

Problem 1: Duals of Duals (10 pts)

Consider the following optimization problem.

$$\min_{x,y \in \mathbb{R}} x^2 + y^2$$

such that

$$2x + y \leq 1$$

$$x \geq 0$$

$$y \geq 0$$

1. Is Slater's condition satisfied for this optimization problem?
2. Use the method of Lagrange multipliers to construct a dual of this optimization problem.
3. Use the method of Lagrange multipliers to construct a dual of your dual from (2) of this optimization problem. Did you recover the primal problem?

Problem 2: Projections onto Convex Hulls (30 pts)

For this problem, you will implement the Frank-Wolfe algorithm using `scipy.linprog` in the Python `scipy` package to help you solve the per-iteration subproblems. Recall that, given $x^{(1)}, \dots, x^{(M)} \in \mathbb{R}^n$, their convex hull is the set of all x that can be written as a convex combination of these points.

1. Use the Frank-Wolfe algorithm to compute the projection of a query point q onto the convex hull of the given $x^{(1)}, \dots, x^{(M)} \in \mathbb{R}^n$. That is, you should solve the following optimization problem.

$$\min_{\lambda \in \mathbb{R}^M} \frac{1}{2} \left\| q - \sum_{m=1}^M \lambda_m x^{(m)} \right\|_2^2$$

such that

$$\sum_{m=1}^M \lambda_m = 1$$

$$\lambda \geq 0$$

Your Python function should take as input the x 's, the query point q , an initial value for λ that satisfies the constraints, and the tolerance ϵ for the Frank-Wolfe convergence condition and return the best function value found during the iterative procedure.

2. Use the method of Lagrange multipliers to construct a dual of this optimization problem.

Problem 3: Convex Envelopes (60 pts)

Consider a collection of points $x^{(1)}, \dots, x^{(M)} \in \mathbb{R}^n$ with corresponding function values $y^{(1)}, \dots, y^{(M)} \in \mathbb{R}$. The convex envelope of these points is the convex function $f_{env} : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $f_{env}(x^{(m)}) \leq y^{(m)}$ for all m and for any other convex function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $g(x^{(m)}) \leq y^{(m)}$ for all m , $f_{env}(x) \geq g(x)$ for all $x \in \mathbb{R}^n$.

1. For a finite point set, is the convex envelope differentiable? Explain.
2. Recall that, for any convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, and any $x, x' \in \mathbb{R}^n$, $f(x) \geq f(x') + w^T(x - x')$, where w is a subgradient of f at x' . Using this, we can formulate the problem of evaluating the convex envelope of our collection of points at a query point $x \in \mathbb{R}^n$ as a convex optimization problem:

$$f_{env}(x) = \sup_{w \in \mathbb{R}^n, y \in \mathbb{R}} y$$

such that

$$y^{(m)} \geq y + w^T(x^{(m)} - x), \text{ for all } m \in \{1, \dots, M\}.$$

- (a) Explain why this optimization problem is unbounded for certain choices of $x \in \mathbb{R}^n$.
- (b) To fix the unboundedness, we can add an additional constraint that $\|w\|_2^2 \leq \gamma^2$ for some given $\gamma \geq 0$. In Python, implement projected gradient descent to solve the optimization problem under this additional constraint. Your Python function should take as input the x 's, the y 's, γ , the query point x , and the number of iterations of projected gradient ascent to perform and return the best function value found during the iterative procedure starting from $w = 0$ and $y = \min_m y^{(m)}$. Hint: you can do the projection analytically if you reformulate the optimization problem to eliminate the linear constraints.
- (c) Construct a dual of the optimization problem in (b) using the method of Lagrange multipliers.
- (d) In Python, implement the Frank-Wolfe algorithm to maximize your dual in (c). Your Python function should take as input the x 's, the y 's, γ , the query point x , a feasible initial point for the Lagrange multipliers, a tolerance ϵ that terminates the Frank-Wolfe algorithm whenever the convergence criteria from class is met, and an upper bound `max_it` on the number of iterations, and returns the best function value found during the iterative procedure. Hint: you can analytically eliminate the Lagrange multiplier corresponding to the constraint $\|w\|_2^2 \leq \gamma^2$.