Problem 1 [10 points = 5 + 5]

Consider a sample $\mathbf{X} = \{X_j : \ 1 \leq j \leq 9\}$ of size $n = 9$ from a population with C.D.F.

$$F(x) = \frac{x}{5} \ \text{ for } \ (0 \leq x \leq 5)$$

Let $X[k]$ denote the k^{th} order statistic for $1 \leq k \leq 9$

1. Derive the density function of $X[4]$
2. Evaluate expectation and variance of $X[4]$

Solution

Problem 2 [10 points = 5 + 5]

Consider a sample $\mathbf{X} = \{X_j : \; 1 \leq j \leq 9\}$ of size $n = 9$ from a population with C.D.F.

$$F(x) = \frac{x}{5} \;\text{ for }\; (0 \leq x \leq 5)$$

Let $X[k]$ denote the k^{th} order statistic for $1 \leq k \leq 9$. Set

$$W = \frac{X[4]}{X[9]}$$

1. Derive the density function for W
2. Evaluate expectation and variance of W

Solution

Problem 3 [10 points]

A continuous random variable Y has density function

$$f^Y(y) = \frac{3^5}{24} \cdot y^4 \cdot \exp(-3y) \quad \text{for} \quad (y > 0)$$

Given $Y = y$, a random variable W has density

$$f^{W|Y}(w|y) = y \cdot \exp(-y \cdot w) \quad \text{for} \quad (w > 0)$$

Derive the **marginal** density for W

Solution

Problem 4: [10 points = 5 + 5]

A continuous random variable Y has density function

$$f^{Y}(y) = \frac{3^5}{24} \cdot y^4 \cdot \exp(-3y) \quad \text{for} \quad (y > 0)$$

Given $Y = y$, a random variable W has density

$$f^{W|Y}(w|y) = y \cdot \exp(-y \cdot w) \quad \text{for} \quad (w > 0)$$

1. Derive **marginal** expectation, $\mathbf{E}[W]$
2. Determine **marginal** variance of W.

Solution

Problem 5 [10 points]

A continuous random variable Y has density function

$$f^{Y}(y) = \frac{3^5}{24} \cdot y^4 \cdot \exp(-3y) \quad \text{for} \quad (y > 0)$$

Given $Y = y$, a random variable W has density

$$f^{W|Y}(w|y) = y \cdot \exp(-y \cdot w) \quad \text{for} \quad (w > 0)$$

Derive the **conditional** density for $(Y|W = w)$

Solution

Problem 6 [10 points]

Assume that continuous random variables, (T, W) are independent and such that

$$T \sim \textbf{Gamma}\,[4,\ 2] \ \text{ and } \ W \sim \textbf{Gamma}\,[6, 2]$$

Derive the density function for the ratio,

$$Y = \frac{T}{W}$$

Solution

Problem 7 [10 points = 5 + 5]

Assume that continuous random variables, (T, W) are independent and such that

$$T \sim \textbf{Gamma}\,[4,\ 2] \;\; \text{and} \;\; W \sim \textbf{Gamma}\,[6, 2]$$

Determine expectations for

$$X = \frac{W}{T} \;\; \text{and} \;\; Y = \frac{T}{W}$$

Solution

Problem 8 [10 points = 5 + 5]

Assume that $\mathbf{Z} = \{Z_j : \; j \geq 1\}$ are independent random variables sharing the standard normal distribution.
An integer-valued variable M does not depend on any of Z_j and has the distribution such that

$$\mathbf{P}[M \geq m] = \left(\frac{1}{4}\right)^{m-1} \quad \text{for} \;\; m \geq 1.$$

Consider a random sum,

$$S = \sum_{j=1}^{3M+1} |Z_j|$$

1. Evaluate expectation, $\mathbf{E}[S]$
2. Derive the variance, $\mathbf{Var}[S]$

Solution

Problem 9 [10 points = 5 + 5]

Assume that $\mathbf{Z} = \{Z_j : \ j \geq 1\}$ are independent random variables sharing the standard normal distribution.
An integer-valued variable M does not depend on any of Z_j and has the distribution such that

$$\mathbf{P}[M \geq m] = \left(\frac{1}{4}\right)^{m-1} \quad \text{for} \quad m \geq 1.$$

Consider a random sum,

$$S = \sum_{j=1}^{3M+1} (Z_j)^2$$

1. Evaluate expectation, $\mathbf{E}[S]$
2. Derive the variance, $\mathbf{Var}[S]$

Solution

Problem 10 [10 points = 6 + 4]

Assume that a continuous random variable Y has its density function defined for $y > 0$ as follows:

$$f^Y(y) = \frac{3^6}{120} \cdot y^5 \cdot \exp(-3y)$$

Given $Y = y$, an integer-valued variable N has conditional distribution as follows:

$$f^{N|Y}(n|y) = \mathbf{P}[N = n|Y = y] = \frac{y^n}{n!} \cdot \exp(-y)$$

1. Derive the **conditional** density of $(Y|N = 2)$
2. Evaluate $\mathbf{E}[Y|N = 2]$

Solution