Problem Set No. 8

Fall 2019

Issued: December 14, 2019 **Due:** December 19, 2019

Problem #1:
Say whether each of the following statements is true/possible or false/impossible and give a brief justification. One word answers will receive no credit.

(a) If $X(t)$ is a zero mean wide-sense stationary process, with autocorrelation function $R_X(\tau) = \max(1 - |\tau|, 0)$,it is mean-squared continuous.

(b) A mean-square differentiable zero mean wide-sense stationary Gaussian process, $X(t)$, and its derivative, $X'(t)$ are uncorrelated (and hence independent).

(c) Consider wide-sense stationary process $x(t)$ with mean 2 and auto covariance $K_x(\tau) = 8e^{-4|\tau|}$, that is input into a linear system with transfer function $H(s) = \frac{1}{(s+2)(s+1)}$, the output $Y(t)$ has mean $m_Y(t) = 4$.

(d) Assume that a wide-sense stationary process $X(t)$ has auto covariance $K_X(\tau) = 5e^{-|\tau|}$. Then, for such a process,

$$E[X(t)] = m_X = \lim_{T\to\infty} \int_{-T}^{T} X(s)ds$$

where the limit is interpreted in the mean square sense.

(e) Consider a zero-mean real-valued $X(t)$ stochastic process that is wide sense stationary. Then, it is possible for the autocovariance function of $X(t)$ to be

$$K_X(\tau) = \begin{cases} 1 & |\tau| \leq 1 \\ 0 & \text{elsewhere} \end{cases}$$

(Hint: consider the power spectral density)

(f) Consider a wide-sense stationary process $U(t)$ that is input into a causal linear system, with transfer function $H(s) = \frac{1}{s-1}$ to obtain output $Y(t)$. Then, the output $Y(t)$ is also wide-sense stationary.

(g) Suppose we have a zero-mean, mean-square continuous stochastic process $X(t)$ on the interval $[0,1]$ with auto covariance $K_X(t_1,t_2) = e^{-|t_1-t_2|}$. Let $f(t), g(t)$ be functions so that $f(t) = 1$, $g(t) = 3(1-2t)$ for $t \in [0,1]$. Note that

$$\int_0^1 f^2(t)dt = 1; \quad \int_0^1 g^2(t)dt = 1; \quad \int_0^1 f(t)g(t)dt = 0$$

i.e. the functions f, g are orthonormal. Define the random coefficients

$$C_1 = \int_0^1 f(t)X(t)dt; \quad C_2 = \int_0^1 g(t)X(t)dt$$

Then, the coefficients C_1, C_2 satisfy $E[C_1C_2] = 0$.

(h) For any wide-sense stationary process $X(t)$, if $\int_{-\infty}^{\infty} |K(\tau)|\, d\tau < \infty$, then the process is completely ergodic.

(i) It is possible for a stochastic process $X(t)$ to have an autocovariance function $K_X(t_1,t_2)$ with $K_X(1,1) = 1$, $K_X(2,2) = 3$, $K_X(1,2) = 2$.

(j) Given two processes $X(t)$ and $Y(t)$ which are independent, their cross-covariance function $K_{XY}(t_1,t_2)$ is always zero.

Problem #2:

We are going to construct a process $M(t)$ that resembles a random walk, except the jumps will be different from ± 1. Specifically, this will be a discrete time process, where $M(0) = 0$, and

$$M(n+1) = M(n) + (n+1)J(n+1) \tag{1}$$

where $J(n)$ are independent, identically distributed Gaussian random variables with mean 0 and variance 1. Note that we can write this in closed form, as

$$M(n) = \sum_{k=1}^{n} kJ(k)$$

Basically, the process $M(t)$ will change values at each discrete time, but the jumps will be getting bigger on average, linearly with the time index. Thus, we can view this as a random walk with random, time-dependent jumps.

(a) Compute $m_M(n) = E[M(n)]$ for a given integer n.

(b) Compute $E[M(n)^2]$. You can leave your answer as a sum.

(c) From the recursive equation for $M(n)$ in (1) above, compute a recursive equation for $m_M(n) = E[M(n)]$ and a recursive equation for its variance $\Sigma(n) = Var(M(n))$.

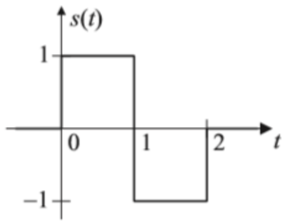
(d) Is $M(n)$ an independent increments process? Explain why or why not.

(e) Compute the autocorrelation function $R_M(m, n) = E[M(m)M(n)]$. You can leave your answer as a sum. Be sure to consider the cases $m > n$ and $m < n$.

(f) Is $M(n)$ a Gaussian process? Explain.

Problem #3:

A continuous-time digital communication system works as follows. Either nothing is transmitted (bit $= 0$, hypothesis H_0) or the signal $As(t)$ is transmitted (bit $= 1$, hypothesis H_1), where $s(t)$ is shown in the figure. The receiver observes $Y(t), t \in [0, 2]$, composed of the transmitted signal corrupted by additive Gaussian noise $W(t)$ with power spectral density $S_W(\omega) = 1$. The two possible transmitted signals are equally likely a priori, as described below.



$$Y(t) = \begin{cases} W(t) & \text{under hypothesis } H_0 \\ As(t) + W(t) & \text{under hypothesis } H_1 \end{cases}$$

(a) Suppose $A = 1$. What is a good orthonormal basis $\{\phi_i(t)\}$ to use in analyzing this problem? Explain.

(b) Design a detector to achieve the minimum probability of error in detecting the correct hypothesis. Specify the required processing of the data $Y(t)$ and associated decision regions.

(c) What is the probability of error for your detector in (b) in terms of $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\alpha^2/2} d\alpha$?

(d) Suppose instead that A is a Gaussian random variable, not dependent on time, and independent of $W(t)$, with $A \sim N(0, 1)$. What is the optimal minimum probability of error receiver for this case? Specify the required processing of the data $Y(t)$ and associated decision regions.

(e) For your detector in part (d) above, write an expression for the probability of error.

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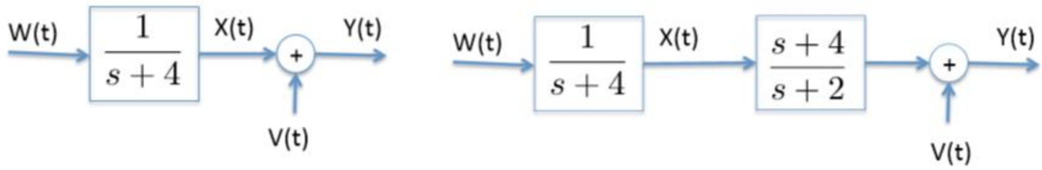

Problem #4:

Consider the system shown in Figure 1 above on the left. Assume that the input process $W(t)$ is white noise, with autocovariance $K_W(\tau) = \delta(\tau)$. The input process $V(t)$ is also white noise, independent of the process $W(t)$, with autocovariance $K_V(\tau) = \frac{1}{9}\delta(\tau)$.

(a) Compute the power spectral densities for $S_X(\omega)$ and $S_Y(\omega)$.

(b) Compute the cross-power spectral density for $S_{YX}(j\omega)$.

(c) Find the noncausal linear filter $H_{nc}(j\omega)$ that minimizes the mean square error in estimating $X(t)$ based on the process $Y(t)$.

(d) Find the variance of the error in the noncausal estimate of $X(t)$ based on observing the process $Y(t)$.

(e) Consider now the system on the right. Assume now that **the autocovariance of the process** $V(t)$ **is** $\mathbf{K_V}(\tau) = \frac{\mathbf{1}}{\mathbf{5}}\delta(\tau)$. Find the non-causal linear filter $H^1_{nc}(j\omega)$ that minimizes the mean square error in estimating $X(t)$ based on the process $Y(t)$.

(f) Compute the variance of the error of the non-causal estimate of $X(t)$ in the previous part.