Out line of Lecture Notes I

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~~Summation~~ operator.

Single Summation

Examples: ~~Single~~

$$\sum_{i=1}^{3} i = 1 + 2 + 3 = 6$$

$$\sum_{i=1}^{2} 2i = 2(1) + 2(2) = 6$$

$$\sum_{i=1}^{4} X^i = X^1 + X^2 + X^3 + X^4$$

$$\sum_{i=1}^{3} X_i = X_1 + X_2 + X_3$$

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + --- + x_{n-1} + x_n$$

$$\sum_{i=1}^{n} 2 = 2 + 2 + --- + 2 = 2n$$

$$\sum_{i=1}^{n} c = c + c + c + --- + c = nc$$ where c is a constant

$$\sum_{i=1}^{n} 2x_i = 2x_1 + 2x_2 + --- + 2x_n = 2(x_1 + x_2 + --- + x_n)$$
$$= 2\sum_{i=1}^{n} x_n$$

$$\sum_{i=1}^{4} \rho^i x^{i-1} = \rho x^0 + \rho^2 x + \rho^3 x^2 + \rho^4 x^3$$
$$= \rho + \rho^2 x + \rho^3 x^2 + \rho^4 x^3$$

$$\sum_{i=1}^{3} x_i + \sum_{i=1}^{3} y_i = x_1 + x_2 + x_3 + y_1 + y_2 + y_3$$
$$= (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3)$$
$$= \sum_{i=1}^{3} (x_i + y_i)$$

$$\sum_{i=1}^{3} x_i + \sum_{i=1}^{5} y_i = x_1 + x_2 + x_3 + y_1 + y_2 + y_3 + y_4 + y_5$$
$$= (x_1 + y_1) + (x_2 + y_2) + (x_3 + y_3) + y_4 + y_5$$
$$= \sum_{i=1}^{3} (x_i + y_i) + \sum_{i=4}^{5} y_i$$

$$\sum_{i=1}^{3} (x_i y_i) = x_1 y_1 + x_2 y_2 + x_3 y_3$$

$$\sum_{i=1}^{3} (x_i) \sum_{i=1}^{3} (y_i) = (x_1 + x_2 + x_3)(y_1 + y_2 + y_3)$$
$$= x_1 y_1 + x_1 y_2 + x_1 y_3$$
$$+ x_2 y_1 + x_2 y_2 + x_2 y_3$$
$$+ x_3 y_1 + x_3 y_2 + x_3 y_3$$
$$= \underbrace{x_1 y_1 + x_2 y_2 + x_3 y_3} + x_1 y_2 + x_1 y_3$$
$$+ x_2 y_1 + x_2 y_3 + x_3 y_1 + x_3 y_2$$
$$= \sum_{i=1}^{3} x_i y_i + x_1 y_2 + x_1 y_3 + x_2 y_2$$
$$+ x_2 y_3 + x_3 y_1 + x_3 y_2$$

Thus $\sum_{i=1}^{3} (x_i) \sum_{i=1}^{3} (y_i) \neq \sum_{i=1}^{3} x_i y_i$

$$\sum_{i=1}^{5} x_i = \underbrace{x_1 + x_2 + x_3}_{} + \underbrace{x_4 + x_5}_{}$$

$$= \sum_{i=1}^{3} x_i + \sum_{i=4}^{5} x_i$$

if you let $k = 3$

then $\displaystyle\sum_{i=1}^{5} x_i = \sum_{i=1}^{k} x_i + \sum_{i=k+1}^{5} x_i$

Double Summation

Examples: [redacted]

$$\sum_{i=1}^{2} \sum_{j=1}^{3} x_i y_j$$

* first some over either "i" or "j". I will sum over "j" first.
* Then Sum over the other index, in my Case "i".

$$\sum_{i=1}^{2} \left(x_i y_1 + x_i y_2 + x_i y_3 \right) = \sum_{i=1}^{2} \left(x_i (y_1 + y_2 + y_3) \right)$$

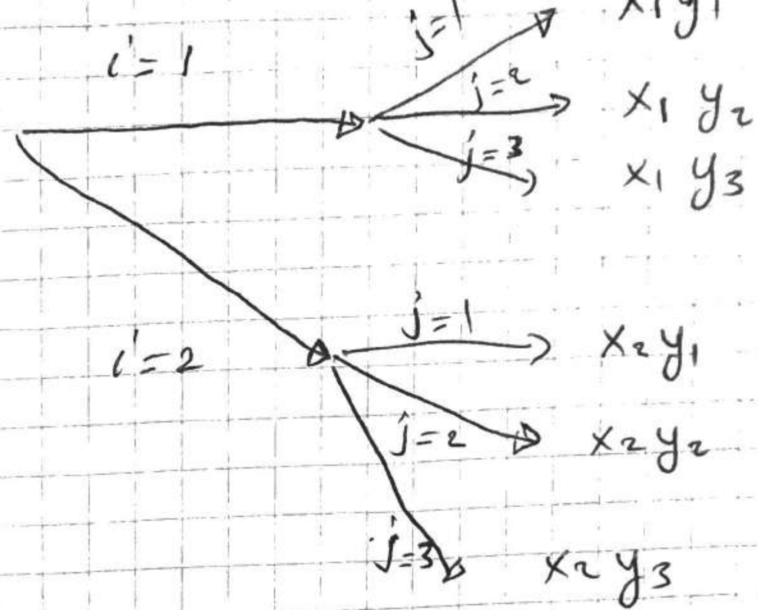
$$= x_1 (y_1 + y_2 + y_3) + x_2 (y_1 + y_2 + y_3)$$

$$= (x_1 + x_2)(y_1 + y_2 + y_3)$$

<u>OR</u> $= x_1 y_1 + x_1 y_2 + x_1 y_3 + x_2 y_1 + x_2 y_2 + x_2 y_3$

$$\sum_{i=1}^{2} \sum_{j=1}^{3} x_i y_i$$

$\cancel{\sum x_i y_i}$



add the branches of the tree you get the Same answer:

$$\sum_{i=1}^{2} \sum_{j=1}^{3} x_i y_j = x_1 y_1 + x_1 y_2 + x_1 y_3 + x_2 y_1 + x_2 y_2 + x_2 y_3$$

Summation Properties and definition

The Summation symbol $(\sum)$ instructs us to sum the elements of a sequence.

$$\sum_{i=1}^{n} x_i$$

n → stopping point or upper limit of Summation

x_i → element.

$\sum$ → Summation Sign

$i=1$ → starting point or lower limit of summation

i → index of Summation.

Properties

1) $\sum_{i=1}^{n} c = nc$; c is a constant term

2) $\sum_{i=1}^{n} c x_i = c \sum_{i=1}^{n} x_i$; c is a constant term.

3) $\sum_{i=1}^{n} x_i = \sum_{i=1}^{k} x_i + \sum_{i=k+1}^{n} x_i$

4) $\sum_{i=1}^{n} x_i = \sum_{i=1}^{k} x_i + \sum_{i=k+1}^{h} x_i + \sum_{i=h+1}^{n} x_i$

where $1 < k < h < n$

5) $\sum_{i=1}^{n} (x_i \pm y_i) = \sum_{i=1}^{n} (x_i) \pm \sum_{i=1}^{n} (y_i)$

6) $\sum_{i=1}^{n} (x_i y_i) \neq \sum_{i=1}^{n} (x_i) \sum_{i=1}^{n} (y_i)$

7) $\sum_{i=1}^{n} \left(\frac{x_i}{y_i} \right) \neq \frac{\sum_{i=1}^{n} (x_i)}{\sum_{i=1}^{n} (y_i)}$

Expectation operator $E(.)$.

~~Ed~~ $\boxed{\text{Properties}}$

1) $E(x+y) = E(x) + E(y)$

2) $E(x-y) = E(x) - E(y)$

3) $E(x+y+z) = E(x) + E(y) + E(z)$

4) $E(xy) \neq E(x)\,E(y)$ ~~unless~~ if x and y are ~~in~~dependent

5) $E(xy) = E(x)\,E(y)$ if x and y are independent

we will assume throughout this course that x and y are dependent

6) $E(c) = c$

7) $E(cx) = c\,E(x)$

8) Any expectation is a constant term, i.e $E(x)$ as a whole is a constant term, although x is a variable.

$\boxed{9)\ E(x^2) \neq E(x)^2}$

Uses of $E(x)$ in Statistics

$E(x)$ represents the population mean.

More generally $E(\cdot)$ operators represent the moments of the distribution.

$E(x)$ / First moment / mean

$E(x^2)$ / Second (uncentered) moment)

$E\left[\left(X - E(x)\right)^2\right]$ second centered moment or the population variance.

It's worth reminding you that the sample average is defined by $\bar{x} = \frac{1}{n} \sum x_i$

and the population average (or the mean) is defined by $E(x)$.

So $E(x)$ can be seen as the population counterpart of the sample average

Similarly we know that the sample variance is defined by

$$\frac{1}{n}\sum(x_i-\bar{x})^2.$$

$E(x-E(x))^2$ is the population counterpart of the sample variance.

Statistical applications of the summation ~~notation~~ and the expectation operators.

(*) Show that $E(x-E(x))^2 = E(x^2)-E(x)^2$ (these are two equivalent definitions of the population variance).

$$E[(x-E(x))^2] = E[x^2+E(x)^2-2xE(x)]$$

$$= E(x^2)+E(E(x)^2)-E(2xE(x))$$

~~since $E(x)$ is constant, and $E(c)=c$, then~~

~~$E(\underbrace{E(x)}_{c}^2) = E(x)^2$~~

$$= E(x^2) + E(E(x)^2) - 2E(xE(x))$$

$$= E(x^2) + E(x)^2 - 2E(x)E(x)$$

$$= E(x^2) + E(x)^2 - 2E(x)^2$$

$$= E(x^2) - E(x)^2 \blacksquare$$

Notes: 1) $E(x)$ is a constant term
and $E(c) = c$, thus $E(E(x)) = E(x)$

2) $E(x \underbrace{E(x)}_{\text{this is constant,}})$ and $E(xc) = cE(x)$

thus $E(xE(x)) = E(x)E(x) = E(x)^2$

Show that $\frac{1}{n}\sum(x_i - \bar{x})^2 = \frac{1}{n}\sum x_i^2 - \bar{x}^2$

Recall $\bar{x} = \frac{1}{n}\sum x_i$. and its a constant term.

$$\frac{1}{n}\sum(x_i - \bar{x})^2 = \frac{1}{n}\sum\left(x_i^2 + \bar{x}^2 - 2x_i\bar{x}\right)$$

$$= \frac{1}{n}\sum x_i^2 + \frac{1}{n}\sum \bar{x}^2 - \frac{1}{n}\sum 2x_i\bar{x}.$$

$$= \frac{1}{n}\sum x_i^2 + \frac{n\bar{x}^2}{n} - \frac{1}{n}2\bar{x}\sum x_i$$

$$= \frac{1}{n}\sum x_i^2 + \bar{x}^2 - 2\bar{x}\underbrace{\frac{1}{n}\sum x_i}_{=\bar{x}}$$

$$= \frac{1}{n}\sum x_i^2 + \bar{x}^2 - 2\bar{x}\,\bar{x}$$

$$= \frac{1}{n}\sum x_i^2 + \bar{x}^2 - 2\bar{x}^2$$

$$= \frac{1}{n}\sum x_i^2 - \bar{x}^2 \quad \blacksquare$$

Notes: since $\bar{x}$ is constant so is $\bar{x}^2$, using $\sum c = nc$ we get $\sum \bar{x}^2 = n\bar{x}^2$

thus $\frac{1}{n}\sum \bar{x}^2 = \frac{n\bar{x}^2}{n} = \bar{x}^2$.

since $\bar{x}$ is constant, using $\sum c x_i = c\sum x_i$, we get

$$\frac{1}{n}\sum \underbrace{2\bar{x}}_{c} x_i = \frac{2\bar{x}}{n}\sum x_i$$

and since $\frac{\sum x_i}{n} = \bar{x}$

then
$$\frac{1}{n}\sum 2\bar{x}\,x_i = 2\bar{x}\underbrace{\frac{\sum x_i}{n}}_{\bar{x}} = 2\bar{x}\bar{x} = 2\bar{x}^2.$$

Matrix Algebra

definition

a Matrix is a ~~reg~~ rectunglar array of elements.

$$\begin{array}{cl} A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} & \begin{array}{l} \rightarrow \text{first row} \\ \rightarrow \text{second row} \end{array} \\ \begin{array}{cc} \downarrow & \downarrow \\ \text{first Column} & \text{Second Column} \end{array} & \end{array}$$

a_{11} is the first element in the first row and the first Column.

generally a_{ij} is th element positioned in the i'th row and jth column.

~~dimensions of a matrix is ta~~

Dimension of a matrix is ~~nº of rows~~ nº of rows x nº of Columns. $(r \times c)$

Thus matrix A has dimensions 2x2.

a vector is a special matrix which has either one row or one column.

a scalar is a special matrix with 1x1 dimension (one element).

~~Types of matricies~~ ~~Consider~~

1) Square matrix n° of rows is equal to n° of columns (r=c)

~~2) Identity Diagonal matrix~~

~~2) Symmetric matrix a square matrix~~

2) Symmetric matrix : $a_{ij} = a_{ji} \quad \forall j \neq i$.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

for A to be symmetric :

$$a_{12} = a_{21}$$
$$a_{13} = a_{31}$$
$$a_{23} = a_{32}$$

Example $A = \begin{bmatrix} 1 & 9 & 4 \\ 9 & 2 & 3 \\ 4 & 3 & 3 \end{bmatrix}$

A Symmetric matrix is a <u>square matrix.</u>

3) Diagonal matrix is a square - symmetric matrix where off diagonal elements are equal to zero

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

4) identity matrix : special Case of diagonal matricies where the diagonal elements are equal to 1.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

~~(5) Transpose matrix of a matrix.~~

5) Transpose of a matrix, an operator that flips the matrix over its diagonal. (Notation: A' or A^T)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

first row became first column and second row became Second Colomn.

Matrix operations;

A + B

$$\overset{A}{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}} \overset{\pm}{\pm} \overset{B}{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}}$$

$$= \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} \end{bmatrix}$$

AB

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

$$= \begin{bmatrix} [a_{11} \ a_{12}] \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} & [a_{11} \ a_{12}] \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} \\ [a_{21} \ a_{22}] \begin{bmatrix} b_{11} \\ b_{21} \end{bmatrix} & [a_{21} \ a_{22}] \begin{bmatrix} b_{12} \\ b_{22} \end{bmatrix} \end{bmatrix}$$

Recall $\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} ac + b \end{bmatrix}$

The $AB = \begin{bmatrix} a_{11} b_{11} + a_{12} b_{21} & a_{11} b_{12} + a_{12} b_{22} \\ a_{21} b_{11} + a_{22} b_{21} & a_{21} b_{12} + a_{22} b_{22} \end{bmatrix}$

Example:

$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 7 \\ 1 & 2 \end{bmatrix}$$

$2 \times 2 \quad 2 \times 2$

For the multiplication to exist, no of columns of first matrix should be equal to number of rows of the second matrix.

$$\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}$$

$2 \times 2 \quad 2 \times 1$

$$\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

3×2 $\qquad$ 2×3

$$= \begin{bmatrix} 3 & 2 & 5 & 6 \\ 4 & 6 & 10 & 13 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

3×3

For instance the first element (3), was ~~calcu~~ obtained as follows:

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1 \times 1) + (2 \times 1) = 3$$

Multiplying a matrix by a scalar

$$\lambda A = \lambda \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \lambda a_{11} & \lambda a_{12} \\ \lambda a_{21} & \lambda a_{22} \end{bmatrix}$$

Determinants and Inverse

The 2×2 Case

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

2×2

determinant of A, which is ~~det~~ denoted by $|A|$ is equal to:

$$|A| = \underbrace{a_{11}\,a_{22}}_{\text{product of diagonal elements}} - \underbrace{a_{21}\,a_{12}}_{\text{product of the off-diagonal elements.}}$$

Example $E = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$$|E| = (1\times 4) - (2\times 3) = 4-6 = -2.$$

Inverse of the 2x2 ~~m~~ matrix

Inverse of A, which is usually denoted by A^{-1}, ~~is d~~ can be obtained as follows:

$$A^{-1} = \underbrace{\frac{1}{|A|}}_{\text{determinant of } A} \underbrace{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}$$

Notice we switched the positions of the diagonal elements and, we premultiplied the off-diagonal elements by -1.

Example:
$$E = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$E^{-1} = \frac{1}{\underbrace{|E|}} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

we computed this already ($|E| = -2$)

$$= \frac{1}{-2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{4}{2} & \frac{-2}{-2} \\ \frac{-3}{-2} & \frac{1}{-2} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix}$$

3x3 Case

first, let's review how to obtain the Minors and cofactors matricies.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

3x3

To find ~~the~~ ~~first~~ a Minor (M_{ij}), cancel out the row and the column intersecting at "ij" position, then find the determinant of the remaining elements.

~~for ii~~ Thus, to find M_{11},

first row   first column

$$\begin{bmatrix} \cancel{a_{11}} & \cancel{a_{12}} & \cancel{a_{13}} \\ \cancel{a_{21}} & a_{22} & a_{23} \\ \cancel{a_{31}} & a_{32} & a_{33} \end{bmatrix} \qquad M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$= a_{22}\,a_{33} - a_{23}\,a_{32}$$

Similarly $M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$

$$M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$M_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}$$

and so on,

The Minors matrix is equal to

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

Example: $A = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

$M_{11} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = (2\times 2) - (1\times 1) = 3$

$M_{12} = \begin{vmatrix} 5 & 1 \\ 3 & 2 \end{vmatrix} = 7$

$M_{13} = \begin{vmatrix} 5 & 2 \\ 3 & 1 \end{vmatrix} = -1$

$M_{21} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1$

$\vdots$

$M = \begin{bmatrix} 3 & 7 & -1 \\ -1 & -1 & 1 \\ -2 & -4 & 2 \end{bmatrix}$

Cofactors matrix:

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

Consider the previous matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$

for C_{11}, $i=1$, $j=1$

$$\Rightarrow C_{11} = (-1)^{1+1} M_{11} = (-1)^2 M_{11} = M_{11} = 3$$

$$C_{12} = (-1)^{1+2} M_{12} = (-1)^3 M_{12} = -M_{12} = -7$$

$$C_{13} = (-1)^{1+3} M_{13} = (-1)^4 M_{13} = M_{13} = -1$$

$$\vdots$$

$$C = \begin{bmatrix} 3 & -7 & -1 \\ 1 & -1 & -1 \\ -2 & 4 & 2 \end{bmatrix}$$

- an easier way to remember how to find the Cofactors matrix is to

alternate the signs of the Minors matrix starting by (+).

$$C = \begin{bmatrix} (+)\, M_{11} & (-)\, M_{12} & (+)\, M_{13} \\ (-)\, M_{21} & (+)\, M_{22} & (-)\, M_{23} \\ (+)\, M_{31} & (-)\, M_{32} & (+)\, M_{33} \end{bmatrix}$$

Recall $M = \begin{bmatrix} 3 & 7 & -1 \\ -1 & -1 & 1 \\ -2 & -4 & 2 \end{bmatrix}$

Alternate the Signs to get C

$$C = \begin{bmatrix} +3 & -7 & +(-1) \\ -(-1) & +(-1) & -(1) \\ +(-2) & -(-4) & +(2) \end{bmatrix} = \begin{bmatrix} 3 & -7 & -1 \\ 1 & -1 & -1 \\ -2 & 4 & 2 \end{bmatrix}$$

Adjoint matrix is the transpose of the Cofactors matrix

$$Adj\,(A) = C' = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

$$Adj\,(A) = \begin{bmatrix} 3 & 1 & -2 \\ -7 & -1 & 4 \\ -1 & -1 & 2 \end{bmatrix}$$

Determinant of 3x3 Case.

$$|A| = \sum_{j=1}^{3} a_{1j} (C_{1j})$$

$$= a_{11} C_{11} + a_{12} C_{12} + a_{13} C_{13}$$

$$= a_{11} M_{11} - a_{12} M_{12} + a_{13} M_{13}$$

Example: $A = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 3 \\ 1 & 2 & 2 \end{bmatrix}$

$$M_{11} = \begin{vmatrix} 0 & 3 \\ 2 & 2 \end{vmatrix} = -6$$

$$M_{12} = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1$$

$$M_{13} = \begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} = 2$$

$$|A| = 1(-6) - 0(-1) + 2(2)$$

$$= -6 + 4 = -2.$$

You are allowed to use other methods to find the determinant.

Inverse (3×3 Case)

$$A^{-1} = \frac{1}{|A|}\,\text{adj}(A)$$

Consider again the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 5 & 2 & 1 \\ 3 & 1 & 2 \end{bmatrix}$$

$$\cancel{|A| =}\ |A| = 1\times 3 - 0(7) + 1(-1)$$
$$= 2$$

we already derived the adj(A)

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 3 & 1 & -2 \\ -7 & -1 & 4 \\ -1 & -1 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 3/2 & 1/2 & -1 \\ -7/2 & -1/2 & 2 \\ -1/2 & -1/2 & 1 \end{bmatrix}$$

~~Ex~~ <u>Practice</u>: find the inverse of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 3 & 4 \end{bmatrix}$$

<u>answer</u>: $A^{-1} = \begin{bmatrix} 2 & 3/2 & -1/2 \\ 0 & 1 & 0 \\ -1 & -3/2 & 1/2 \end{bmatrix}$.

Matrix Properties:

1) if A is symmetric then $A' = A$

2) if $A' = A$, then A is symmetric

3) $AB \neq BA$ (there are few exceptions though)

4) $ABC = (AB)C = A(BC)$

5) $AI = A$

6) $IA = A$

7) $(A')' = A$

8) $(A^{-1})^{-1} = A$

9) $(A^{-1})' = (A')^{-1}$

10) $A^{-1}A = AA^{-1} = I$

11) $(AB)^{-1} = B^{-1}A^{-1}$

12) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

13) $|A^{-1}| = \frac{1}{|A|}$

14) $|I| = 1$

15) $|A'| = |A|$

16) $|AB| = |A| . |B|$

17) $|A+B| \neq |A| + |B|$

18) trace of A = sum of diagonal elements.

19) $tr(ABC) = tr(CAB) = tr(BCA)$

20) if $|A| = 0$, A is not invertible

Thus, the inverse doesnot exist.

we call a non-invertible matrix "Singular".

21) if $AA = A$, then A is called idempotent matrix.

22) $(ABC)' = C'B'A'$ 33

Examples on the use of these properties: (application to projection matrices)

Example 1:

Consider the projection matrix

$$P = X(X'X)^{-1}X'$$

and the off-projection matrix

$$M = (I - X(X'X)^{-1}X').$$

a) Show that P and M are symmetric.

$$
\begin{aligned}
P' &= [X(X'X)^{-1}X']' \\
&= (X')'((X'X)^{-1})'X' \\
&= X((X'X)')^{-1}X' \\
&= X((X')'(X')')^{-1}X' \\
&= X(X'X)^{-1}X' = P
\end{aligned}
$$

Since $P' = P$ then P is symmetric.

$$M = (I - \underbrace{X(X'X)^{-1}X'}_{\text{Notice this is equal to } P}) = (I - P)$$

$$M' = (I - P)' = (I' - P')$$

Identity matrix is always symmetric (Can you show it?).

then $M' = I - P'$

but we just showed that $P' = P$

$$\Rightarrow M' = \underbrace{(I - P)}_{M}$$

Since $M' = M$, then M is

Symmetric.

b) Show that P and M are idempotent

$$PP = (X(X'X)^{-1}X')(X(X'X)^{-1}X')$$

$$= X\,\cancel{(X'X)^{-1}X'X}\,(X'X)^{-1}X'$$

$$= X(X'X)^{-1}X' = P$$

Since $PP = P$, then P is idempotent.

$$M = (I - P)$$

$$MM = (I-P)(I-P) = II - IP - PI + PP$$

$II = I$
$IP = P$
$PI = P$
And we just showed above that $PP = P$

Thus, $MM = I - P - P + P = I - P = M$

Since $MM = M$, M is idempotent.

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c) Show that $P\cancel{X} = X$

$$PX = (X(X'X)^{-1}X')X$$

$$= X(\cancel{X'X})^{-1}\cancel{X'X} = X$$

d) Show that $MX = 0$

$$MX = (I - P)X = IX - PX$$

$$= X - X = 0$$

e) Show that $MP = 0$

$$MP = (I - P)P = IP - PP$$

$$= P - P = 0$$

f) Show that $M + P = I$

$$M = (I - P)$$

$$M + P = (I - P) + P = I.$$

Example 2:

Verify that $(AB)^T = B^T A'$ using the following matricies

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 1 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 0 & 2 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 4 & 0 \\ 3 & 2 \end{bmatrix}.$$

$$A' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad B' = \begin{bmatrix} 4 & 1 & 0 \\ 3 & 1 & 2 \end{bmatrix}$$

$$B'A' = \begin{bmatrix} 4 & 1 & 0 \\ 3 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 3 & 2 \end{bmatrix}$$

which is equal to $(AB)'$

Solving System of equations using

1) Matrix Inverse Method
2) Cramer's Rule

1) Matrix Inverse Method

Assume you have the following system of equations

$$3x_1 + 2x_2 = 20$$
$$x_1 - x_2 = 5$$

- write the system in the matrix form

$$\underbrace{\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 20 \\ 5 \end{bmatrix}}_{d}$$

matrix of Coefficients

using the matrix notation,

$$A\ x = d$$

we are interested in Solving for x.

$$Ax = d$$

$$\underbrace{A^{-1}A}x = A^{-1}d \quad \text{(premultiply both sides by } A^{-1} \text{ to isolate } x)$$

$$\underbrace{Ix} = A^{-1}d$$

$$\boxed{x = A^{-1}d}$$

$$x = A^{-1}d$$

$$= \frac{1}{|A|}\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 20 \\ 5 \end{bmatrix}$$

$$|A| = -3 - 2 = -5$$

$$\Rightarrow x = \frac{1}{-5}\begin{bmatrix} -1 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 20 \\ 5 \end{bmatrix}$$

$$= -\frac{1}{5}\begin{bmatrix} -20-10 \\ -20+15 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-30}{-5} \\ \frac{-5}{-5} \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

Recall $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$. Thus $\boxed{\begin{matrix} x_1 = 6 \\ x_2 = 1 \end{matrix}}$

More generally:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = d_1 \\ a_{21}x_1 + a_{22}x_2 = d_2 \end{cases}$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}}_{d}$$

$$\cancel{A^{-1}A}\, x = A^{-1}d$$

$$\boxed{x = A^{-1}d}$$

Although I presented the ~~definition~~ method in the 2×2 case it ~~it can be easily gene~~ applies to any square matrix.

3×3 Case

$$a_{11} x_1 + a_{12} x_2 + a_{13} x_3 = d_1$$

$$a_{21} x_1 + a_{22} x_2 + a_{23} x_3 = d_2$$

$$a_{31} x_1 + a_{32} x_2 + a_{33} x_3 = d_3$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$A x = d$$

$$\Rightarrow \boxed{x = A^{-1} d}$$

Cramer's Rule:

Consider the Same System of equation (the 2x2 case)

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$X_1 = \frac{|A_1|}{|A|}$$

where $A_1 = A$ with the first Column is replaced by d.

Thus $A_1 = \begin{bmatrix} d_1 & a_{12} \\ d_2 & a_{22} \end{bmatrix}$

Thus $X_1 = \dfrac{\begin{vmatrix} d_1 & a_{12} \\ d_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \dfrac{(d_1 a_{22}) - (a_{12} d_2)}{(a_{11} a_{22}) - (a_{21} a_{12})}$

$$X_2 = \frac{|A_2|}{|A|}$$

where $A_2 = \begin{bmatrix} a_{11} & d_1 \\ a_{21} & d_2 \end{bmatrix}$

$$X_2 = \frac{\begin{vmatrix} a_{11} & d_1 \\ a_{21} & d_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}} = \frac{(d_2 a_{11} - d_1 a_{21})}{(a_{11} a_{22} - a_{12} a_{21})}$$

Consider the System

$$3x_1 + 2x_2 = 20$$

$$x_1 - x_2 = 5$$

$$\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 20 \\ 5 \end{bmatrix}$$

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$$X_1 = \frac{\begin{vmatrix} 20 & 2 \\ 5 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix}} = \frac{-20-10}{-3-2}$$

$$= \frac{-30}{-5} = 6$$

$$X_2 = \frac{\begin{vmatrix} 3 & 20 \\ 1 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 1 & -1 \end{vmatrix}} = \frac{15-20}{-5} = \frac{-5}{-5} = 1$$

which are the same answers obtained by the Matrix inverse method.

Economic application

- The demand for Coffee ~~is given~~ and Tea are give by:

$$D_c = 100 - 2p_c + 0.5p_t$$
$$D_t = 120 - p_t + 0.75p_c$$

and the Supply by:

$$S_c = 10 + p_c$$
$$S_t = 5 + 2p_t$$

Solve for equilibrium prices and quantities. ~~using matrix inverse method and Cramer's~~

Coffee: Supply = Demand

$$\Rightarrow 100 - 2p_c + 0.5p_t = 10 + p_c$$
$$\Rightarrow \boxed{3p_c - 0.5p_t = 90}$$

Tea: Supply = Demand $120 - p_t + 0.75p_c = 5 + 2p_t$

$$\Rightarrow \boxed{3p_t - 0.75p_c = 115}$$

Now we have a system of equations

$$\begin{cases} 3P_c - 0.5P_t = 90 \\ -0.75P_c + 3P_t = 115 \end{cases}$$

you can either use matrix inverse method or cramer's Rule.

Matrix inverse method

$$\underbrace{\begin{bmatrix} 3 & -0.5 \\ -0.75 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} P_c \\ P_t \end{bmatrix}}_{P} = \underbrace{\begin{bmatrix} 90 \\ 115 \end{bmatrix}}_{d}$$

$$\cancel{\begin{bmatrix} P_c \\ P_t \end{bmatrix} = \Big[}$$

$$P = A^{-1}d$$

$$P = \frac{1}{9 - (0.75)(0.5)} \begin{bmatrix} 3 & 0.5 \\ 0.75 & 3 \end{bmatrix} \begin{bmatrix} 90 \\ 115 \end{bmatrix}$$

$$P = \frac{1}{8.625}\begin{bmatrix} 270 + 57.5 \\ 67.5 + 345 \end{bmatrix}$$

$$P = \begin{bmatrix} P_c \\ P_t \end{bmatrix} = \begin{bmatrix} \frac{327.5}{8.625} \\ \frac{412.5}{8.625} \end{bmatrix} = \begin{bmatrix} 37.9710 \\ 47.826 \end{bmatrix}$$

$P_c = 37.9710$

$P_t = 47.826$

~~More general~~

Two Commodities market model

if two goods are substitutes, then equating the Supply to the demand of each good will yeild a system of the following form:

$$c_1 P_1 + c_2 P_2 = -c_0$$
$$\gamma_1 P_1 + \gamma_2 P_2 = -\gamma_0$$

where $c_1, c_2, \gamma_1, \gamma_2, c_0, \gamma_0$ are all known parameters and P_1 and P_2 are the variables (& P_1 and P_2 are the prices of the goods).

$$\begin{bmatrix} c_1 & c_2 \\ \gamma_1 & \gamma_2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} -c_0 \\ -\gamma_0 \end{bmatrix}$$

Matrix inverse method.

$$\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ \gamma_1 & \gamma_2 \end{bmatrix}^{-1} \begin{bmatrix} -c_0 \\ -\gamma_0 \end{bmatrix}$$

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \left\{ \frac{1}{c_1\gamma_2 - c_2\gamma_1} \begin{bmatrix} \gamma_2 & -c_2 \\ -\gamma_1 & c_1 \end{bmatrix} \begin{bmatrix} -c_0 \\ -\gamma_0 \end{bmatrix} \right)$$

$$= \frac{1}{c_1\gamma_2 - c_2\gamma_1} \begin{bmatrix} c_2\gamma_0 - \gamma_2 c_0 \\ \gamma_1 c_0 - \gamma_0 c_1 \end{bmatrix}$$

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} \dfrac{c_2\gamma_0 - \gamma_2 c_0}{c_1\gamma_2 - c_2\gamma_1} \\ \\ \dfrac{\gamma_1 c_0 - \gamma_0 c_1}{c_1\gamma_0 - c_2\gamma_1} \end{bmatrix}$$

Note that prices can't be negative or equal to zero, thus we have to impose the restrictions

$$\frac{c_2\gamma_0 - \gamma_2 c_0}{c_1\gamma_2 - c_2\gamma_1} > 0$$

and $$\frac{\gamma_1 c_0 - \gamma_0 c_1}{c_1\gamma_0 - c_2\gamma_1} > 0$$

Practice: (Solve the system using Cramer's Rule).

National accounts Application.

$$Y = C + A_0$$

$$C = a + b(Y - T)$$

$$T = d + tY$$

Y, C and T are National income, consumption and Tax revenues respectively.

~~Rewrite the system~~

$$\begin{cases} Y - C = A_0 \\ C - b(Y - T) = a \\ T - tY = d \end{cases}$$

$$\begin{cases} Y - C = A_0 \\ -bY + C + bT = a \\ -tY + T = d \end{cases}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{bmatrix} \begin{bmatrix} Y \\ C \\ T \end{bmatrix} = \begin{bmatrix} A_0 \\ a \\ d \end{bmatrix} \quad (4)$$

I will use cramer's Rule

practice (Repeat the exercise using matrix inverse method).

$$Y = \frac{\begin{vmatrix} A_0 & -1 & 0 \\ a & 1 & b \\ d & 0 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{a - bd + A_0}{1 - b + bt}$$

$$C = \frac{\begin{vmatrix} 1 & A_0 & 0 \\ -b & a & b \\ -t & d & 1 \end{vmatrix}}{\begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix}} = \frac{a - bd + A_0 b - bt A_0}{1 - b + bt}$$

$$T = \frac{\begin{vmatrix} 1 & -1 & A_0 \\ -b & 1 & a \\ -t & 0 & d \end{vmatrix}}{1 - b + bt} = \frac{d(1-b) + t(a + A_0)}{1 - b + bt}.$$