the mean value of the resulting current using the concepts of conditional probability.

Answer: 1.5958

Exercise 2–8.2

A traveling wave tube has a mean-time-to-failure of 4 years. Given that the TWT has survived for 4 years, find the conditional probability that it will fail between years 4 and 6.

Answer: 0.3935

2–9 Examples and Applications

The preceding sections have introduced some of the basic concepts concerning the probability distribution and density functions for a continuous random variable. Before extending these concepts to more than one variable, it is desirable to consider a few examples illustrating how they might be applied to simple engineering problems.

As a first example, consider the elementary voltage-regulating circuit shown in Figure 2–27(a). It employs a Zener diode having an idealized current–voltage characteristic as shown in Figure 2–27(b). Note that current is zero until the voltage reaches the breakdown value ($V_z = 10$) and from then on is limited by the external circuit, while the voltage across the diode remains constant. Such a circuit is often used to limit the voltage applied to solid-state devices. For example, the R_L indicated in the circuit may be a transistorized amplifier designed to work at 9 V and that is damaged if the voltage exceeds 10 V. The supply voltage, V_S, is from a power supply whose nominal voltage is 12 V, but whose actual voltage contains a sawtooth ripple and, hence, is a random variable. For purposes of this example, it will be assumed that this random variable has a uniform distribution over the interval from 9 to 15 V.

Zener diodes are rated in terms of their ability to dissipate power as well as their breakdown voltage. It will be assumed that the average power rating of this diode is $W_z = 3$ W. It is then desired to find the value of series resistance, R, needed to limit the mean dissipation in the Zener diode to this rated value.

When the Zener diode is conducting, the voltage across it is $V_z = 10$, and the current through it is

$$I_z = \frac{V_s - V_z}{R} - I_L \quad \text{and} \quad V_s > \frac{V_z(R + R_L)}{R_L} = \frac{10(R + 10)}{10}$$

where the load current, I_L, is 1 A. The power dissipated in the diode is

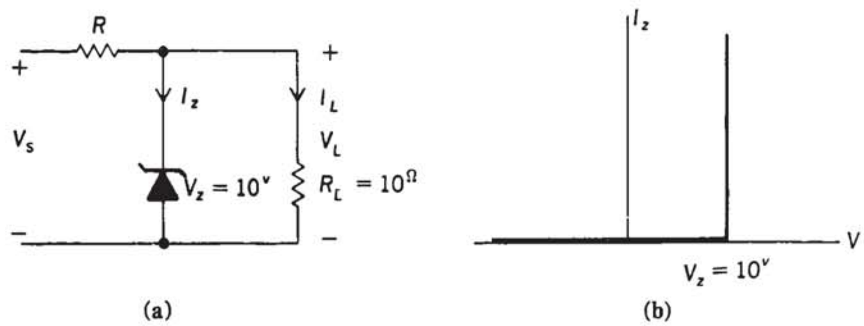


Figure 2–27 Zener diode voltage regulator: (a) voltage-regulating circuit and (b) Zener diode characteristic.

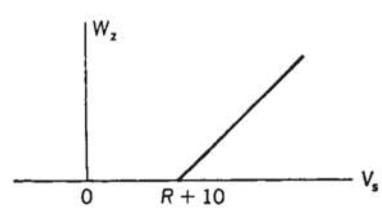
$$
\begin{aligned}
W_z = V_z I_z &= \frac{V_z(V_s - V_z)}{R} - I_L V_z \\
&= \frac{10V_s - 100}{R} - 10 \qquad V_s > R + 10
\end{aligned}
$$

A sketch of this power as a function of the supply voltage V_s is shown in Figure 2–28, and the probability density functions of V_s and W_z are shown in Figure 2–29. Note that the density function of W_z has a large delta function at zero, since the diode is not conducting most of the time, but is uniform for larger values of W since W_z and V_s are linearly related in this range. From the previous discussion of transformations of density functions in Section 2–3, it is easy to show that

$$
\begin{aligned}
f_W(w) &= F_V(R+10)\,\delta(w) + \frac{R}{10} f_V\left(\frac{Rw}{10} + R + 10\right) \qquad 0 \le w \le \frac{50}{R} - 10 \\
&= 0 \qquad \text{elsewhere}
\end{aligned}
$$

where $F_V(\cdot)$ is the distribution function of V_s. Hence, the area of the delta function is simply the probability that the supply voltage V_s is *less* than the value that causes diode conduction to start.

Figure 2–28 Relation between diode power dissipation and supply voltage.



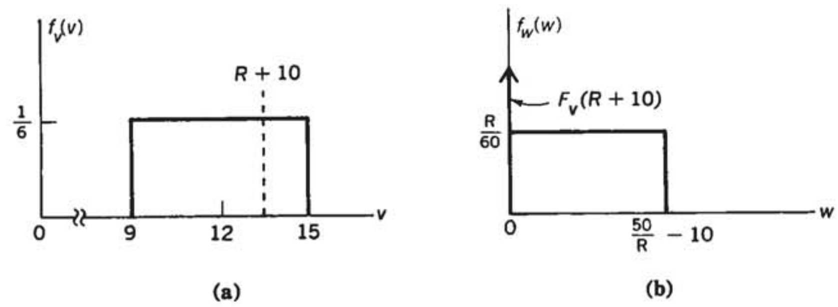


Figure 2–29 Probability density functions for supply voltage and diode power dissipation: (a) probability density function for V_s and (b) probability density function for W_z.

The mean value of diode power dissipation is now given by

$$E[W_z] = \overline{W}_z = \int_{-\infty}^{\infty} w f_W(w)\, dw$$

$$= \int_{-\infty}^{\infty} w F_V(R+10)\,\delta(w)\, dw + \int_0^{\infty} w \left(\frac{R}{10}\right) f_V\left(\frac{Rw}{10} + R + 10\right) dw$$

The first integral has a value of zero (since the delta function is at $w = 0$) and the second integral can be written in terms of the uniform density function

$$[f_V(v) = \frac{1}{6}, \qquad 9 < v \le 15], \text{ as}$$

$$\overline{W}_Z = \int_0^{(50/R)-10} w \left(\frac{R}{10}\right)\left(\frac{1}{6}\right) dw = \frac{(5-R)^2}{1.2R}$$

Since the mean value of diode power dissipation is to be less than or equal to 3 watts, it follows that

$$\frac{(5-R)^2}{1.2R} \le 3 \qquad 0 < R \le 5$$

from which

$$R \ge 2.19\,\Omega$$

It may now be concluded that any value of R greater than 2.19 Ω would be satisfactory from the standpoint of limiting the mean value of power dissipation in the Zener diode to 3 W. The actual choice of R would be determined by the desired value of output voltage at the nominal supply voltage of 12 V. If this desired voltage is 9 V (as suggested above) then R must be

$$R = \frac{3}{\frac{9}{10}} = 3.33\,\Omega$$

which is greater than the minimum value of 2.19 Ω and, hence, would be satisfactory.

As another example, consider the problem of selecting a multiplier resistor for a dc voltmeter as shown in Figure 2–30. It will be assumed that the dc instrument produces full-scale deflection when 100 μA is passing through the coil and has a resistance of 1000 Ω. It is desired to select a multiplier resistor R such that this instrument will read full scale when 10 V is applied. Thus, the nominal value of R to accomplish this (which will be designated as R^*) is

$$R^* = \frac{10}{10^{-4}} - 1000 = 9.9 \times 10^4\,\Omega$$

However, the actual resistor used will be selected at random from a bin of resistors marked $10^5\,\Omega$. Because of manufacturing tolerances, the actual resistance is a random variable having a mean of 10^5 and a standard deviation of 1000 Ω. It will also be *assumed* that the actual resistance is a Gaussian random variable. (This is a customary assumption when deviations around the mean are small, even though it can never be precisely true for quantities that must be always positive, like the resistance.) On the basis of these assumptions it is desired to find the probability that the resulting voltmeter will be accurate to within 2%.[6]

The smallest value of resistance that would be acceptable is

$$R_{\min} = \frac{10 - 0.2}{10^{-4}} - 1000 = 9.7 \times 10^4$$

while the largest value is

$$R_{\max} = \frac{10 + 0.2}{10^{-4}} - 1000 = 10.1 \times 10^4$$

The probability that a resistor selected at random will fall between these two limits is

$$P_c = \Pr[9.7 \times 10^4 < R \leq 10.1 \times 10^4] = \int_{9.7\times10^4}^{10.1\times10^4} f_R(r)\,dr \qquad (2\text{–}51)$$

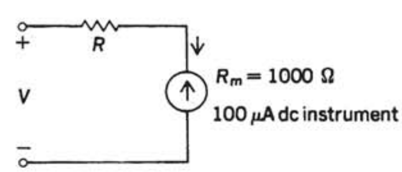


Figure 2–30 Selection of a voltmeter resistor.

[6]This is interpreted to mean that the error in voltmeter reading due to the resistor value is less than or equal to 2% of the full scale reading.

where $f_R(r)$ is the Gaussian probability density function for R and is given by

$$f_R(r) = \frac{1}{\sqrt{2\pi}(1000)} \exp\left[-\frac{(r-10^5)^2}{2(10^6)}\right]$$

The integral in (2–51) can be expressed in terms of the standard normal distribution function, $\Phi(\cdot)$, as discussed in Section 2–5. Thus, P_c becomes

$$P_c = \Phi\left(\frac{10.1 \times 10^4 - 10^5}{10^3}\right) - \Phi\left(\frac{9.7 \times 10^4 - 10^5}{10^3}\right)$$

which can be simplified to

$$\begin{aligned} P_c &= \Phi(1) - \Phi(-3) \\ &= \Phi(1) - [1 - \Phi(3)] \end{aligned}$$

Using the tables in Appendix D, this becomes

$$P_c = 0.8413 - [1 - 0.9987] = 0.8400$$

Thus, it appears that even though the resistors are selected from a supply that is nominally incorrect, there is still a substantial probability that the resulting instrument will be within acceptable limits of accuracy.

The third example considers an application of conditional probability. This example considers a traffic measurement system that is measuring the speed of all vehicles on an expressway and recording those speeds in excess of the speed limit of 70 miles per hour (mph). If the vehicle speed is a random variable with a Rayleigh distribution and a most probable value equal to 50 mph, it is desired to find the mean value of the excess speed. This is equivalent to finding the conditional mean of vehicle speed, given that the speed is greater than the limit, and subtracting the limit from it.

Letting the vehicle speed be S, the conditional distribution function that is sought is

$$F[s|S > 70] = \frac{\Pr\{S \le s, S > 70\}}{\Pr\{S > 70\}} \tag{2–52}$$

Since the numerator is nonzero only when $s > 70$, (2–52) can be written as

$$\begin{aligned} F[s|S > 70] &= 0 & s \le 70 \\ &= \frac{F(s) - F(70)}{1 - F(70)} & s > 70 \end{aligned} \tag{2–53}$$

where $F(\cdot)$ is the probability distribution function for the random variable S. The numerator of (2–53) is simply the probability that S is between 70 and s, while the denominator is the probability that S is greater than 70.

The conditional probability density function is found by differentiating (2–53) with respect to s. Thus,

$$f(s|S>70)=0 \qquad s\le 70$$
$$=\frac{f(s)}{1-F(70)} \qquad s>70$$

where $f(s)$ is the Rayleigh density function given by

$$f(s)=\frac{s}{(50)^2}\exp\left[-\frac{s^2}{2(50)^2}\right] \qquad s\ge 0 \tag{2–54}$$
$$=0 \qquad s<0$$

These functions are sketched in Figure 2–31.

The quantity $F(70)$ is easily obtained from (2–54) as

$$F(70)=\int_0^{70}\frac{s}{(50)^2}\exp\left[-\frac{s^2}{2(50)^2}\right]ds=1-\exp\left[-\frac{49}{50}\right]$$

Hence,

$$1-F(70)=\exp\left[-\frac{49}{50}\right]$$

The conditional expectation is given by

$$E[S|S>70]=\frac{1}{\exp[-49/50]}\int_{70}^{\infty}\frac{s^2}{(50)^2}\exp-\left[\frac{s^2}{2(50)^2}\right]ds$$
$$=70+50\sqrt{2\pi}\exp\left[\frac{49}{50}\right]\left\{1-\Phi\left(\frac{7}{5}\right)\right\}$$
$$=70+27.2$$

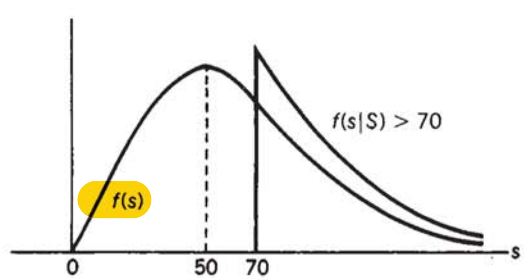


Figure 2–31 Conditional and unconditional density functions for a Rayleigh-distributed random variable.

Thus, the mean value of the excess speed is 27.2 miles per hour. Although it is clear from this result that the Rayleigh model is not a realistic one for traffic systems (since 27.2 miles per hour excess speed is much too large for the actual situation), the above example does illustrate the general technique for finding conditional means.

The final example in this section combines the concepts of both discrete probability and continuous random variables and deals with problems that might arise in designing a satellite communication system. In such a system, the satellite normally carries a number of traveling wave tubes in order to provide more channels and to extend the useful life of the system as the tubes begin to fail. Consider a case in which the satellite is designed to carry 6 TWTs and it is desired to require that after 5 years of operation there is a probability of 0.95 that at least one of the TWTs is still good. The quantity that we need to find is the mean time to failure (MTF) for each tube in order to achieve this degree of reliability. In order to do this we need to use some of the results discussed in connection with Bernoulli trials in Sec. 1–10. In this case, let k be the number of good TWTs at any point in time and let p be the probability that any TWT is good. Since we want the probability that at least one tube is good to be 0.95, it follows that

$$\Pr(k \geq 1) = 0.95$$

or

$$\sum_{k=1}^{6} p_6(k) = 1 - p_6(0) = 1 - \binom{6}{0} p^0 (1-p)^6 = 0.95$$

which can be solved to yield $p = 0.393$. If we assume, as usual, that the lifetime of any one TWT follows an exponential distribution, then

$$\int_5^{\infty} \frac{1}{\overline{T}} e^{-\tau/\overline{T}}\, d\tau = 0.393$$

$$\overline{T} = 5.353$$

Thus, the mean time to failure for each TWT must be at least 5.353 years in order to achieve the desired reliability.

A second question that might be asked is "How many TWTs would be needed to achieve a probability of 0.99 that at least one will be still functioning after 5 years?" In this case, n is unknown but, for TWTs having the same MTF, the value of p is still 0.393. Thus,

$$1 - p_n(0) = 0.99$$

$$\binom{n}{0} p^0 (1-p)^n = 0.01$$

This may be solved for n to yield $n = 9.22$. However, since n must be an integer, this tells us that we must use at least 10 traveling wave tubes to achieve the required reliability.