \frac{1}{s} \right) = \frac{K_{ss}}{1 + K_{ss} K_p} \end{aligned} \quad (3.10)$$

In conclusion in order to minimize the deviation of $y(t \rightarrow \infty)$ from $r = 0$ a large value for coefficient K_p is needed.

Proportional-Integral control

For proportional-integral control (PI-control) it holds that $K_d = 0$. A control law including an integral term eliminates steady-state error which is the major drawback of P-control. It

is for this reason that the PI controller finds widespread use in industry. A typical example is *speed governors* used for regulation of the rotational speed (rpm, revolutions per second) of engines, turbines, motors etc.

PI-control is defined by the following transfer function.

$$\begin{aligned} u(t) &= K_p \cdot e(t) + K_i \cdot \int_0^t e(\xi) d\xi \\ &\Updownarrow \\ U(s) &= \left(K_p + \frac{K_i}{s} \right) \cdot E(s) = \frac{K_p \cdot s + K_i}{s} \cdot E(s) \end{aligned} \quad (3.11)$$

The effect of PI-control is now investigated in the case of an LTI-SISO process with a generic transfer function defined as a ratio of two polynomials $p_n(s)$ and $p_0(s)$ with degree m and n respectively and $m \leq n$.

$$G(s) = \frac{p_n(s)}{p_0(s)} \quad (3.12)$$

With PI-control, equation (3.1) yields

$$Y(s) = \frac{(K_i + K_p \cdot s) \cdot p_n(s)}{s \cdot p_0(s) + (K_i + K_p \cdot s) \cdot p_n(s)} \cdot R(s) + \frac{s \cdot p_n(s)}{s \cdot p_0(s) + (K_i + K_p \cdot s) \cdot p_n(s)} \cdot D(s) \quad (3.13)$$

As an example, consider the transfer equation in equation (3.8). Then, equation (3.13) yields

$$Y(s) = \frac{K_{ss} \cdot (K_i + K_p \cdot s)}{s \cdot (1 + \tau s) + K_{ss} \cdot (K_i + K_p \cdot s)} \cdot R(s) + \frac{K_{ss} \cdot s}{s \cdot (1 + \tau s) + K_{ss} \cdot (K_i + K_p \cdot s)} \cdot D(s) \quad (3.14)$$

Assuming, as in the case of P-control, that $R(s) = 0$ and applying the Final Value Theorem for $d(t) = u_{step}(t) \Leftrightarrow D(s) = 1/s$, the following is obtained.

$$y(t \rightarrow \infty) = \lim_{s \rightarrow 0} (s \cdot Y(s)) = \frac{K_{ss} \cdot 0}{0 \cdot (1 + \tau \cdot 0) + K_{ss} \cdot (K_i + K_p \cdot 0)} = 0 \quad (3.15)$$

As can be seen, PI-control introduces a zero at $s = 0$ in the closed-loop transfer function connecting the disturbance to the output signal, resulting to elimination of the steady-state error.

A word of caution is needed here concerning closed-loop stability. The closed-loop characteristic polynomial as can be seen in equation (3.13) is as follows.

$$p_c(s) = s \cdot p_0(s) + (K_i + K_p \cdot s) \cdot p_n(s) \quad (3.16)$$

For the closed-loop system to be stable all the roots of the polynomial above must lie in the left-hand complex plane. However, the degree of this polynomial is $(n+1)$, i.e. higher by one than the open-loop, process characteristic polynomial, $p_0(s)$. This is the price paid for the improvement in system's behavior after introducing PI-control.

In the specific case of the transfer function in equation (3.8), the closed-loop characteristic polynomials in the cases when P and PI-control is employed are as follows.

$$\text{P-control: } p_{c1}(s) = \tau s + K_{ss}K_p + 1 \quad (3.17)$$

$$\text{PI-control: } p_{c2}(s) = \tau s^2 + (K_{ss}K_p + 1)s + K_{ss}K_i \quad (3.18)$$

In the case that $K_i = 0$ the polynomial in equation (3.18) in conjunction with the zero-pole cancellation taking place in equation (3.14) for $s = 0$, equation (3.18) coincides with the one in (3.17) i.e. the one valid for P-control. Another interesting case is when $(K_{ss}K_p + 1) = 0 \Leftrightarrow K_p = -1/K_{ss}$, when polynomial $p_{c2}(s)$ has zero first-order term. In this case the closed-loop system poles are as follows.

$$p_{1,2} = \pm \sqrt{K_{ss}K_i/\tau} \quad (3.19)$$

In conclusion, if $K_i > 0$ (and $K_{ss} > 0$) the closed-loop system becomes unstable with one pole in the right-hand complex plane, whilst if $K_i \leq 0$ the closed-loop system is critically stable with two purely imaginary, conjugate poles. The analysis above demonstrates the importance of the proportional term in PI-control for guaranteeing closed-loop system stability.

The reason for which the PI-control law eliminates steady-state error is because the transient effect is not terminated before the output signal coincides with the setpoint. This is achieved due to the accumulation of error in the integrator, which, therefore, acts as a memory element in the closed-loop system. In effect, as long as the error integral is nonzero a persisting control action remains. The tradeoff for this advantage is that the output may be driven to damped or even sustained oscillations.

Proportional-Integral-Differential control

This is the full form of the P-I-D control law. A differential term (D-term) is employed in order to allow us to deal with some of the problems attributed to the use of the I-term. One of the most important is the prohibitive overshoot observed in the closed-loop system response when PI-control is applied. For example, in the figure below the response of a closed-loop system with PI control is shown exhibiting 70% overshoot above the setpoint value assumed by the output signal when steady-state is restored.

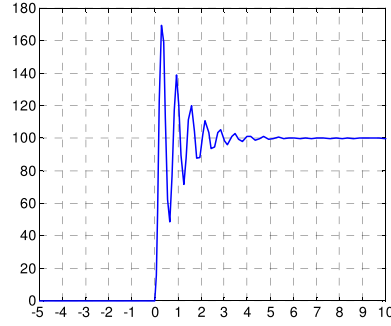


Figure 3-8: Response of system with PI-control exhibiting prohibitive overshoot.

Employing the D-term in the control law incorporates prediction capabilities in the controller. Indeed, in a linear system if the error's rate of change de/dt is large, e.g. when a step disturbance signal is applied, it is safe to expect the output signal to demonstrate excessive overshoot. In this respect, the D-term incorporates the slope of the error signal, $e(t)$, allowing to the controller to predict the imminent overshoot and to appropriately react in order to prevent it by generating a component in the control signal proportional to de/dt .

The effect of the D-term to the response of the closed-loop system will be now investigated when the process transfer function is of the 1st-order as in equation (3.8). The controller transfer function to be used in this case is shown below.

$$U(s) = \left(K_p + \frac{K_i}{s} + K_d s \right) \cdot E(s) = \frac{K_i + K_p s + K_d s^2}{s} \cdot E(s) \quad (3.20)$$

The general relation (3.1) for the closed-loop response yields

$$Y(s) = \frac{K_{ss}(K_i + K_p s + K_d s^2) \cdot R(s) + K_{ss} s \cdot D(s)}{(\tau + K_{ss} K_d) s^2 + (1 + K_{ss} K_p) s + K_{ss} K_i} \quad (3.21)$$

As can be readily seen in the above, all three coefficients of the closed-loop, 2nd-order characteristic polynomial may be set as desired by simply adjusting the gain of each one of the terms present in the PID control law.

A very interesting case of PID control, partially because of its widespread use in conventional ship course-keeping autopilots, is PD-control, i.e. when the I-term is absent from the control law ($K_i = 0$). Because of the missing I-term, steady-state error is exhibited in this case as well. Indeed, when $K_i = 0$ equation (3.21) yields

$$Y(s) = \frac{K_{ss}(K_p + K_d s)}{(\tau + K_{ss} K_d) s + (1 + K_{ss} K_p)} \cdot R(s) + \frac{K_{ss}}{(\tau + K_{ss} K_d) s + (1 + K_{ss} K_p)} \cdot D(s) \quad (3.22)$$

One can easily see that, by applying the Final Value Theorem of the Laplace Transform, for $R(s)=0$ and a step change in the disturbance signal, $d(t)=u_{step}(t)$, the steady-state response of the system, $y(t \rightarrow \infty)$, is the same with the one when simply P-control is employed, i.e.:

$$y(t \rightarrow \infty) = \frac{K_{ss}}{1 + K_{ss} K_p}$$

It is finally noted that commonly in practice use of D-term is avoided. Although some issues in the realization of the full PID control law do exist, the main drawback is the amplification of noise due to the nonzero D-term. Indeed, consider the following relationship, demonstrating the effect of real-time signal differentiation, in the natural frequency (Fourier) domain.

$$\mathcal{L}\left\{\frac{d}{dt}y(t)\right\} = s \cdot Y(s) \stackrel{s=j\omega}{=} (j\omega) \cdot Y(j\omega) = \omega \cdot |Y(j\omega)| \cdot \exp(\angle Y(j\omega) + \pi/2) \quad (3.23)$$

It is evident that high-frequency (HF, large ω), but otherwise weak in terms of power or amplitude, noise components inherent in the output signal, $y(t)$, are amplified because of differentiation. This enhancement is manifested by the ω -factor in the frequency response. Therefore, the differential error signal, de/dt , is many times so noisy that cannot be practically used, at least for control applications.

3.4 The Root Locus Technique

The Root Locus technique was developed by W.R. Evans for the design and comparative assessment of LTI-SISO systems. The methodology for the construction of the locus (i.e. the subset of the complex plane) of the closed-loop system poles, when a parameter (gain) of the controller varies, is shown in the present section. In specific, referring to Figure 3-1 and forcing the disturbance, d , to be identically zero, a pure analog control law, i.e. a gain K , will be used, as shown in Figure 3-9. The process transfer function will be presumed known in the factorized form given below.

$$G(s) = \frac{(s+z_1)(s+z_2)\dots(s+z_m)}{(s+p_1)(s+p_2)\dots(s+p_n)} = \frac{s^m + b_1s^{m-1} + \dots + b_m}{s^n + a_1s^{n-1} + \dots + a_n} \quad (3.24)$$

Note that for the Root Locus technique to be applied, all **poles** and **zeros** of the **open-loop**, process transfer function must be known or determined *in advance*.

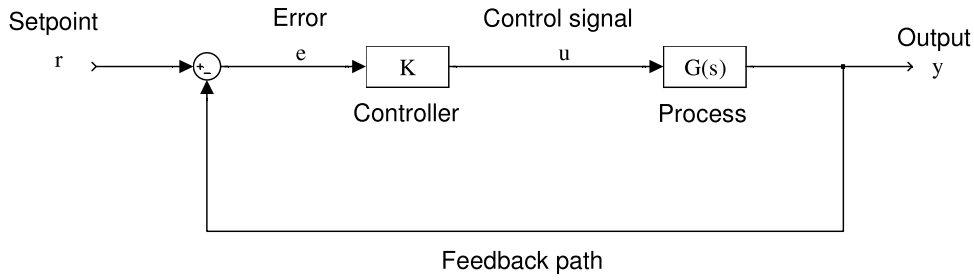


Figure 3-9: Closed-loop block diagram for Root Locus analysis.

In effect the closed-loop transfer function assumes the following form.

$$H(s) = \frac{K \cdot G(s)}{1 + K \cdot G(s)} = \frac{K \cdot (s + z_1)(s + z_2) \dots (s + z_m)}{(s + p_1)(s + p_2) \dots (s + p_n) + K \cdot (s + z_1)(s + z_2) \dots (s + z_m)} \quad (3.25)$$

The closed-loop characteristic polynomial, $p_c(s)$, is the denominator of the above and complying to our standard assumption that $n > m$ for the process transfer function, $G(s)$, the order of $p_c(s)$ will be equal to n .

The Root Locus (or simply the locus) of a closed-loop transfer function as in equation (3.25) is the location of all points on the complex plane satisfying the following for any $K \in (0, +\infty)$.

$$p_c(s) = (s + p_1)(s + p_2) \dots (s + p_n) + K \cdot (s + z_1)(s + z_2) \dots (s + z_m) = 0 \quad (3.26)$$

The locus of complex plane points that satisfy equation (3.26) for $K \in (-\infty, 0)$ is referred to as the Complementary Root Locus.

In the sequel, some fundamental properties of the Root Locus of polynomials with real coefficient are presented. These properties assist greatly in the graphical construction of the locus. Therefore, it is not necessary to determine mathematically the polynomial roots as functions of K . If this was not feasible, the locus determination would be an extremely tedious task or even impossible for polynomials of order higher than four. Proof of the lemmas given below as it can be sought after in standard control systems literature.

1. The departure points of the locus, i.e. those for $K = 0$, are the poles of $G(s)$.
2. The arrival points of the locus, i.e. those for $K \rightarrow \pm\infty$, are the zeros of $G(s)$.
3. The number of discrete, component loci is n .
4. The Root Locus and the Complementary Root Locus are symmetrical with respect to the real axis.

5. For $|s| \rightarrow +\infty$ the Root Locus asymptotically converges towards straight lines with slopes as follows.

$$\theta_i = \frac{2i+1}{n-m} \cdot \pi, \quad i = 0, 1, \dots, (n-m)-1 \quad (3.27)$$

Similarly, The Complementary Root Locus asymptotically converges towards straight lines with slopes as follows.

$$\theta_i = \frac{2i}{n-m} \cdot \pi, \quad i = 0, 1, \dots, (n-m)-1 \quad (3.28)$$

6. All $2(n-m)$ asymptotes in equations (3.27) and (3.28) for the Root Locus and the Complementary Root Locus pass through the same point on the real axis. This point is given below

$$\sigma_0 = -\frac{a_1 - b_1}{n-m} = -\frac{\sum_{i=1}^n p_i - \sum_{i=1}^m z_i}{n-m} \quad (3.29)$$

7. a) A real axis segment is part of the Root Locus provided that an **odd** cumulative number of poles and zeros of $G(s)$ are located on its right-hand side.
- b) A real axis segment is part of the Root Locus provided that an **even** cumulative number of poles and zeros of $G(s)$ are located on its right-hand side.
8. The *breakpoints* (corner points) of the locus are the repeated (multiple) roots of the closed-loop characteristic polynomial $p_c(s)$; those can be determined as the roots of the following equation.

$$\frac{d}{ds}[K \cdot G(s)] = 0 \Rightarrow \frac{d}{ds}G(s) = 0 \quad (3.30)$$

9. a) The Root Locus angle of departure, ϑ_j , from the j -th pole of $G(s)$, $(-p_j)$, is given by employing the following condition for $s \rightarrow (-p_j)$.

$$1 + K \cdot G(s) = 0 \Rightarrow \begin{cases} |K \cdot G(s)| = 1 \\ \angle K \cdot G(s) = \pm(2v+1)\pi, v = 0, 1, 2, \dots \end{cases}$$

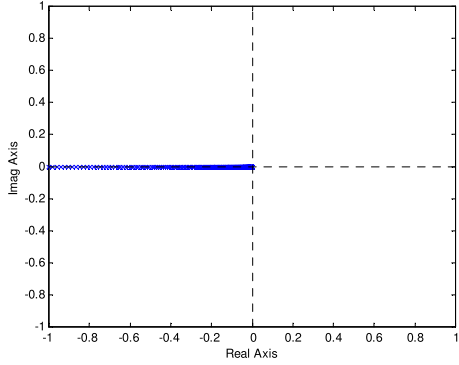
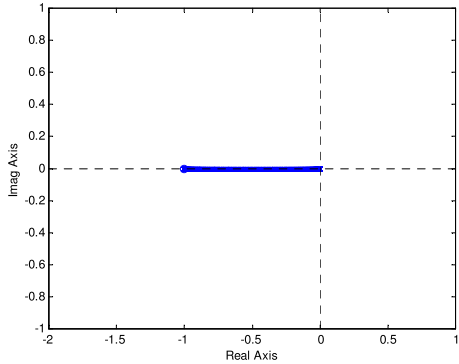
The second (phase-related) of the above yields for $s \rightarrow (-p_j)$:

$$\lim_{s \rightarrow (-p_j)} [\angle K \cdot G(s)] = \sum_{i=1}^m \angle(z_i - p_j) - \sum_{\substack{i=1 \\ i \neq j}}^n \angle(p_i - p_j) - \vartheta_j = \pm(2v+1)\pi \quad (3.31)$$

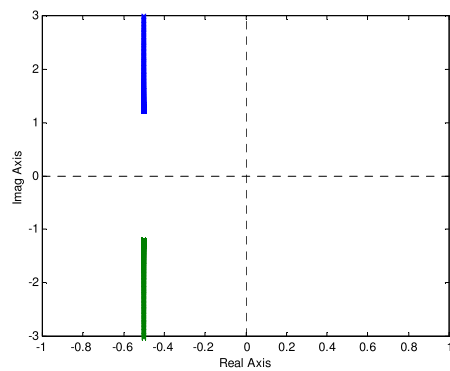
b) The Root Locus angle of arrival, φ_j , at the j -th zero of $G(s)$, $(-z_j)$, is given by the same condition as the one used in 9-a but applied for $s \rightarrow (-z_j)$.

$$\begin{aligned} \lim_{s \rightarrow (-z_j)} [\angle K \cdot G(s)] &= \varphi_j + \sum_{\substack{i=1 \\ i \neq j}}^m \angle(z_i - z_j) - \sum_{i=1}^n \angle(p_i - z_j) = \\ &= \pm(2v+1)\pi, v = 0, 1, 2, \dots \end{aligned} \quad (3.32)$$

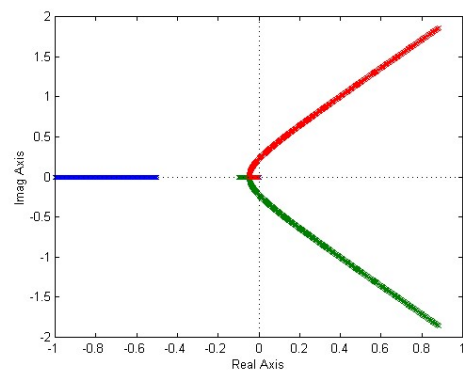
The properties presented previously are now applied to the construction of the root locus of various examples of process transfer function, $G(s)$, as in equation (3.25).

$G(s)$	Root Locus
$\frac{1}{s}$	
$\frac{(s+1)}{s}$	

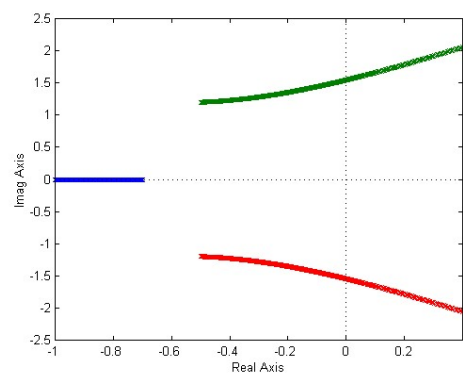
$$\frac{1}{(s^2 + s + 1.69)}$$



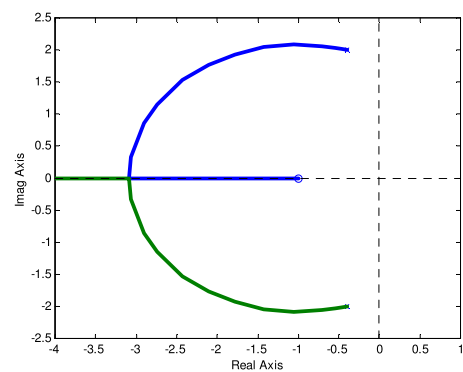
$$\frac{1}{s(s + 0.1)(s + 0.5)}$$



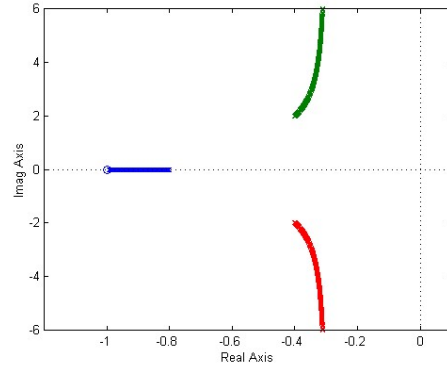
$$\frac{1}{(s + 0.7)(s^2 + s + 1.69)}$$



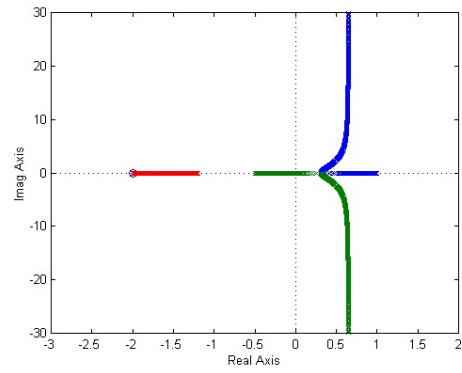
$$\frac{(s + 1)}{(s^2 + 0.8s + 4.16)}$$



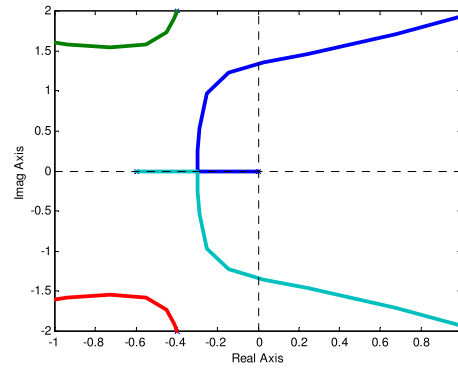
$$\frac{(s+1)}{(s+0.8)(s^2+0.8s+4.16)}$$



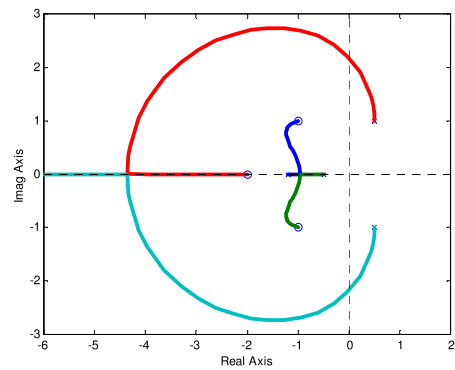
$$\frac{(s+2)}{(s+0.5)(s+1.2)(s-1)}$$



$$\frac{1}{s(s+0.6)(s^2+0.8s+4.16)}$$



$$\frac{(s+2)(s^2+2s+2)}{(s+0.5)(s+1.2)(s^2-s+1.25)}$$



4. COURSE-KEEPING AUTOPILOTS

4.1 The vessel in the control loop

Surface vessel motions

A model describing the motions of a typical surface ship is needed. Such model should be simple, concise as well as intuitive and fit for use in the control synthesis process.

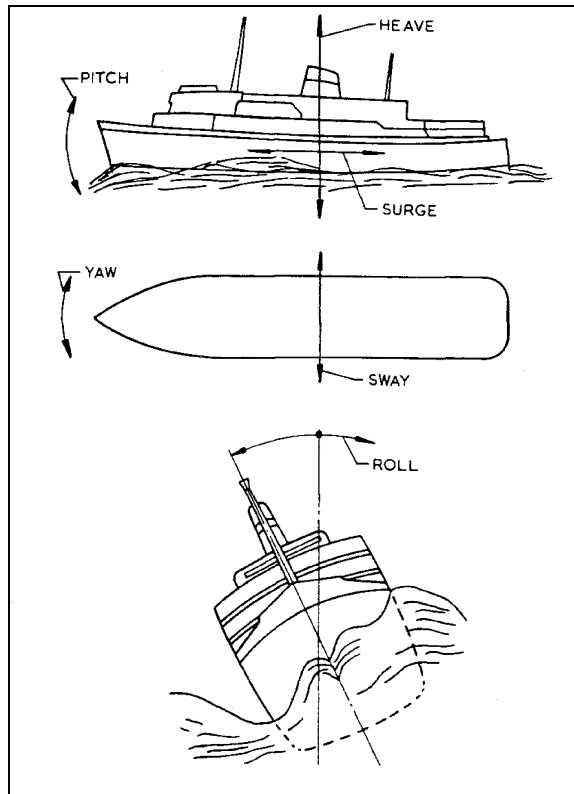


Figure 4-1: Surface ship motions.

The motions of a surface ship can be decomposed into six degrees of freedom when the vessel is considered as a rigid body (beam). In Figure 5-1 the degrees of freedom of a ship's motion are shown and named.

The dynamic response of a vessel is modeled by a set of differential equations with respect to time. These equations concern the rate of change of linear and angular momentum of the vessel. With respect to a body-fixed (therefore non-inertial in general) frame of reference, as used in ship dynamics, the equations of motion are given below.

$$\begin{aligned} \frac{d}{dt} \left\{ \int_{m_s} (\vec{v}_s + \vec{\omega}_s \times \vec{r}) dm \right\} &= \Sigma \vec{F}_{ext} \\ \frac{d}{dt} \left\{ \int_{m_s} \vec{r} \times (\vec{v}_s + \vec{\omega}_s \times \vec{r}) dm \right\} + m_s (\vec{v}_s \times \vec{v}_G) &= \Sigma \vec{M}_{ext} \end{aligned} \quad (4.1)$$

In the above m_s is the vessel's mass, $\vec{v}_s$ the linear velocity of the origin O of the body-fixed reference frame (wrt to an inertial reference frame), $\vec{\omega}_s$ the angular velocity of the origin O of the body-fixed reference frame (wrt to an inertial reference frame) and $\vec{v}_G$ the translational motion of the ship's center of mass. Vector $\vec{r}$ stands for the position vector of an arbitrary mass infinitesimal element dm shipboard with respect to the body-fixed frame. The two vector sums on the right-hand side of the differential equations above are the total force and moment applied externally to the vessel. The spatial integrals on the left hand side are the linear and angular momentum of the vessel. Finally, note the inertial terms due to the fact that a non-inertial reference frame is used moving at velocity $\vec{v}_s$ with respect to an inertial one. The body-fixed and inertial reference frames used are shown in Figure 4-2. In the same figure, the symbols used to denote the translational and rotational velocities are shown as well as the ones used to denote the external forces and moments. On the axes of the inertial frame the symbols used for the infinitesimals of rotation and translation are given.

Choosing a body-fixed instead of an inertial reference frame, is based on the following reasoning: a) Acceleration measurements onboard provide with the absolute, inertial values but decomposed to the body-fixed reference frame instead of the inertial one; b) The geometric and inertia characteristics of the vessel are time-invariant with respect to the body-fixed frame. This fact mathematically is expressed as $\dot{\vec{r}} = 0$. Also, it can be used to swap spatial integration with time differentiation in equation (4.1).

$$\begin{aligned} \int_{m_s} \frac{d}{dt} \{ \vec{v}_s + \vec{\omega}_s \times \vec{r} \} dm &= \Sigma \vec{F}_{ext} \\ \int_{m_s} \frac{d}{dt} \{ \vec{r} \times (\vec{v}_s + \vec{\omega}_s \times \vec{r}) \} dm + m_s (\vec{v}_s \times \vec{v}_G) &= \Sigma \vec{M}_{ext} \end{aligned} \quad (4.2)$$

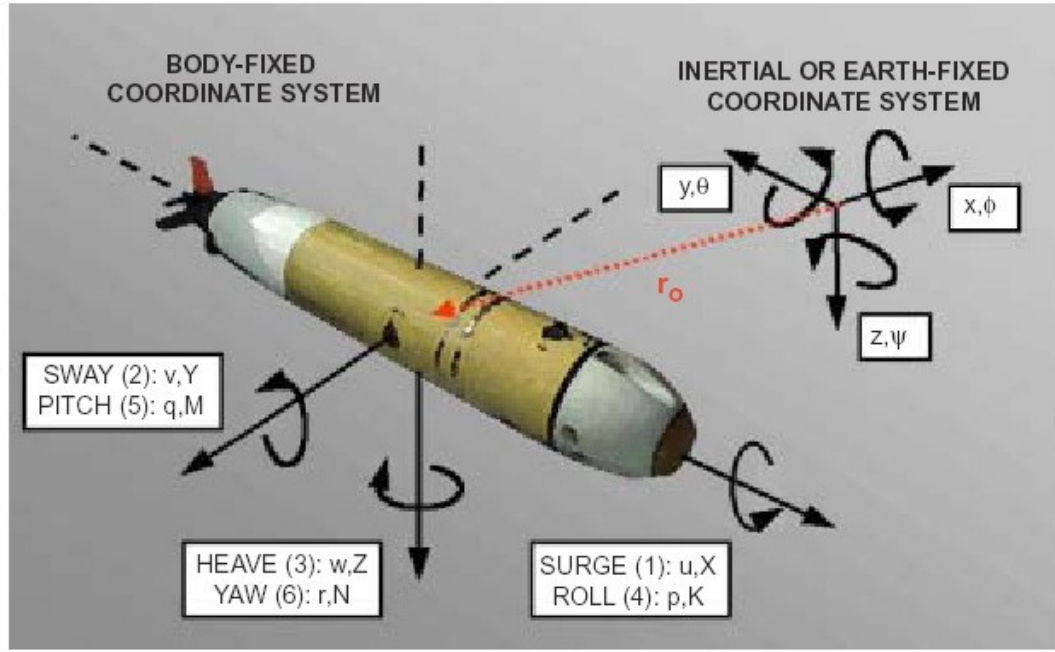


Figure 4-2: Inertial and body-fixed reference frame used in the analysis.

Formulation of the linearized state equations

Using the above, as well as the linearization procedure presented in Chapter 1, the generic surface vessel equations given previously are simplified in the sense of linearization. The linearized equations are derived by neglecting higher-order, nonlinear terms in the original ship motion equations. Such terms are those including powers higher than one of $\vec{v}_s$ or $\vec{\omega}_s$ or cross-products thereof or their time derivatives. Furthermore, linearization is applied by considering perturbations around a stable equilibrium point. The derived set of linear ordinary differential equations is valid for small-amplitude variations of $\vec{v}_s$ and $\vec{\omega}_s$ around the equilibrium point. Therefore, the linearized differential equations hold for variations $\delta\vec{v}_s$ and $\delta\vec{\omega}_s$. However, in order to simplify the expressions the δ -notation is dropped, since the classical linear feedback control problem is formulated and solved on the basis of the linearized plant equations.

In the sequel the 3×3 following inertia tensor, $\tilde{I}_s$, which is time invariant with respect to the body-fixed frame, is needed.

$$\tilde{I}_s = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix} \quad (4.3)$$

The scalar inertial entries in tensor $\tilde{I}_s$ are given for I_{xx} and I_{xy} . For the other entries, definitions are analogous.

$$I_{xx} = \int_V (y^2 + z^2) dm, \quad I_{xy} = - \int_V xy dm \quad (4.4)$$

For most practical autopilot designs, the following simplifying assumption that the body-fixed frame has been selected so as $\vec{r}_G = (x_G \ 0 \ 0)^T$.

The small-signal analysis for autopilot design in the case of ocean-going vessels moving in a time-averaged sense forward at constant speed U , can be applied if the following decomposition scheme is considered.

$$\vec{v}_s = \begin{bmatrix} U+u \\ v \\ w \end{bmatrix} = \begin{bmatrix} U \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} u \\ v \\ w \end{bmatrix} \text{ with } \frac{dU}{dt} = \omega_s \times U, \text{ i.e. } \dot{U} = 0 \quad (4.5)$$

The equilibrium point around which the linearization is applied has to be evidently $\vec{U} = (U \ 0 \ 0)^T$

After analyzing and linearizing the equations, the following set of 1st-order, linear differential equations are derived.

$$\begin{aligned} \text{surge:} \quad & m_s \dot{u} = X \\ \text{sway:} \quad & m_s [\dot{v} + Ur + x_G \dot{r}] = Y \\ \text{heave:} \quad & m_s [\dot{w} - Uq - x_G \dot{q}] = Z + Z_{hs} \\ \text{roll:} \quad & I_{xx} \dot{p} = K + K_{hs} \\ \text{pitch:} \quad & I_{yy} \dot{q} + m_s x_G (-\dot{w} + Uq) = M + M_{hs} \\ \text{yaw:} \quad & I_{zz} \dot{r} + m_s x_G (\dot{v} + Ur) = N \end{aligned} \quad (4.6)$$

The above are derived on the assumption of a hull symmetric with respect to plane xz. Subscript *hs* indicates a hydrostatic component in a force or moment. Hydrostatic components are in general dependent upon displacement variables like translational ones x, y, z , as well as rotational ones ϕ, θ, ψ ; however, hydrostatic components cannot depend on translational or rotational velocities. A force or moment component that depends on the vessel's translational (u, v, w) or rotational velocities (p, q, r) or higher-order time derivatives (like e.g. accelerations) is called hydrodynamic.

As seen in equation (4.6), after linearization surge, sway and yaw become decoupled of roll, pitch and heave. Furthermore, note that the forces and moments affecting the surge-sway-heave motions are purely hydrodynamic. At this point, expressions for X, Y and N

are needed. These excitation signals may originate: a) From the ocean-hull interaction; b) From control surface actuation. The most common control surface of surface vessels is the rudder. Other control surfaces include anti-roll stabilizing fins etc. c) From perturbation of random nature that is either difficult or impossible or even too complicated to be modeled.

$$\begin{aligned} X(t) &= X_R(t) + X_H(t) + X_0(t) \\ Y(t) &= Y_R(t) + Y_H(t) + Y_0(t) \\ N(t) &= N_R(t) + N_H(t) + N_0(t) \end{aligned} \quad (4.7)$$

In the above, subscript R indicates the component due to rudder action, H the one due to the ocean-hull interaction and 0 a random signal due to modeling errors and simplifications.

The rudder components evidently depend exclusively on the deflection of the rudder from the vertical xz symmetry plane of the vessel's hull.

$$X_R(t) = X_R(\delta_R(t)), Y_R(t) = Y_R(\delta_R(t)), N_R(t) = N_R(\delta_R(t)) \quad (4.8)$$

For small (in the absolute sense) deflection values of angle δ_R the following relationships can be employed.

$$X_R = X_\delta \cdot \delta_R, Y_R = Y_\delta \cdot \delta_R, N_R = N_\delta \cdot \delta_R \quad (4.9)$$

In the above hydrodynamic derivatives X_δ , Y_δ and N_δ are defined as follows.

$$X_\delta = \frac{\partial X_R}{\partial \delta_R}, Y_\delta = \frac{\partial Y_R}{\partial \delta_R}, N_\delta = \frac{\partial N_R}{\partial \delta_R} \quad (4.10)$$

In a similar manner, the components originating from ocean-hull interaction are dependent solely on kinematical variables of the vessel.

$$X_H = X_H(u, \dot{u}), Y_H = Y_H(v, \dot{v}, r, \dot{r}), N_H = N_H(v, \dot{v}, r, \dot{r}) \quad (4.11)$$

By using the small-signal analysis concept, the following expressions can be obtained.

$$\begin{aligned} X_H &= X_u \cdot u + X_{Du} \cdot \dot{u} \\ Y_H &= Y_v \cdot v + Y_{Dv} \cdot \dot{v} + Y_r \cdot r + Y_{Dr} \cdot \dot{r} \\ N_H &= N_v \cdot v + N_{Dv} \cdot \dot{v} + N_r \cdot r + N_{Dr} \cdot \dot{r} \end{aligned} \quad (4.12)$$

In the above the hydrodynamic derivatives X_u , X_{Du} , Y_v , Y_{Dv} , Y_r , Y_{Dr} , N_v , N_{Dv} , N_r , N_{Dr} are defined similarly to the ones related to the rudder in equation (4.10).

4.2 Surface vessel state-space model

On the basis of the decoupled model of surge, sway and yaw in equations (4.6) as well as the expressions (4.7), (4.9) and (4.12) for the exogenous forces and moments applied to a vessel, the following set of differential equations can be obtained.

$$\begin{aligned} (m_s - X_{Du}) \cdot \dot{u} &= X_u \cdot u + X_\delta \cdot \delta_R + X_0(t) \\ (m_s - Y_{Dv}) \cdot \dot{v} + (m_s x_G - Y_{Dr}) \cdot \dot{r} &= Y_v \cdot v + (Y_r - m_s U) \cdot r + Y_\delta \cdot \delta_R + Y_0(t) \\ (I_{zz} - N_{Dr}) \cdot \dot{r} + (m_s x_G - N_{Dv}) \cdot \dot{v} &= N_v \cdot v + (N_r - m_s x_G U) \cdot r + N_\delta \cdot \delta_R + Z_0(t) \end{aligned} \quad (4.13)$$

In matrix form and after omitting the equation for surge (which is not coupled with the ones for yaw and sway), the following is obtained.

$$\begin{aligned} \Theta \cdot \dot{x}_\varnothing &= A_\varnothing \cdot x_\varnothing + b_\varnothing \cdot \delta_R + d_\varnothing \\ \Downarrow \\ \begin{bmatrix} m_s - Y_{Dv} & m_s x_G - Y_{Dr} \\ m_s x_G - N_{Dv} & I_{zz} - N_{Dr} \end{bmatrix} \cdot \begin{bmatrix} \dot{v} \\ \dot{r} \end{bmatrix} &= \begin{bmatrix} Y_v & Y_r - m_s U \\ N_v & N_r - m_s x_G U \end{bmatrix} \cdot \begin{bmatrix} v \\ r \end{bmatrix} + \begin{bmatrix} Y_\delta \\ N_\delta \end{bmatrix} \cdot \delta_R + \begin{bmatrix} Y_0 \\ N_0 \end{bmatrix} \end{aligned} \quad (4.14)$$

The correspondence between the brief and extended versions of equation (4.14) is obvious.

Assuming that matrix Θ in (4.14) is invertible the following are obtained.

$$\Theta \cdot \dot{x}_\varnothing = A_\varnothing \cdot x_\varnothing + b_\varnothing \cdot \delta_R + d_\varnothing \Leftrightarrow \dot{x}_\varnothing = \Theta^{-1} A_\varnothing \cdot x_\varnothing + \Theta^{-1} b_\varnothing \cdot \delta_R + \Theta^{-1} d_\varnothing \quad (4.15)$$

$$\dot{x}_\varnothing = \Theta^{-1} A_\varnothing \cdot x_\varnothing + \begin{bmatrix} b & \Theta^{-1} \end{bmatrix} \cdot \begin{bmatrix} \delta_R \\ d_\varnothing \end{bmatrix}, b = \Theta^{-1} b_\varnothing \quad (4.16)$$

The above is the dynamic equation of a system with 2 states and 3 scalar input signals one of which is the control action (rudder deflection angle δ_R); the other two inputs are the random disturbances Y_0 and N_0 . For reasons made clear later on, the state vector, $x_\varnothing$, in (4.16) has to be enriched by the yaw angle, ψ , as follows.

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & \Theta^{-1} \cdot A_\varnothing \\ 0 \end{bmatrix} \cdot x + \begin{bmatrix} 0 & 0 & 0 \\ b & \Theta^{-1} \end{bmatrix} \cdot \begin{bmatrix} \delta_R \\ d_\varnothing \end{bmatrix} \Leftrightarrow \dot{x} = A \cdot x + B \cdot u \text{ with } x = \begin{bmatrix} \psi \\ v \\ r \end{bmatrix} \quad (4.17)$$

By employing the Laplace transform the following relation is obtained.

$$x(s) = (sI_3 - A)^{-1} \cdot \begin{bmatrix} 0 \\ b \end{bmatrix} \cdot \delta_R(s) + (sI_3 - A)^{-1} \cdot d(s) \text{ where } d = \begin{bmatrix} 0 \\ \Theta^{-1} d_\varnothing \end{bmatrix} \quad (4.18)$$

In many commonplace cases of autopilot design, the single feedback signal is the yaw angle, $\psi(t)$, through means of either a magnetic compass or a gyrocompass. In these cases

the input to the controller (autopilot) is the yaw error, i.e. the difference obtained by subtracting ψ from the heading setpoint.

In effect, the dynamical equation (4.18) is complemented by the following output, algebraic one connecting the state vector x to the scalar output signal $\psi(t)$ (or $\psi(s)$ in the complex frequency domain).

$$y = c \cdot x = [1 \quad 0 \quad 0] \cdot x = \psi \quad (4.19)$$

Since x is of order 3, the transfer function from the control action δ_R to the output signal ψ is of order 3 as well. Because of the special form of matrix A (exhibiting a full zero column) one of the eigenvalues and open-loop plant poles is 0. The full characteristic polynomial is given by the following.

$$p_0(s) = s \cdot |sI_2 - \Theta^{-1} \cdot A_\varnothing| \quad (4.20)$$

Combining (4.20) in the generic matrix inversion formula the following transfer function from the control action, δ_R , to yaw angle, ψ , is obtained.

$$H_N(s) = \frac{\psi(s)}{\delta_R(s)} = \frac{\kappa_0 + (s + \kappa_1)}{p_0(s)} = \frac{k + k\tau_0 s}{s \cdot (\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2)s + 1)} \quad (4.21)$$

The transfer function above connecting δ_R with ψ is known as the Nomoto equation. For most practical cases of displacement hulls, the factor $|sI_2 - \Theta^{-1} \cdot A_\varnothing| = (\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2)s + 1)$ exhibits two purely real roots denoted as $-1/\tau_1$ and $-1/\tau_2$.

The full expression for yaw angle, ψ , including the disturbance signals Y_0 and N_0 , in the complex frequency domain, has the following form.

$$\psi(s) = H_N(s)\delta_R(s) + T_N(s)N_0(s) + T_Y(s)Y_0(s) \quad (4.22)$$

Similarly to the Nomoto equation transfer function, the expressions for transfer functions $T_N(s)$ and $T_Y(s)$ can be obtained.

$$T_N(s) = \frac{k_N + k_N \tau_N s}{s \cdot (\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2)s + 1)} \quad \text{and} \quad T_Y(s) = \frac{k_Y + k_Y \tau_Y s}{s \cdot (\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2)s + 1)} \quad (4.23)$$

4.3 PID Autopilots

Many practical, specifically low-end autopilots, employ the PID control law, given below.

$$\begin{aligned}
\delta_R(t) &= -K_p \cdot \psi(t) - K_i \cdot \int_0^t \psi(\xi) d\xi - K_d \cdot \frac{d}{dt} \psi(t) \\
&\quad \Downarrow \\
\delta_R(s) &= -K(s) \cdot \psi(s) = -\left(K_p + \frac{K_i}{s} + K_d \cdot s \right) \cdot \psi(s) = -\frac{K_i + K_p \cdot s + K_d \cdot s^2}{s} \cdot \psi(s)
\end{aligned} \tag{4.24}$$

Substituting the above in equation (4.22) the following closed-loop expressions can be obtained.

$$\psi(s) = G_N(s)N_0(s) + G_Y(s)Y_0(s) = \frac{T_N(s)}{1 + H_N(s)K(s)}N_0(s) + \frac{T_Y(s)}{1 + H_N(s)K(s)}Y_0(s) \tag{4.25}$$

In effect, transfer functions $G_N(s)$ and $G_Y(s)$ from the yaw moment and sway force disturbance have to be of the 4th order according to equations (4.21) and (4.23). However, for many practical applications the following 1st-order open-loop transfer functions consist a good approximation.

$$\tilde{H}_N(s) = \frac{k}{s \cdot (\tau_1 s + 1)}, \quad \tilde{T}_N(s) = \frac{k_N}{s \cdot (\tau_1 s + 1)} \quad \text{and} \quad \tilde{T}_Y(s) = \frac{k_Y}{s \cdot (\tau_1 s + 1)} \tag{4.26}$$

For the above approximants to be valid, it must hold that $\tau_1 \gg \tau_2$ and in effect for $s \rightarrow 0$, $(\tau_1 \tau_2 s^2 + (\tau_1 + \tau_2)s + 1) \approx (\tau_1 s + 1)$. Similar assumptions are needed for the time constants in the numerator of the full-order transfer functions. For most practical displacement hull forms, $\tau_1 \gg \tau_2$ and therefore the so-called dominant pole technique is applicable.

In effect

$$\tilde{G}_N(s) = \frac{\frac{k_N}{s \cdot (\tau_1 s + 1)}}{1 + \frac{K_i + K_p \cdot s + K_d \cdot s^2}{s} \cdot \frac{k_N}{s \cdot (\tau_1 s + 1)}} = \frac{k_N s}{\tau_1 s^3 + (k_N K_d + 1)s^2 + k_N K_p s + k_N K_i} \tag{4.27}$$

The expression for $\tilde{G}_Y(s)$ is similar and as seen the reduced order transfer functions will be of the 3rd-order. Further reduction (simplification) can be achieved by intervening in the controller structure.

Such intervention is the design assumption that $K_i \ll K_p, K_d$. This assumption is based on the fact that the integral term is included in the PID control law only for eliminating the long-term, near-steady-state effects of the Y and N disturbance signals. That is why the I-term constant can assume relatively small values. Furthermore as seen in equation (4.27) the I-term constant appears exclusively in the constant (zero-order) term of the closed-loop

characteristic polynomial. Therefore, small I-term constant values will give a closed-loop pole in the vicinity of 0. Such a pole, however, will cancel out with the closed-loop zero at 0. In effect, the following simplified closed-loop transfer function can be obtained for small values of K_i .

$$\tilde{G}'_N(s) = \frac{k_N}{\tau_1 s^2 + (k_N K_d + 1)s + k_N K_p} \quad (4.28)$$

Finally, note that for small, purely positive values of K_i the pole $s=0$ moves inside the left-hand complex plane and the closed-loop system becomes asymptotically stable.

In conclusion, two of the poles of $\tilde{G}_N(s)$ can be determined as ones of $\tilde{G}'_N(s)$, whilst the third one, that happens to be the most dominant since it is closer to the imaginary axis at least for small values of K_i , is the one that coincides with the origin for $K_i = 0$.

In effect, the poles of $\tilde{G}'_N(s)$ define whether the response of the vessel to yaw moment disturbance exhibits oscillations or not. Indeed if the value of the damping constant, ζ , of the 2nd-order characteristic polynomial $\tilde{p}'_N(s) = \tau_1 s^2 + (k_N K_d + 1)s + k_N K_p$ is below 0.7 then oscillations in the response are to be expected. In effect, proper selection of constants K_p , K_d can be made in order to eliminate excessive oscillatory behavior in closed-loop response. On the other hand, all controller constants, but predominantly K_i , contribute to obtain a closed-loop system that is adequately stable. Similar conclusions hold for transfer function $\tilde{G}_Y(s)$, too.

4.4 State observers and use in autopilots

A rather important problem to apply the P(I)D control law arising in the common marine autopilot system, is the implementation of the D-term. In the marine autopilot, the D-term corresponds to the yaw rate.

Indeed, the only signal available for feedback is the heading $\psi(t)$ and its differentiation is not recommended in order to avoid amplifying the noise in the control loop. A solution to this problem in practice is to employ a system called a *state-vector observer* (or *estimator*).

In specific, a state observer allows reconstruction on the basis of the measurement (output) vector value. In the case of marine autopilots, the state vector is defined within the framework of the sway-yaw as introduced in the notes. The state observer then obtains the form below:

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{A}} \cdot \hat{\mathbf{x}} + \hat{\mathbf{B}} \cdot \delta_R + \mathbf{L} \cdot \psi \quad (4.29)$$

where $\hat{x}$ is the state vector estimate (output of the state observer) and $\hat{A}$, $\hat{B}$ & L matrices of appropriate dimensions. Since they define the state observer equations, they need to be determined.

In the 60s, Luenberger proposed a method to establish the observer matrices based on the pole placement principle introduced in the notes. Prerequisites for Luenberger's method are:

- 1) The system under investigation upon which state observation is to be applied, has to be **observable**.
- 2) The measurement signals (in the case of the autopilot this is the heading error) need to be reasonably free of any noise or interference.

In the case that the second of the prerequisites is not met it is possible to construct a state observer by employing optimal filtering techniques. The observer derived in this technique is known from signal theory as Kalman filter.

The Luenberger approach, that will be developed here is based on obtaining a dynamic equation for the state prediction error $\hat{e}$, defined as follows:

$$\hat{e} = x - \hat{x} \Rightarrow \dot{\hat{e}} = \dot{x} - \dot{\hat{x}} \quad (4.30)$$

By substituting in the above $\dot{\hat{x}}$ from eq. (4.29) as defined previously, for $\dot{x}$ the system's dynamic equation from the state equations and ψ from the output equation the following is readily obtained:

$$\begin{aligned} \dot{\hat{e}} &= A \cdot x + \begin{bmatrix} 0 \\ b \end{bmatrix} \cdot \delta_R + d - \left[\hat{A} \cdot (x - \hat{e}) + \hat{B} \cdot \delta_R + Lc \cdot x \right] \\ &\quad \Downarrow \\ \dot{\hat{e}} &= \hat{A} \cdot \hat{e} + (A - Lc - \hat{A}) \cdot x + \left(\begin{bmatrix} 0 \\ b \end{bmatrix} - \hat{B} \right) \cdot \delta_R + d \end{aligned} \quad (4.31)$$

By setting:

$$\hat{A} = A - Lc \quad \text{and} \quad \hat{B} = \begin{bmatrix} 0 \\ b \end{bmatrix} \quad (4.32)$$

Equation (4.31) for the state estimation error becomes:

$$\dot{\hat{e}} = \hat{A} \cdot \hat{e} + d = (A - Lc) \cdot \hat{e} + d \quad (4.33)$$

Using the Laplace Transform:

$$\hat{e}(s) = (sI_3 - A + Lc)^{-1} \cdot d(s) \quad (4.34)$$

The equation above is essentially a dynamic equation whose poles can be placed ad hoc by appropriately selecting the values of matrix L . Therefore, the state estimate $\hat{x}$ is given by the following:

$$\dot{\hat{x}} = A \cdot \hat{x} + \begin{bmatrix} 0 \\ b \end{bmatrix} \cdot \delta_r + L \cdot (\psi - c \cdot \hat{x}) \quad (4.35)$$

Moreover, a good choice for the entries L the estimation error $\hat{e}$ tends to coincide with random vector d . It is noted here that the size of d decides the accuracy of the estimation for x since the estimate $\hat{x}$ cannot approximate x any better.

Also note that Equation (4.35) is essentially the state-space dynamic equation of the system augmented by term $L \cdot (\psi - c \cdot \hat{x})$. This term is a correction since with a properly set L allows generating estimates $\hat{x}$ so that model (4.35) of the system approximates the actual system dynamics.

A Luenberger state observer obviously generates not just yaw rate r needed in PID autopilots but for the entire state vector including the sway velocity v . Therefore it can be used to implement advanced control schemes as well like e.g. pole placement with full state feedback as presented in the notes.

5. POLE PLACEMENT OF LTI SYSTEMS

5.1 Introduction

Modern control design techniques rely heavily on the system's description in state space. The reason for this is the need to develop control systems for SISO as well as MIMO systems.

In the remaining of this chapter LTI systems are considered. For the open-loop system (referred to as plant or process) a state-space description must be provided of the following general form.

$$\begin{aligned}\dot{x} &= A \cdot x + B \cdot u \\ y &= C \cdot x\end{aligned}\tag{5.1}$$

In the above, u stands for the m -dimensional input vector (excitations), x for the n -dimensional state vector and y for the p -dimensional output vector (measurements) of the LTI system at hand. By manipulating equation (5.1) the following open-loop transfer function is obtained.

$$Y(s) = H(s) \cdot U(s) = C \cdot (sI_n - A)^{-1} \cdot B \cdot U(s)\tag{5.2}$$

Input-output decoupling

Consider the case below where the open-loop transfer function matrix $H(s)$ is square and diagonal, i.e. the process has the same number of inputs and outputs ($p=m$).

$$H(s) = \begin{bmatrix} H_{11}(s) & 0 & \dots & 0 \\ 0 & H_{22}(s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & H_{mm}(s) \end{bmatrix}\tag{5.3}$$

It is straightforward that the LTI-MIMO system is equivalent to m in number LTI-SISO systems with scalar transfer functions those appearing as the diagonal, nonzero elements of the transfer function matrix, $H(s)$. This case is referred to by the term *input-output decoupling*.

It is evident that an input-output decoupled system as in equation (5.3) is only nominally MIMO because each and every one of the output signals is connected exclusively to a single input. In effect, for each and everyone of the m SISO systems a separate controller may be designed with techniques relevant to LTI-SISO systems.

Full state feedback

Given the description in equation (5.1) the control law is of the following form.

$$u = L \cdot r + F \cdot x \quad (5.4)$$

In the above, r is a new input vector applied to the closed-loop system with dimension m_r . The controller design problem consists of determining the $m \times m_r$ matrix L and the $m \times n$ matrix F so that the closed-loop system demonstrates specified characteristics. The block diagram of this control scheme is shown in the figure below.

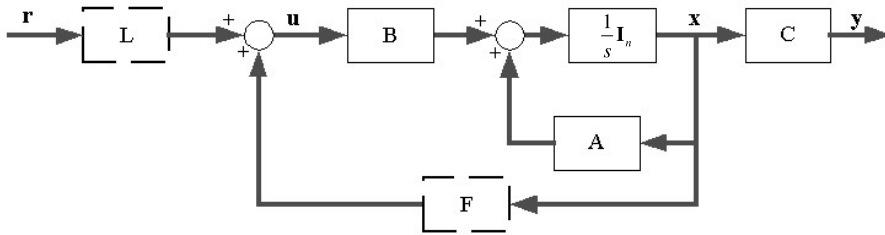


Figure 5-1: Block diagram of a system with full state feedback.

A typical example of closed-loop system specifications is input-output decoupling. In this case the transfer function matrix, $H(s)$, is square but not diagonal. Therefore, a control law of the general form in equation (5.4) is sought after that can render the closed-loop transfer function matrix diagonal. In effect, in the case of input-output decoupling it holds that $m=m_r=p$. By substitution of the control law in equation (5.4) in the state equations (5.1) and then application of the Laplace transform, the following is obtained.

$$\left. \begin{array}{l} \dot{x} = (A+BF) \cdot x + BL \cdot r \\ y = C \cdot x \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} sX(s) = (A+BF) \cdot X(s) + BL \cdot R(s) \\ Y(s) = C \cdot X(s) \end{array} \right\} \quad (5.5)$$

After manipulation the above yields

$$Y(s) = G(s) \cdot R(s) = C \cdot (sI - A - BF)^{-1} \cdot BL \cdot R(s) \quad (5.6)$$

Therefore, the control design problem is to determine matrices F and L in the control law so that the closed-loop transfer function matrix $G(s)$ is diagonal. If this is achieved each and everyone component of the closed-loop system's input vector r is connected exclusively to one of the components of the output vector. Therefore, if further improvement of the system behavior is desired, m outer and decoupled SISO control loops may be employed.

5.2 Pole Placement Design

Generalities

As mentioned in section 1.4 the poles of a multivariable LTI system are the eigenvalues of the system's matrix A and therefore can be computed as the roots of the characteristic polynomial.

$$|sI - A| = 0 \quad (5.7)$$

The poles of a system determine its dynamics and mainly stability and transient behavior. This is the reason why in many situations full state feedback is employed in order to relocate the poles of the open-loop system to new locations on the complex plane, specified for the closed-loop system. In effect, the related control problem is referred to as either *pole placement* or *eigenvalue assignment*. Solving the pole placement problem, allows for a closed-loop system with significantly improved characteristics. A great success story of pole placement with full state feedback is to convert an unstable system, i.e. an open-loop plant with one or more poles in the right-hand complex plane, to a stable closed-loop system.

A prerequisite for the applicability of linear, full state feedback to a specific pole placement problem is that the open-loop plant is controllable according to the important theoretical result by Wohnam (1967). In specific, if a system is state-controllable then all of its poles may be relocated arbitrarily on the complex plane. In the case that the system is uncontrollable, pole placement through state feedback is still applicable but only for a subset of the open-loop poles; the poles in this subset are the plant's controllable poles.

Given an LTI system as in equation (5.1) the closed-loop characteristic polynomial with full state feedback can be directly obtained by use of equation (5.6).

$$p_c(s) = |sI - A - BF| = s^n + a_{n-1} \cdot s^{n-1} + \dots + a_1 \cdot s + a_0 \quad (5.8)$$

In the above, the coefficients of polynomial $p_c(s)$ are functions of the elements in the $m \times n$ feedback matrix F . Furthermore, for the pole placement problem it can be assumed that $m = m_r$ and $L = I_m$.

The specifications in a pole placement problem consist of the desired locations of the closed-loop poles on the complex plane. As mentioned before, physically realizable systems, i.e. with real parameters, have either purely real poles or pairs of complex conjugate poles. Therefore, in the factorized form of the characteristic polynomial of any physically realizable system, a purely real pole p corresponds to a factor of the form $(s - p)$ whilst a pair of complex conjugate poles p and p^* corresponds to the 2nd-order factor $(s - p) \cdot (s - p^*) = s^2 - 2 \cdot \text{Re}(p) \cdot s + |p|^2$. By multiplicatively combining all such

factors corresponding to the desired closed-loop poles and algebraically manipulating their product, a polynomial in reduced form like $\hat{p}_c(s) = s^n + \hat{a}_{n-1} \cdot s^{n-1} + \dots + \hat{a}_1 \cdot s + \hat{a}_0$ is obtained.

Then, one has to equate the coefficients of the same powers of s of polynomials $p_c(s)$ and $\hat{p}_c(s)$. By doing that, a linear algebraic system of n equations is obtained, which if solved provides for the constants in the feedback matrix F of the controller.

$$a_v(F_{ij}; i = 1, \dots, m, j = 1, \dots, n) = \hat{a}_v, v = 1, \dots, n \quad (5.9)$$

In general, the number of equations of the algebraic system above is smaller than the number of unknowns. This means that after successfully solving the algebraic system (5.9), there remain $(m-1)n$ (if $m > 1$) free design parameters in the controller, to be used in order to meet objectives other than pole placement.

LTI-SISO systems in companion canonical form

Pole placement design is now investigated in the case of an LTI-SISO system with scalar transfer function of the form shown below.

$$H(s) = \frac{Y(s)}{U(s)} = \frac{b_0}{s^n + a_{n-1} \cdot s^{n-1} + \dots + a_1 \cdot s + a_0} \quad (5.10)$$

Further, the state-space realization in companion canonical form as in equation (1.48) is considered. The full state feedback matrix F for this realization is, therefore, of dimensions $1 \times n$ and with a general form as follows.

$$F = [f_0 \quad f_1 \quad \dots \quad f_{n-1}] \quad (5.11)$$

The closed-loop transfer function matrix assumes the form below.

$$\begin{aligned} A+BF &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \cdot [f_0 \quad f_1 \quad f_2 \quad \dots \quad f_{n-1}] \\ &\Downarrow \\ A+BF &= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_0 - a_0 & f_1 - a_1 & f_2 - a_2 & \dots & f_{n-1} - a_{n-1} \end{bmatrix} \end{aligned} \quad (5.12)$$

The characteristic polynomial can then be calculated as the determinant of matrix $(sI - A - BF) = (sI - (A + BF))$. A straightforward calculation is done if the determinant is deployed with respect to the last row, which yields the following closed-loop characteristic polynomial.

$$p_c(s) = |sI - A - BF| = s^n + (a_{n-1} - f_{n-1}) \cdot s^{n-1} + \dots + (a_1 - f_1) \cdot s + (a_0 - f_0) \quad (5.13)$$

Then, by employing the procedure described earlier one obtains polynomial $\hat{p}_c(s)$ out of the specified pole locations of the closed-loop system. By equating the coefficients of the same powers of polynomials $p_c(s)$ and $\hat{p}_c(s)$, the following n in number design equations are obtained.

$$f_i = a_i - \hat{a}_i, \quad i = 0, \dots, n-1 \quad (5.14)$$

Useful conclusions can be made if the pole placement controller design is compared to the Root Locus technique where output, instead of state feedback, is employed.

LTI-MIMO systems

It is better to investigate this case through an example. Consider the following multivariable system of the general form (5.1) with 2 inputs, 3 outputs and 3 states.

$$A = \begin{bmatrix} 1 & 2.4 & 1 \\ -3 & 0.5 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad C = I_3 \quad (5.15)$$

It can be shown that this system is controllable

The open-loop plant transfer function matrix of the LTI system at hand is as follows.

$$H(s) = \begin{bmatrix} \frac{s^2 + 0.9s - 1.9}{s^3 - 2.5s^2 + 7.2s - 5.6} & \frac{s^2 - 0.5s + 1.4}{s^3 - 2.5s^2 + 7.2s - 5.6} \\ \frac{s^2 - 5s + 4}{s^3 - 2.5s^2 + 7.2s - 5.6} & \frac{-2s}{s^3 - 2.5s^2 + 7.2s - 5.6} \\ \frac{2s - 2.1}{s^3 - 2.5s^2 + 7.2s - 5.6} & \frac{s^2 - 0.5s + 4.2}{s^3 - 2.5s^2 + 7.2s - 5.6} \end{bmatrix}_3 \quad (5.16)$$

The open-loop characteristic polynomial of this system is, therefore,

$$p_0(s) = s^3 - 2.5s^2 + 7.2s - 5.6 \quad (5.17)$$

The open-loop poles, i.e. the roots of the polynomial above, are

$$s_0 = 0.9807, s_{1,2} = 0.7596 \pm j2.2656 \quad (5.18)$$

The locations on the complex plane of the open-loop poles are shown in Figure 5-2 whilst the open-loop plant's step response is shown in Figure 5-3. As can be seen the system is unstable and its step response demonstrates oscillations.

In effect, a feedback control law of the form (5.4) is sought after so that the poles of the closed-loop system are stable (i.e. demonstrate negative real part). Furthermore, in order to mitigate oscillations, the closed-loop poles are specified to be purely real, i.e. demonstrate zero imaginary part. The desired closed-loop poles are specified to be the following ones.

$$p_1 = -0.5, p_2 = -1.0, p_3 = -1.5 \quad (5.19)$$

These poles yield the following desired characteristic polynomial.

$$\hat{p}_c(s) = (s + p_1) \cdot (s + p_2) \cdot (s + p_3) = s^3 + 3s^2 + 2.75s + 0.75 \quad (5.20)$$

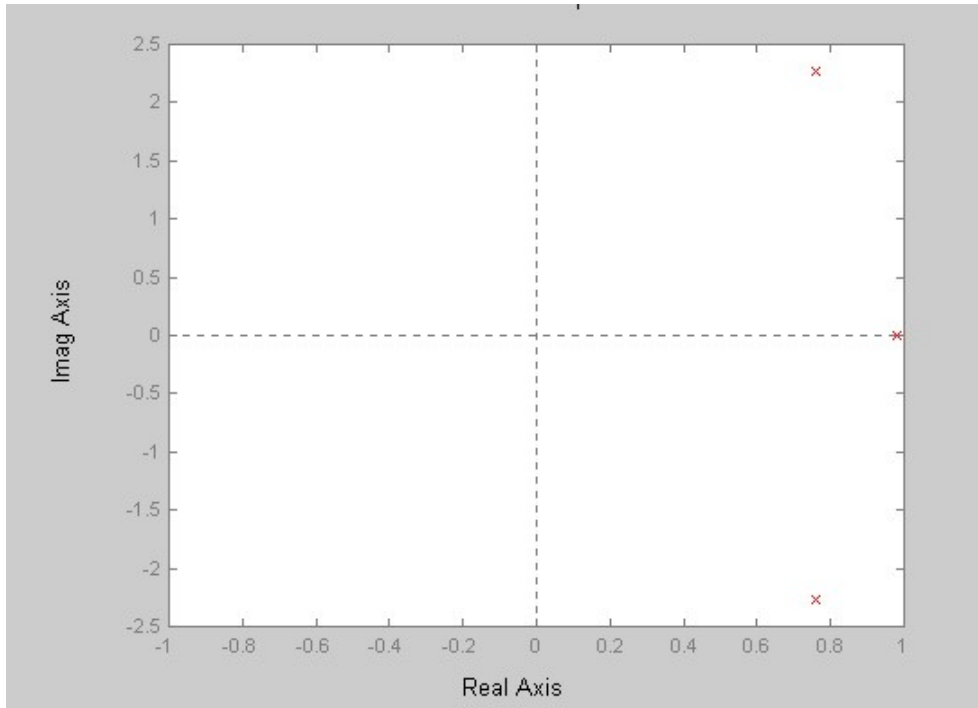


Figure 5-2: Open-loop pole locations of plant (5.15).

The closed-loop matrix $(A+BF)$, parameterized by the elements of the 3×2 matrix F in the control law, is given by

$$A+BF = \begin{bmatrix} 1+F_{11}+F_{21} & 2.4+F_{12}+F_{22} & 1+F_{13}+F_{23} \\ -3+F_{11} & 0.5+F_{12} & 1+F_{13} \\ 1+F_{21} & 1+F_{22} & 1+F_{23} \end{bmatrix} \quad (5.21)$$

Then, the determinant $|sI - A - BF|$ is calculated. This determinant is the closed-loop characteristic polynomial $p_c(s)$ of order three. The coefficients of $p_c(s)$ are parameterized by the elements of F , F_{ij} . By equating the coefficients of polynomials $p_c(s)$ and $\hat{p}_c(s)$ and solving the resulting linear algebraic system of equations, a matrix F is computed as follows.

$$F = \begin{bmatrix} 0.3021 & 1.4785 & -0.617 \\ 2.4378 & 0.9649 & 1.2816 \end{bmatrix} \quad (5.22)$$

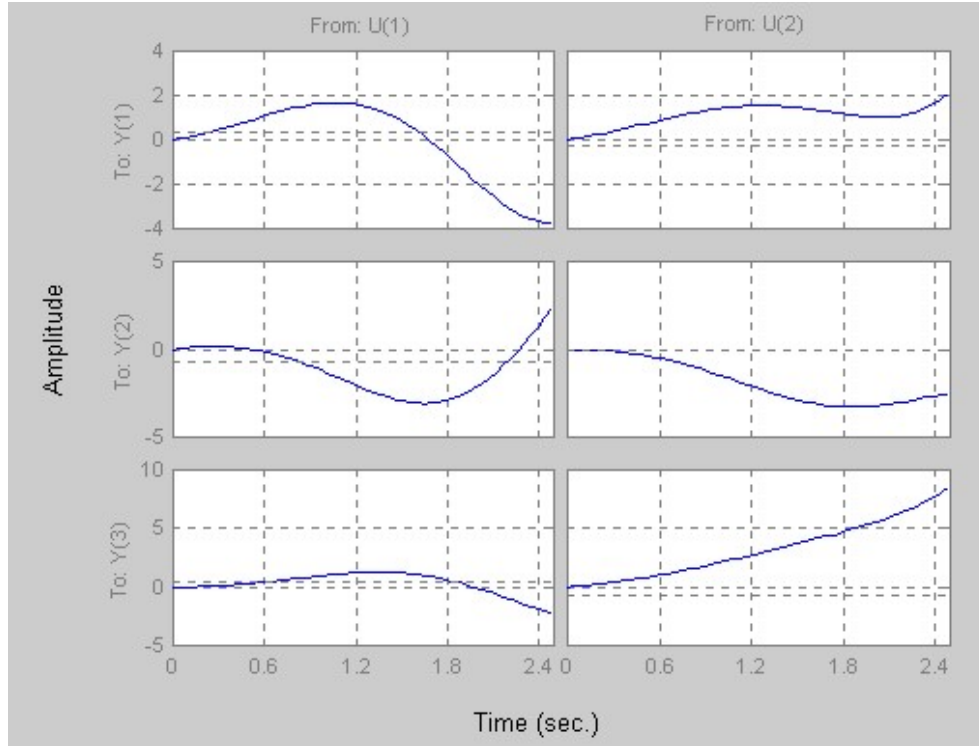


Figure 5-3: Open-loop step response of plant (5.15).

The closed-loop transfer function matrix of the system with feedback matrix F as in equation (5.22) is as follows.

$$H(s) = \begin{bmatrix} \frac{s^2 + 1.217s + 0.2183}{s^3 + 3s^2 + 2.75s + 0.75} & \frac{s^2 + 1.595s + 0.4767}{s^3 + 3s^2 + 2.75s + 0.75} \\ \frac{s^2 - 1.281s - 2.282}{s^3 + 3s^2 + 2.75s + 0.75} & \frac{-1.685s - 1.549}{s^3 + 3s^2 + 2.75s + 0.75} \\ \frac{-1.403s - 1.399}{s^3 + 3s^2 + 2.75s + 0.75} & \frac{s^2 + 1.281s + 0.03623}{s^3 + 3s^2 + 2.75s + 0.75} \end{bmatrix}_3 \quad (5.23)$$

The step response of the closed-loop system is shown in Figure 5-4. As can be seen, it complies with the requirements of stability and mitigated oscillations.

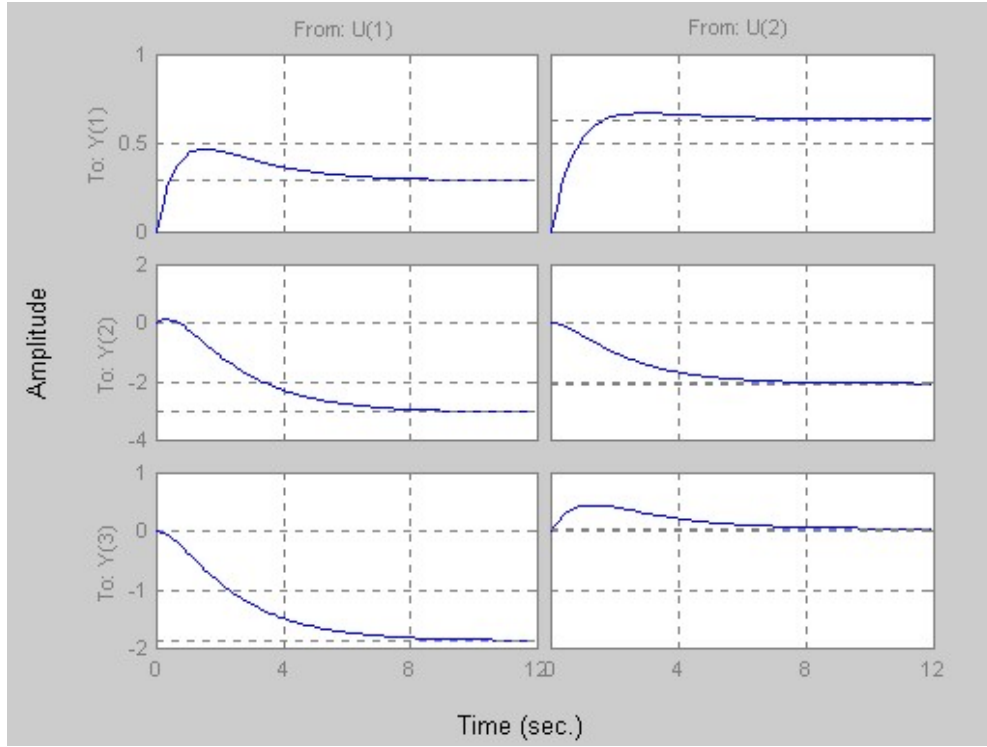


Figure 5-4: Closed-loop step response of system with transfer function (5.23).

In effect, the poles of $\tilde{G}'_N(s)$ define whether the response of the vessel to yaw moment disturbance exhibits oscillations or not. Indeed if the value of the damping constant, ζ , of the 2nd-order characteristic polynomial $\tilde{p}'_N(s) = \tau_1 s^2 + (k_N K_d + 1)s + k_N K_p$ is below 0.7 then oscillations in the response are to be expected. In effect, proper selection of constants K_p , K_d can be made in order to eliminate excessive oscillatory behavior in closed-loop response. On the other hand, all controller constants, but predominantly K_i , contribute to obtain a closed-loop system that is adequately stable. Similar conclusions hold for transfer function $\tilde{G}_Y(s)$, too.

6. DISCRETE-TIME SYSTEMS

6.1 Digital (Discrete-Time) Control Systems

The progress in integrated circuit manufacturing (VLSI) combined with the need for high complexity and accuracy control algorithms without development cost or time increase are the basic factors that contributed in the continuously increasing use of digital control systems in industry and technology.

A digital control system is a system where the basic mathematical operations and functions, including the control law, are implemented as an embedded program onboard real-time, digital computer hardware. Since the computer hardware employed is digital the signals that can be processed or generated cannot be continuous-time i.e. defined for any instant in time. Instead the computer processes and generates tables (i.e. “vectors”) of values for the input and output signal respectively. Such tables hold the values of the corresponding signals at discrete time instants and are referred to as discrete-time signals. The time instants for which a discrete-time signal is known or defined are in most practical cases equally spaced; this scheme is known as uniform sampling and the spacing interval between two successive samples of the signal is mentioned as sampling interval or period.

The systems, e.g. real-time digital hardware, processing discrete-time signals are known as discrete-time systems. The main advantages in using computers for vehicle, process or system control instead of analog (continuous-time) controllers are:

- 1) (Re)use of general purpose hardware for implementing a wide variety of control algorithms instead of custom-built, proprietary control systems that may become obsolete causing higher acquisition and development cost.
- 2) Changes in the control law or algorithm can be easily implemented as changes in the control software through reprogramming without need to modify the hardware.
- 3) Easy implementation of adaptive or time-varying control algorithms thanks to programming.
- 4) Very small to zero sensitivity to environmental conditions, e.g. ambient temperature, in contrast to e.g. analog electric circuitry.

- 5) Given, well-defined accuracy in computation determined by the computer's word length.
- 6) User-friendly Human-Machine Interface (HMI).
- 7) Network connectivity.

However, there are some disadvantages in using discrete-time control systems the most important of which is the need for converter or adapter circuits or interface systems at their input and output in order to be able to interact with the plant or process under control, which is most often a continuous-time system. Such interfaces are the Analog-to-Digital (A/D) and Digital-to-Analog (D/A) converters.

6.2 Signal sampling

Modeling of ideal sampler

Consider a continuous-time signal $x(t)$, which is driven through a sampling circuit (sampler) operating at a constant sampling rate (frequency), $f_s = 1/T_s$, as shown in the figure below.

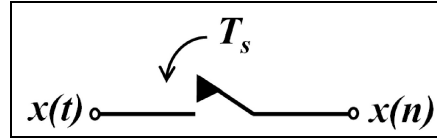


Figure 6-1: Analog signal sampler.

The sampler's output is a sequence of entries of the sampled signal, $x(nT_s)$, where n is an integer and T_s the sampling period. Sequence $x(n) = x(nT_s)$ is a discrete-time signal.

We will now represent the discrete-time signal as well as the sampling process in a continuous-time framework. Such representation can be derived on the basis of the figure below.

A pulsed continuous-time signal $x_\Delta(t)$ that corresponds to sequence $x(nT_s)$ can be generated from the original continuous-time signal $x(t)$ (dashed line), if the latter is multiplied by the following *sampling pulse train* $P_\Delta(t)$:

$$P_\Delta(t) = \sum_{n=-\infty}^{+\infty} p_\Delta(t - nT_s) \quad (6.1)$$

where:

$$p_\Delta(t) = \begin{cases} 1/\Delta, & t \in [0, \Delta] \\ 0, & t \notin [0, \Delta] \end{cases}, \quad 0 < \Delta \ll T_s \quad (6.2)$$

In effect:

$$x_{\Delta}(t) = x(t) \cdot P_{\Delta}(t) \quad (6.3)$$

However, the following holds:

$$\lim_{\Delta \rightarrow 0} p_{\Delta}(t) = \delta(t) \Rightarrow \lim_{\Delta \rightarrow 0} P_{\Delta}(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \quad (6.4)$$

In the above ‘ideal’ case, the sampling pulse train is an impulse train or Dirac comb. In effect, equation (6.4) becomes:

$$x_{\delta}(t) = x(t) \cdot \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{+\infty} x(nT_s) \delta(t - nT_s) = \sum_{n=-\infty}^{+\infty} x(n) \delta(t - nT_s) \quad (6.5)$$

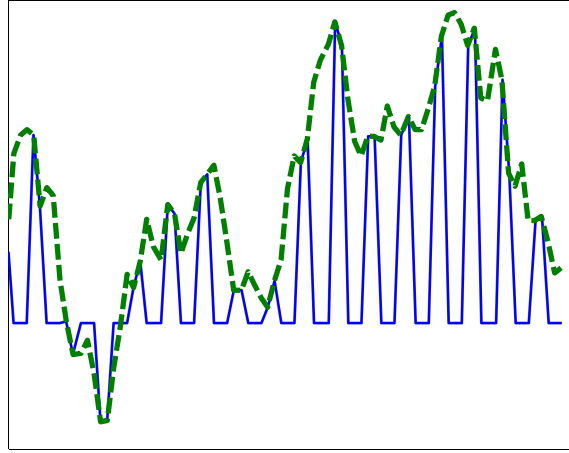


Figure 6-2: Sampling of continuous-time waveform at uniform rate.

Above the following fact for Dirac’s impulse delta has been used: $\chi(t)\delta(t) = \chi(0)\delta(t)$, which implies the assumption that signal $\chi(t)$ assumes non-infinite values.

Hence, the ideal sampler operation consists of multiplicatively applying the input signal by a Dirac comb with T_s . Now we will determine the spectrum of signal $x_{\delta}(t)$ at the output of the sampler by employing the Fourier Transform.

Fourier Series Expansion and Fourier Transform

Any periodic signal $x(t)$ with period T can be expanded to a complex Fourier series according to the following:

$$x(t) = \sum_{n=0}^{+\infty} c_n \exp(jn \frac{2\pi}{T} t) \quad (6.6)$$

Coefficients c_n of the series are calculated according to:

$$c_n = \frac{1}{T} \int_0^T x(t) \exp(-jn \frac{2\pi}{T} t) dt \quad (6.7)$$

The above can be extended to the case of aperiodic (non-periodic) signals assuming that period T is infinite. That is how the *Fourier Transform* emerges:

$$X(\omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{+\infty} x(t) \exp(-j\omega t) dt, \quad \omega = \frac{2\pi}{T} \quad (6.8)$$

The *Inverse Fourier Transform* is defined by:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\omega) \exp(j\omega t) d\omega \quad (6.9)$$

To generalize formalism we are talking referring to Fourier transform of any signal, periodic or not. In the case of periodic signals the Fourier Transform is directly connected to coefficients c_n of its series expansion as follows:

$$X(\omega) = 2\pi \sum_{n=-\infty}^{+\infty} c_n \delta(\omega - n \frac{2\pi}{T}) \quad (6.10)$$

The above can be derived directly by plugging it in (6.8) and generating (6.6). The Fourier Transform, just like the Laplace Transform, from which the former can be defined by setting $s = j\omega$, is a linear integral transform. Its properties are identical with those of the Laplace Transform, and can be sought in literature. An important property is that of multiplication-convolution duality:

$$\begin{aligned} x(t) = x_1(t) \otimes x_2(t) &\triangleq \int_{-\infty}^{+\infty} x_1(\xi) \cdot x_2(t - \xi) \cdot d\xi \Rightarrow X(\omega) = X_1(\omega) \cdot X_2(\omega) \\ x(t) = x_1(t) \cdot x_2(t) &\Rightarrow X(\omega) = X_1(\omega) \otimes X_2(\omega) \triangleq \int_{-\infty}^{+\infty} X_1(\xi) \cdot X_2(\omega - \xi) \cdot d\xi \end{aligned} \quad (6.11)$$

The above can be directly applied to the Dirac comb describing the function of an ideal sampler as well as its periodicity.

Indeed, the Dirac comb in equation (6.4) is a periodic signal with period T_s . Its Fourier series is given below:

$$c_n = \frac{1}{T_s} \int_0^{T_s} \delta(t) \exp(-jn\omega_s t) dt = \frac{\exp(-jn\omega_s 0)}{T_s} \int_0^{T_s} \delta(t) dt = \frac{1}{T_s}, \quad \omega_s = \frac{2\pi}{T_s} \quad (6.12)$$

Hence, based on (6.10) the spectrum (Fourier Transform) of the Dirac comb is given by:

$$\mathcal{F} \left\{ \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \right\} = 2\pi \sum_{n=0}^{+\infty} \frac{1}{T_s} \delta(\omega - n \frac{2\pi}{T_s}) = \omega_s \sum_{n=0}^{+\infty} \delta(\omega - n\omega_s) \quad (6.13)$$

Nyquist sampling rate – Aliasing

Combining the relationship above for Dirac comb spectrum with equation (6.5) as well as the Fourier Transform duality between multiplication and convolution, one can determine the spectrum of signal $x_\delta(t)$ at the output of an ideal sampler:

$$\begin{aligned} x_\delta(t) &= x(t) \cdot \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \Rightarrow \\ &\Rightarrow X_\delta(\omega) = X(\omega) \otimes \mathcal{F} \left\{ \sum_{n=-\infty}^{+\infty} \delta(t - nT_s) \right\} = \omega_s X(\omega) \otimes \sum_{n=0}^{+\infty} \delta(\omega - n\omega_s) \\ &\Rightarrow X_\delta(\omega) = \omega_s \sum_{n=0}^{+\infty} X(\omega) \otimes \delta(\omega - n\omega_s) = \omega_s \sum_{n=0}^{+\infty} \int_{-\infty}^{+\infty} X(\xi) \delta(\omega - n\omega_s - \xi) d\xi \end{aligned}$$

The following property of Dirac impulse delta can now be employed:

$$\chi(r) \otimes \delta(r - r_0) = \int_{-\infty}^{+\infty} \chi(\xi) \delta(r - r_0 - \xi) d\xi = \chi(r - r_0) \quad (6.14)$$

Eventually one obtains:

$$X_\delta(\omega) = \mathcal{F} \left\{ \sum_{n=-\infty}^{+\infty} x(nT_s) \delta(t - nT_s) \right\} = \omega_s \cdot \sum_{n=0}^{+\infty} X(\omega - n\omega_s) \quad (6.15)$$

Using (6.15) one can assess the effect of sampling on spectrum $X(\omega)$ of the analog, continuous-time signal at the input. Sampling is introducing, therefore, an infinite number of aliases of spectrum $X(\omega)$ of the analog signal, centered at integer multiples of the sampling circular frequency ω_s .

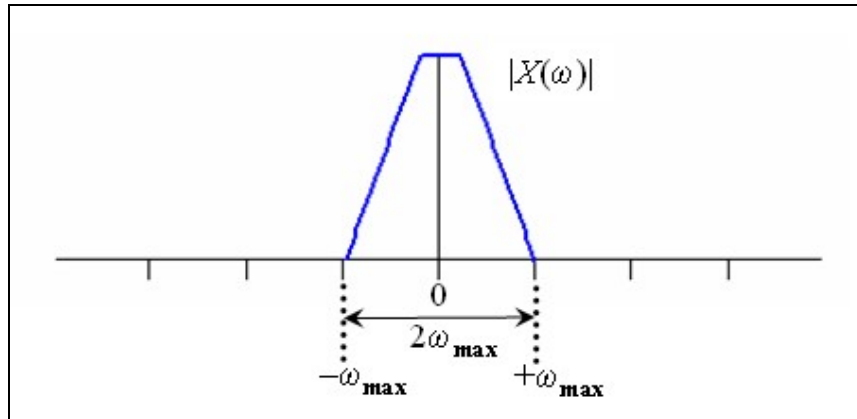


Figure 6-3: Typical spectrum of real low-pass signal.

To visualize this consider a real signal with all its spectral content lying within the low frequency range (low-pass or baseband signal); like e.g. the signal with spectrum as in the figure above. Note that the amplitude spectrum $|X(\omega)|$ of the signal is an odd function of circular frequency ω due to the fact it is assumed to be purely real in the time domain.

The amplitude spectrum of signal $x_s(t)$ is shown in the figure below as generated by sampling of low-pass signal $x(t)$ above at three (3) different sampling rates: $\omega_{s,0} = 2\omega_{\max}$, $\omega_{s,1} = 4\omega_{\max}$, $\omega_{s,2} = \omega_{\max}$.

Using the plot, one can readily derive the following sampling rate criterion attributed to Nyquist (or Shannon sampling theorem) for low-pass signal like the one in the figure above:

$$\omega_s \geq \omega_{s,0} = 2\omega_{\max} \quad (6.16)$$

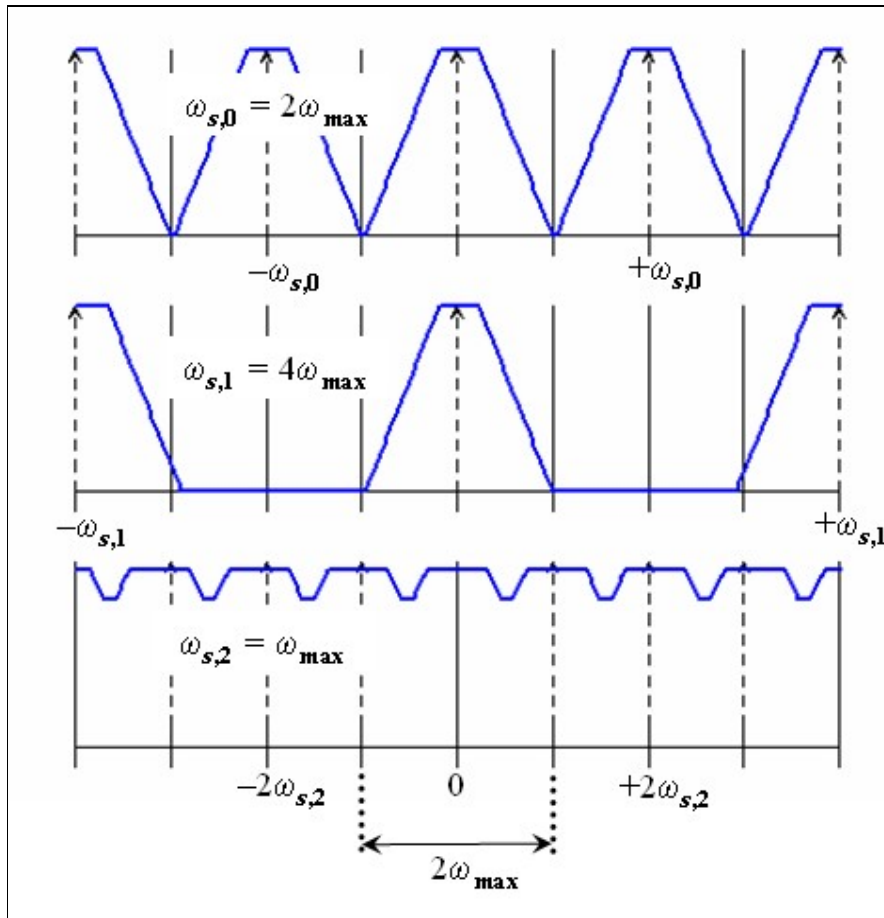


Figure 6-4: Sampled signal spectrum at various sampling rates.

If the condition above holds as seen in equation (6.15) (spectrum periodicity for the sampled signal) as well as in the figure below, an ideal (“brick”) low-pass filter with transfer function as follows can be used to reconstruct the original continuous-time signal:

$$H_{\text{LPF}}(\omega) = \begin{cases} 1, & |\omega| \leq \omega_s / 2 \\ 0, & |\omega| > \omega_s / 2 \end{cases} \quad (6.17)$$

The original spectrum is ideally reproduced without any distortion since it holds that:

$$H_{\text{LPF}}(\omega)X_{\delta}(\omega) = \omega_s X(\omega) \quad (6.18)$$

This is made clearer next when reconstruction of a continuous-time signal based on the values of a discretized version is discussed.

6.3 Analog signal reconstruction using a discrete-time signal

The simplest process to generate a continuous-time signal, $x(t)$, from a discretized sequence (discrete-time signal) $x(n)$ is to employ the Zero-Order-Hold (ZOH) network. This system “holds” the output constant and equal to the one applied to its input for an entire sampling period T_s . In effect, the output of ZOH when driven by a discretized signal is given in the figure below.

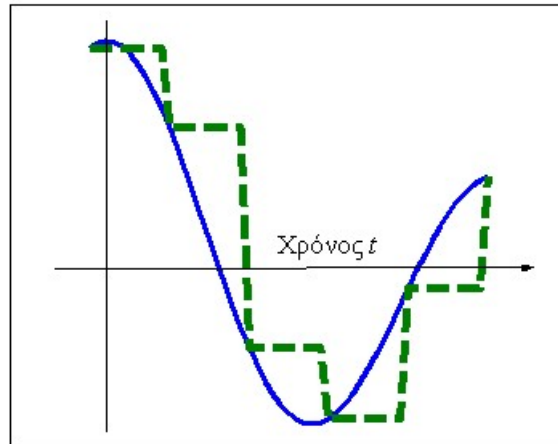


Figure 6-5: Original continuous-time signal (continuous line) and (dashed line).

The mathematical description of a ZOH system is achieved through the following impulse response and transfer function in the complex frequency domain of variable s :

$$h_{\text{ZOH}}(t) = u_{\text{step}}(t) - u_{\text{step}}(t - T_s) \Leftrightarrow H_{\text{ZOH}}(s) = \frac{1 - e^{-sT_s}}{s} \quad (6.19)$$

To better understand the action of the ZOH system, its transfer function in the circular frequency domain ω (Fourier Transform) is given below, as well:

$$H_{\text{ZOH}}(\omega) = \frac{1 - e^{-j\omega T_s}}{j\omega} = T_s e^{-j\omega T_s/2} \frac{\sin\left(\frac{\omega T_s}{2}\right)}{\left(\frac{\omega T_s}{2}\right)} = \frac{e^{-j\pi f/f_s}}{f_s} \text{sinc}\left(\frac{f}{f_s}\right) \quad (6.20)$$

In the above $\text{sinc}(\chi) = \text{sinc}(\pi\chi)/\pi\chi$. In Figure 6-6 the amplitude spectra of the transfer functions of the ZOH system (normalized by setting $f_s = 1$ Hz) and the ideal reconstruction low-pass filter which was introduced with equation (6.17).

Indeed, in the circuit above if resistance R is small (theoretically zero) then time constant τ indicating the time needed for charging the capacitor in the hold circuit is small when compared to sampling period T_s . The voltage across the terminals of the capacitor stays at the value set in the previous sampling instant, when the switch was momentarily switched on, until the next instant the switch is on etc. The differences of the analog memory element, as the switched RC circuit is known, and the model ZOH system are: (a) The ohm resistance value of the circuit is non-zero and therefore the time to charge the capacitor is finite and not zero; (b) The duration of switch's conduction phase is finite and non-zero, and; (c) the ohm resistance applied to the terminals of the capacitor when the switch is off is not infinite (open circuit) but finite eventually leading to discharge of the capacitor.

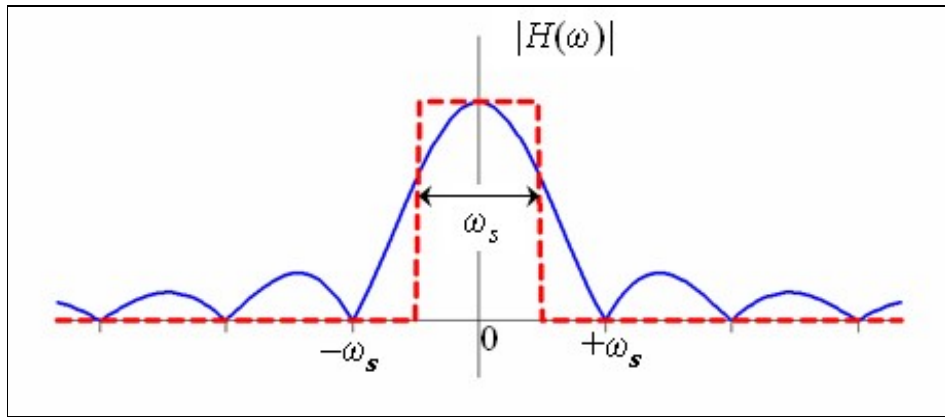


Figure 6-6: Amplitude spectrum of transfer function $H_{\text{ZOH}}(\omega)$ (continuous line) and $H_{\text{LPF}}(\omega)$ (dashed line).

The use of the RC hold circuit is widespread thanks to its simplicity. However, if a more accurate reconstruction with less distortion, without increasing the sampling rate, then hold circuits of order higher than zero (e.g. first-order hold, FOH) so that its transfer function better approximates the ideal “brick” low-pass filter in equation (6.17) and shown in Figure 6-6.

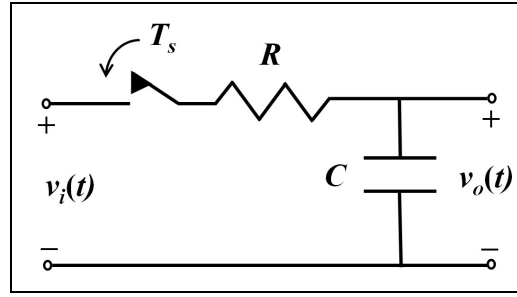


Figure 6-7: Switched RC circuit modeled by the ZOH system.

6.4 The Z-Transform

The single-sided (unilateral) Z-Transform is a linear transform defined as follows for a discrete-time signal $x(n)$:

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=0}^{+\infty} x(n) \cdot z^{-n} \quad (6.21)$$

The Z-Transform is derived from the Laplace Transform of signal $x_\delta(t)$, which is generated by the ideal sampler in Figure 6-1.

$$x_\delta(t) = \sum_{n=-\infty}^{+\infty} x(n)\delta(t - nT_s) \Rightarrow X_\delta(s) = \mathcal{L}\left\{\sum_{n=-\infty}^{+\infty} x(n)\delta(t - nT_s)\right\} = \sum_{n=0}^{+\infty} x(n)e^{-n \cdot sT_s} \quad (6.22)$$

In effect:

$$X(z)\Big|_{z^{-1}=\exp(-sT_s)} = X_\delta(s) \quad (6.23)$$

Variable z^{-1} is used for describing a signal, or system as we'll see later on. Its significance lies in the fact that multiplying by z^{-1} in the transform domain translates to a time shift by one sampling period, T_s , in the discrete-time domain n . Indeed:

$$y(n) = x(n-1) \Leftrightarrow Y(z) = z^{-1}X(z) \quad (6.24)$$

Except the above, here are some more of the most important properties of the Z-Transform.

1) Linearity:

$$\left. \begin{array}{l} X_1(z) = \mathcal{Z}\{x_1(n)\} \\ X_2(z) = \mathcal{Z}\{x_2(n)\} \end{array} \right\} \Rightarrow k_1 \cdot X_1(z) + k_2 \cdot X_2(z) = \mathcal{Z}\{k_1 \cdot x_1(n) + k_2 \cdot x_2(n)\} \quad (6.25)$$

$k_1, k_2 \in \mathbb{R}$

2) Delay or advance by N discrete-time units::

$$\mathcal{Z}\{x(n)\} = X(z) \Rightarrow \begin{cases} \mathcal{Z}\{x(n-N)\} = z^{-N} X(z) \\ \mathcal{Z}\{x(n+N)\} = z^N X(z) - \sum_{k=0}^{N-1} z^{N-k} x(k) \end{cases} \quad (6.26)$$

3) Initial or final value theorems:

$$\mathcal{Z}\{x(n)\} = X(z) \Rightarrow \begin{cases} x(n=0) = \lim_{z \rightarrow \infty} (X(z)) \\ \lim_{n \rightarrow \infty} (x(n)) = \lim_{z \rightarrow 1} ((1-z^{-1})X(z)) \end{cases} \quad (6.27)$$

NOTE: The final value theorem ($n \rightarrow \infty$) holds only if $X(z)$ is an analytic function of complex variable z .

4) Discrete convolution theorem:

$$\left. \begin{aligned} X_1(z) &= \mathcal{Z}\{x_1(n)\} \\ X_2(z) &= \mathcal{Z}\{x_2(n)\} \\ x(n) &= x_1(k) \otimes x_2(k) \triangleq \sum_{k=-\infty}^{+\infty} x_1(k) \cdot x_2(n-k) \end{aligned} \right\} \Rightarrow X(z) = \mathcal{Z}\{x(n)\} = X_1(z) \cdot X_2(z) \quad (6.28)$$

NOTE: Because $x(n)$ needs to be **causal**, i.e. can only depend on values of signals x_1 and x_2 prior or up to instant n , the bounds of the convolutional summation are set as follows

$(x_1(k) \otimes x_2(k))(n) = \sum_{k=0}^n x_1(k) \cdot x_2(n-k)$. It is also noted out here that the commutative and associative properties hold for the convolution operator applied on two discrete-time signals.

Finally, it is pointed out that the change of variable, $z = \exp(sT_s)$, on the basis of which the Z-Transform is obtained from the Laplace Transform of signal $x_s(t)$, defines a single-valued mapping of complex frequency s -plane to the plane of, also complex, delay operator z . Remember that the stability region on the complex plane $s = a + j\omega$, is the left-hand half-plane defined by inequality: $\alpha \leq 0$. In effect, the stability region on the delay operator z -plane is defined by inequality:

$$|z| \leq 1 \quad (6.29)$$

Since:

$$z = e^{sT_s} = e^{aT_s} e^{j\omega T_s} \Rightarrow \begin{cases} |z| = e^{aT_s} \\ \arg(z) = e^{j\omega T_s} \end{cases} \quad (6.30)$$

I.e. the stability region is the unit disc. Similarly, the origin of the s -plane is mapped to point $z = 1 + j0$ on the z -plane, while the imaginary axis of the s -plane is wrapped around the unit circle on the z -plane. Finally, the real s axis is mapped to the real z semi-axis.

6.5 Discrete-time LTI systems

Difference equations

A special class of dynamic systems, widely used in automatic control, is Linear Time Invariant Systems (LTI). Such systems are described in continuous-time by linear differential equations with constant coefficients. In discrete-time, such differential equation is converted to a difference equation corresponding to a discrete-time system described by a transfer function in variable z , or equivalently z^{-1} . The general form of a linear difference equation with input $u(n)$ and output $y(n)$ is the following:

$$y(n) = \sum_{k=0}^M b_k u(n-k) - \sum_{k=1}^N a_k y(n-k) \quad (6.31)$$

To get the iterative process started in order to obtain samples of discrete-time output signal $y(n)$ for $n \geq 0$ N initial conditions are needed: $y(-k)$, $k = 1, \dots, N$. Using (6.31) we can conclude that the n -th sample of the output is determined as a weighed sum of the current plus M past input samples and N past output samples.

Linear Time-Invariant Discrete-time Systems

The generic difference equation (6.31) in the discrete-time domain defines an LTI discrete-time system. In the Z-Transform domain the difference equation becomes the transfer function as follows:

$$\begin{aligned} H(z^{-1}) &= \frac{Y(z)}{U(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + \dots + a_{N-1} z^{-(N-1)} + a_N z^{-N}} \\ &\quad \Updownarrow \\ H(z) &= \frac{b_0 z^N + b_1 z^{N-1} + \dots + b_M z^{N-M}}{z^N + a_1 z^{N-1} + \dots + a_N}, \quad M \leq N \end{aligned} \quad (6.32)$$

A special yet very important case of discrete-time systems are Finite Impulse Response (**FIR**) systems in contrast to the generic **IIR** (Infinite Impulse Response) system defined in (6.31). For a FIR system it holds that $a_k = 0, \forall k > 0$ in (6.31), hence:

$$y(n) = \sum_{k=0}^M b_k u(n-k) \quad \text{and} \quad H(z^{-1}) = b_0 + b_1 z^{-1} + \dots + b_M z^{-M} \quad (6.33)$$

In effect, the n -th output sample is *exclusively* determined as a weighed sum of the current plus M past input samples. In effect, the difference equations seizes to be recursive and that is why in literature FIR systems are also mentioned as **MA** (Moving Average) ones.

In contrast, discrete-time LTI systems that have exclusively poles, i.e. $b_i = 0$, for every $i > 0$ in (6.32), are also known as **AR** (Auto-Regressive), while the generic IIR system is mentioned also as **ARMA** in this framework.

6.6 Closed-loop systems including a control computer

This section aims to conclude the picture of a system controller by a real-time (digital) computer. In this end, the generic block diagram of such a closed-loop system is given in Figure 6-8.

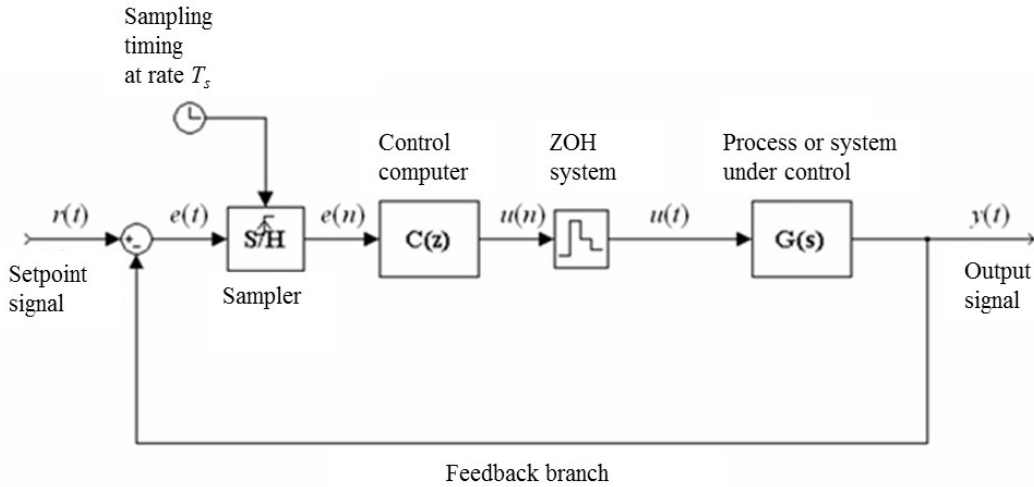


Figure 6-8: Generic block diagram of a closed-loop system with control computer.

As seen the only discrete-time block in the system is the control computer, which is framed by interface subsystems required, i.e. the sampler and the hold circuit.

Given that at least one of the subsystems is discrete-time, we need to determine the *discrete-time transfer function* (i.e. in the Z-Transform domain) connecting the input signal to the output one. Note that both the output and the reference (set-point) signal are sampled at the same sampling instants with the error signal, $e(t)$, i.e. spaced apart by one sampling interval T_s and in sync. The closed-loop discrete-time transfer function in the z -domain, connecting signal $r(n)$ to $y(n)$ is a dynamic relationship, that can be considered full only if the original continuous-time (analog) signals can be reconstructed from their sampled discrete-time versions. In effect sampling interval, T_s , or equivalently circular frequency ω_s , must comply with the Nyquist criterion.

Note that in a mixed-signal the position of the sampler of Figure 6-1 cannot be changed. Therefore regardless of whether the subsystems framing a sampler are LTI or not they cannot be interchanged with it. Furthermore, the intermission of a sampler between two continuous-time systems results in their separation in the Z-Transform domain. This means that if a sampler is intermitted between two analog systems, the conversion of their dynamics to discrete-time equivalent representations needs to be considered separately. This means that the overall discrete-time z -domain transfer function can be derived as the product of the individual ones. In contrast, for a group of continuous-time subsystems between two consecutive samplers one needs to derive in advance the overall continuous-time representation (e.g. transfer function) and then convert that to discrete-time. The concept for both cases is presented also in Figure 6-9.

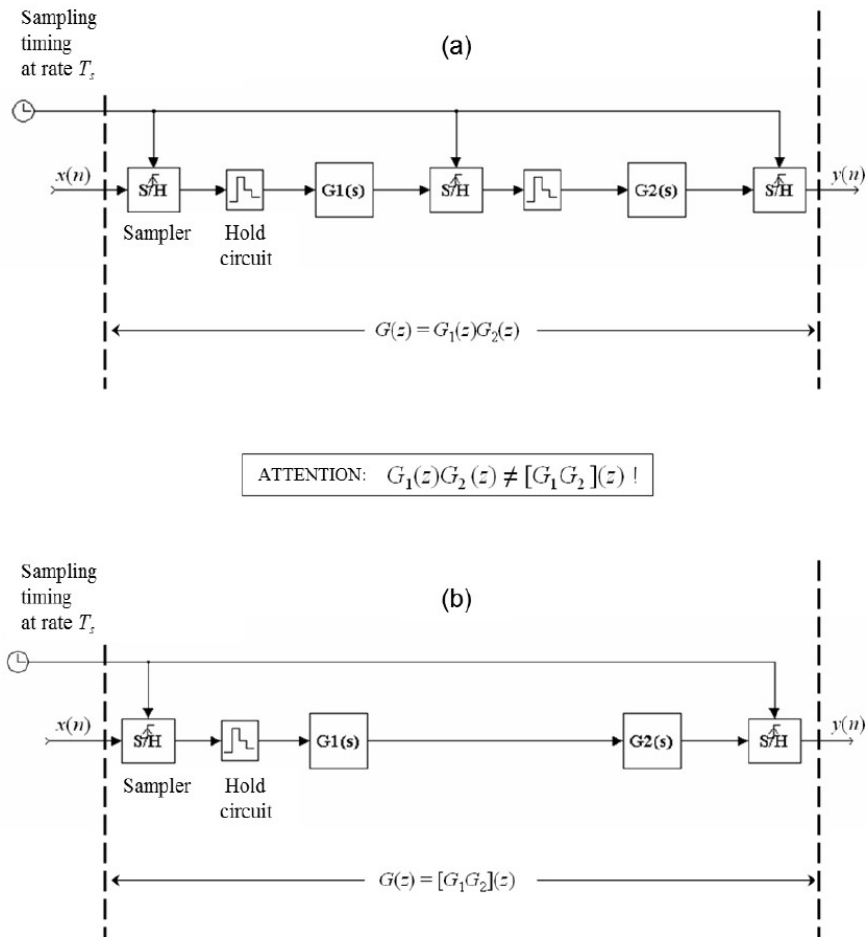


Figure 6-9: Continuous-time blocks with intermittent sampler (a) and without (b).

To simplify notation and without loss of clarity the notation adopted in Figure 6-9 is maintained throughout. With respect to that, the following hold for continuous-time transfer functions $G_1(s)$ and $G_2(s)$:

$$\left. \begin{aligned} [G_1 G_2](z) &= \mathcal{Z}\{H_{\text{ZOH}}(s)G_1(s)G_2(s)\} \\ G_1(z)G_2(z) &= \mathcal{Z}\{H_{\text{ZOH}}(s)G_1(s)\} \cdot \mathcal{Z}\{H_{\text{ZOH}}(s)G_2(s)\} \end{aligned} \right\} \Rightarrow [G_1 G_2](z) \neq G_1(z)G_2(z) \quad (6.34)$$

In the above, notation $X(z) = \mathcal{Z}\{X(s)\} = \mathcal{Z}\{x(t)\}$, $x(t) = \mathcal{L}^{-1}\{X(s)\}$ means that to obtain the Z-Transform of a continuous-time signal that is defined exclusively in the Laplace Transform domain, one needs to (a) first apply the Inverse Laplace Transform to obtain the signal in continuous-time; (b) then sample the continuous-time signal at the sampling rate specified, and; (c) finally, apply the Z-Transform on the discrete-time signal acquired by sampling.

However, one needs to point out that the ZOH circuit in Figure 6-9 has been broken down to its two parts, i.e. the ideal sampler and the analog memory element (typically an RC cell). Mathematically, this breakdown can be expressed as follows.

$$\begin{aligned} H(z) &= \mathcal{Z}\{H_{\text{ZOH}}(s)H(s)\} = \mathcal{Z}\left\{\frac{1-e^{-sT_s}}{s}H(s)\right\} = \mathcal{Z}\left\{\frac{H(s)}{s} - e^{-sT_s} \cdot \frac{H(s)}{s}\right\} = \\ &= \mathcal{Z}\left\{\frac{H(s)}{s}\right\} - \mathcal{Z}\left\{e^{-sT_s} \cdot \frac{H(s)}{s}\right\} = \mathcal{Z}\left\{\frac{H(s)}{s}\right\} - z^{-1} \cdot \mathcal{Z}\left\{\frac{H(s)}{s}\right\} = (1-z^{-1})\mathcal{Z}\left\{\frac{H(s)}{s}\right\} \end{aligned} \quad (6.35)$$

Going back to the case of a mixed processing system, like e.g. the one shown in Figure 6-8, the following conversion procedure can be established

- 1) Identify and name all discrete-time signals as the output of any sampler in the system.
- 2) Employing the rules-of-thumb shown in Figure 6-9 and equations (6.34), develop the continuous-time transfer function between two consecutive samplers..
- 3) The continuous-time transfer functions in step 2 above are then converted to discrete-time ones by directly applying the Z-Transform on their impulse response.

To demonstrate how the methodology above is applied we now go back to Figure 6-9. Evidently, the discrete-time signals are $e(n)$ and $u(n)$. Their Z-transform are connected as follows:

$$U(z) = C(z)E(z) \quad (6.36)$$

In the above, the control computer is evidently modeled as a discrete-time system.

Furthermore, it holds that:

$$Y(z) = [H_{\text{ZOH}}G](z) \cdot U(z) \quad (6.37)$$

For the closed-loop:

$$E(z) = R(z) - [H_{\text{ZOH}}G](z) \cdot U(z) \quad (6.38)$$

By combining the above the following is obtained for the closed-loop in the z -domain:

$$E(z) = \frac{1}{1 + [H_{\text{ZOH}}G](z) \cdot C(z)} \cdot R(z) \quad (6.39)$$

For the system output signal y :

$$Y(z) = \frac{[H_{\text{ZOH}}G](z) \cdot C(z)}{1 + [H_{\text{ZOH}}G](z) \cdot C(z)} \cdot R(z) \quad (6.40)$$

The above is the input-output relationship of the mixed-signal processing system in discrete-time (i.e. transfer functions in the z -domain) connecting discrete-time signals $r(n)$ and $y(n)$.

6.7 Continuous-time system mapping

Introduction

There are many ways (methods) to map a continuous-time system to a discrete-time equivalent. Anyone of these methods is based on preserving some characteristic of the continuous-time system during this procedure; this is why there are more than one ways. However, **stability** (or instability) of a continuous-time system is preserved by employing any of the methods presented here.

The mapping methods considered here can be classified in the following categories:

- 1) Approximate differentiation or integration:
 - Backward difference
 - Forward difference
 - Trapezoidal integration (Tustin transformation)
- 2) Time-domain response matching at sampling instants:
 - Impulse response matching
 - Step response matching
- 3) Dynamic matching, i.e. matching of all poles and zeroes appearing in the transfer function

Approximate differentiation or integration

Consider a continuous-time signal $x(t)$ being sampled with sampling interval T_s and the discrete-time signal $x(n)$ is obtained. Then, the following approximations can be derived.

Approximate derivative with respect to time using backward difference:

$$\dot{x}(t) \approx \frac{x(n) - x(n-1)}{T_s} \Rightarrow s \equiv \frac{1 - z^{-1}}{T_s} \quad (6.41)$$

Approximate derivative with respect to time using forward difference:

$$\dot{x}(t) \approx \frac{x(n+1) - x(n)}{T_s} \Rightarrow s \equiv \frac{z - 1}{T_s} = \frac{1 - z^{-1}}{T_s z^{-1}} \quad (6.42)$$

Approximate integral with respect to time using the trapezoidal rule:

$$\begin{aligned} x_1(nT_s) &= \int_0^{nT_s} x(\xi) d\xi = \int_0^{(n-1)T_s} x(\xi) d\xi + \int_{(n-1)T_s}^{nT_s} x(\xi) d\xi \triangleq x_1((n-1)T_s) + \Delta x_1(n) \\ \Delta x_1(n) &\approx T_s \cdot \frac{x(n) + x(n-1)}{2} \end{aligned} \quad (6.43)$$

But:

$$X_1(s) = \frac{1}{s} X(s) \text{ and } X_1(z) = z^{-1} X_1(z) + T_s \frac{1 + z^{-1}}{2} X(z) \quad (6.44)$$

The above in effect yields:

$$\frac{1}{s} \equiv \frac{T_s}{2} \cdot \frac{1 + z^{-1}}{1 - z^{-1}} \Leftrightarrow s \equiv \frac{2}{T_s} \cdot \frac{1 - z^{-1}}{1 + z^{-1}} \quad (6.45)$$

Equations (6.41), (6.42) and (6.45) mean that when presented with a transfer function $H(s)$ in the complex frequency (s) domain, describing a continuous-time LTI system, it is possible to derive a discrete-time transfer function $H(z)$, or $H(z^{-1})$, defining a discrete-time LTI system by substituting variable s with the approximation desired.

Impulse response matching

The methods presented previously were based on approximating the integration or differentiation operator appearing in the differential (dynamic) equation of an LTI system. However, an LTI system can, also, be described in full by its response to an arbitrary input

signal. Common choices include using either Dirac's delta (impulse) or the unit step signal (Heaviside signal).

In effect, if the impulse or step response of a continuous-time system is sampled, then a discrete-time description can be derived as follows.

Invariance of impulse response:

The impulse delta is defined as follows in discrete-time:

$$\delta(n) = \begin{cases} 0, n \neq 0 \\ 1, n = 0 \end{cases} \Leftrightarrow \Delta(z) = 1 \quad (6.46)$$

Given the transfer function, $H(z)$ of a discrete-time LTI system it is easy to prove that, just like in the continuous-time case, $H(z)$ is the Z-Transform of the system's impulse response.

Indeed, the output $y(n)$ of a discrete-time LTI system is given by the following convolutional sum:

$$y(n) = u(k) \otimes h(k) \triangleq \sum_{k=-\infty}^{+\infty} u(k) \cdot h(n-k) \Rightarrow Y(z) = H(z) \cdot U(z) \quad (6.47)$$

In effect:

$$u(n) = \delta(n) \Rightarrow U(z) = 1 \Rightarrow Y(z) = H(z)$$

Consider a continuous-time LTI system with impulse response $h(t) = \mathcal{L}^{-1}\{H(s)\}$, which is being sampled at rate f_s . This means that continuous-time signal $h_\delta(t) = \sum_{n=-\infty}^{+\infty} h(nT_s)\delta(t-nT_s)$ corresponds to discrete-time signal $h_\delta(n) \equiv h(nT_s)$, i.e. a sample set of $h(t)$.

The Z-transform of $h_\delta(n)$, $H_\delta(z^{-1}) = \sum_{n=0}^{+\infty} h_\delta(n)z^{-n}$, defines a discrete-time LTI system the impulse response of which, by definition, coincides with that of continuous-time transfer function, $H(s)$, at all sampling instants..

Assume, now, that for every positive M there is a positive integer ε such that:

$$\sum_{n=M+1}^{+\infty} h_\delta^2(n) < \varepsilon \quad (6.48)$$

Given the above, one can proceed to impulse response truncation with accuracy ε . This procedure consists of approximating the original IIR system with transfer function

$H_\delta(z^{-1}) = \sum_{n=0}^{+\infty} h_\delta(n) z^{-n}$ by the following FIR one with memory M :

$$\hat{H}_{\text{FIR}}(z^{-1}) = \sum_{n=0}^M h_\delta(n) z^{-n} = \sum_{n=0}^M b_n z^{-n} \quad (6.49)$$

Approximating a continuous-time system with a discrete-time one, which on top might be FIR, comes with a number of advantages especially for real-time applications.

Invariance of step response:

The unit step signal in discrete-time is defined as follows:

$$u_{\text{step}}(n) = \begin{cases} 1, n \geq 0 \\ 0, n < 0 \end{cases} \Leftrightarrow U_{\text{step}}(z) = \sum_{n=0}^{+\infty} z^{-n} = \frac{1}{1 - z^{-1}} \quad (6.50)$$

The Z-Transform of the unit step signal is derived by applying the lemma for the infinite sum of a geometric progression with ratio that in magnitude is less than 1.

Consider now an LTI system with impulse response as follows:

$$y_{\text{step}}(t) = \mathcal{L}^{-1} \{ H(s) U_{\text{step}}(s) \} = \mathcal{L}^{-1} \left\{ \frac{H(s)}{s} \right\} \quad (6.51)$$

We will now determine a discrete-time transfer function $H(z)$ such that the latter's unit step response coincides with that of the continuous-time system above. Mathematically, this is defined as follows:

$$\mathcal{Z}^{-1} \{ H(z) U_{\text{step}}(z) \} = y_{\text{step}}(n) = y_{\text{step}}(nT_s) \quad (6.52)$$

Then, by employing (6.50), one obtains that:

$$H(z) = (1 - z^{-1}) \mathcal{Z} \{ y_{\text{step}}(nT_s) \} \quad (6.53)$$

Since:

$$z^{-1} = \exp(-sT_s) \text{ and } Y_{\text{step}}(s) = \frac{H(s)}{s}$$

We can finally obtain that:

$$H(z) = \mathcal{Z} \{ y_{\text{ZOH}}(nT_s) \} \quad (6.54)$$

In the above, it holds that:

$$y_{ZOH}(t) = h_{ZOH}(t) \otimes h(t) \Leftrightarrow Y_{ZOH}(s) = H_{ZOH}(s)H(s) = \frac{1 - e^{-sT_s}}{s} H(s) \quad (6.55)$$

Pole and zero matching of a transfer function

This method derives from equation $z^{-1} = \exp(-sT_s)$ connecting variable z with complex frequency s . Combining the above with the following factorized form of a continuous-time scalar transfer functions:

$$H(s) = K \frac{(s + z_1)(s + z_2) \dots (s + z_m)}{(s + p_1)(s + p_2) \dots (s + p_n)}, \quad m \leq n \quad (6.56)$$

One can map every pole and every zero of the continuous-time transfer function to a corresponding pole and zero of an equivalent discrete-time transfer function:

$$\begin{aligned} \zeta_i &= -\exp(-T_s z_i), \quad i = 1, \dots, m \\ \pi_i &= -\exp(-T_s p_i), \quad i = 1, \dots, n \end{aligned} \quad (6.57)$$

In effect, the following discrete-time transfer function is derived in its factorized form:

$$H(z) = k \frac{(z+1)^{n-m} (z + \zeta_1)(z + \zeta_2) \dots (z + \zeta_m)}{(z + \pi_1)(z + \pi_2) \dots (z + \pi_n)} \quad (6.58)$$

Factor $(z+1)^{n-m}$ is introduced in the discrete-time transfer function above in correspondence to factor $(\tau s+1)^{n-m}$ with $\tau \rightarrow \pm j0$, which is “hidden” in the continuous-time transfer function (6.56). The intent is to equate numerator degree to that of the denominator. This is achieved by introducing an extra zero at imaginary infinity ($\pm j\infty$) with multiplicity $(n-m)$ using factor $(\tau s+1)^{n-m}$. In the discrete-time transfer function, factor $(\tau s+1)^{n-m}$ is mapped to $(z+1)^{n-m}$, since point $\pm j\infty$ on the s -plane is mapped to -1 on the z -plane.

Finally, gain k in the discrete-time transfer function is determined by equating its value at a certain z with that of the continuous-time transfer function at the equivalent point s . For example, for $z = 1$ the corresponding point is $s = 0$, and therefore k is calculated as follows:

$$H(z)|_{z=1} = H(s)|_{s=0} \Leftrightarrow k \frac{2^{n-m} (1 + \zeta_1)(1 + \zeta_2) \dots (1 + \zeta_m)}{(1 + \pi_1)(1 + \pi_2) \dots (1 + \pi_n)} = K \frac{z_1 z_2 \dots z_m}{p_1 p_2 \dots p_n} \quad (6.59)$$

7. LAPLACE AND Z-TRANSFORM TABLES

7.1 Brief Table of Laplace Transform pairs

Time Function		Laplace Transform
Unit Impulse	$\delta(t)$	1
Unit Step	$\mathbf{1}(t)$	$\frac{1}{s}$
Unit Ramp	t	$\frac{1}{s^2}$
Polynomial	t^n	$\frac{n!}{s^{n+1}}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Sine Wave	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
Cosine Wave	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
Damped Sine Wave	$e^{-at} \sin \omega t$	$\frac{\omega}{(s+a)^2 + \omega^2}$
Damped Cosine Wave	$e^{-at} \cos \omega t$	$\frac{s+a}{(s+a)^2 + \omega^2}$

7.2 Table of some Z-Transform pairs

k th Term of the Time Sequence	z -Transform
1 at k , 0 elsewhere (Kronecker delta sequence)	z^{-k}
1 (unit step sequence)	$\frac{z}{z-1}$
k (unit ramp sequence)	$\frac{z}{(z-1)^2}$
A^k (for complex numbers A)	$\frac{z}{z-A}$
kA^k	$\frac{Az}{(z-A)^2}$
$\frac{(k+1)(k+2)\cdots(k+n-1)}{(n-1)!} A^k$	$\frac{z^n}{(z-A)^n}$