Problem 1 (10 points)

Explain geometrically or algebraically how magnitude of a complex number is equivalent to Pythagorean Theorem.

Problem 2 (6 points total)

Here is a list of complex numbers

$$z_1 = 6 + i\sqrt{13} \qquad z_4 = -20 + i4$$
$$z_2 = \frac{\sqrt{2}}{2} - i\frac{\sqrt{3}}{2} \qquad z_5 = -\sqrt{7} + i\sqrt{3}$$
$$z_3 = -5 + i12 \qquad z_6 = \frac{1}{4} + i\frac{\sqrt{6}}{2}$$

2a Compute the following. (3 points)

$$|z_1| = ?$$
$$|z_2| = ?$$
$$|z_3| = ?$$

2b Compute the following. (3 points)

$$z_3 + z_4 = ?$$
$$(z_2)^2 + z_6 = ?$$
$$\frac{1}{2}z_5 - z_2 = ?$$

Problem 3 (6 points)

Recall Euler's Formula for the imaginary exponential.

$$e^{i\theta} = \cos\theta + i\sin\theta$$

(Additional hints: remember your unit circle formulas and $z = re^{i\theta}$)

3a Convert to complex exponential form. (2 points)

$$z = -\frac{\sqrt{3}}{2} + i\frac{1}{2}$$

3b Convert to complex exponential form. (2 points)

$$z = 2 - 2i$$

3c Convert to Cartesian (aka rectangular) form. (2 points)

$$z = 5e^{i2\pi/3}$$

Problem 4 (8 points)

Compute the following quotients. Leave your answer in Cartesian form.

4a (2 points)

$z = 1 - 3i, \quad w = 1 + 4i, \quad \frac{z}{w} = \quad ?$

4b (2 points)

$z = 1 + 3i, \quad w = 1 - 2i, \quad \frac{z}{w} = \quad ?$

4c (2 points)

$z = -5 + 16i, \quad w = 6 + i, \quad \frac{z}{w} = \quad ?$

4d (2 points)

$z = \frac{1}{3} - \frac{1}{3}i, \quad w = -\frac{1}{6} + \frac{1}{12}i, \quad \frac{z}{w} = \quad ?$

Problem 5 (8 points)

Here is a list of Complex Numbers.

$$z_1 = 1 - 3i, \quad z_2 = 1 + 3i, \quad z_3 = -5 + 16i, \quad z_4 = \frac{1}{3} - \frac{1}{3}i$$

Derive the following expressions in terms of inverse tangent. Then use a computer's or calculator's inverse tangent (typically called *arctan*, *atan*, or tan^{-1}) to find the angles listed below. Use radians. Provide 3 digits after the decimal point. (Rounding up and down are both fine.) Measure angles counterclockwise and give a number between 0 and 2π.

5a (2 points)

What is the angle from z_1 to $w = 1$?

5b (2 points)

What is the angle from z_1 to z_2?

5c (2 points)

What is the angle from z_3 to z_4?

5d (2 points)

What is the angle from z_3 to i?

Problem 6 (4 points)

A function $f(\theta)$ is periodic if after some period T, it repeats. In other words

$$f(\theta + T) = f(\theta) \quad \text{for all} \quad \theta.$$

Letting θ be a real number, is $f(\theta) = e^{i\theta}$ periodic? (2 points) If so, what is its period? (2 points)

Problem 7 (10 points)

Let $w = 10 - 5i$. Express all of the roots for each equation with variable z in terms of an integer k. (Hint: you don't have to write out all solutions explicitly. Use a variable k and say what its highest and lowest values are.)

7a (2 points)

$z^2 = w$

7b (2 points)

$z^3 = w$

7c (2 points)

$z^{10} = w$

7d (2 points)

$z^{50} = w$

7e (2 points)

$z^{6000} = w$

Problem 8 (7 points)

Direct Comparison of solution methods.

8a (3 points)

Solve the following system of linear equations by variable substitution.

$$5x + 6y = 7$$
$$-40x - 46y = -44$$

8b (3 points)

Find the reduced row echelon for of this matrix by Gaussian Elimination.

$$\left[\begin{array}{cc|c} 5 & 6 & 7 \\ -40 & -46 & -44 \end{array}\right]$$

8c (1 point)

Why do you think computers prefer Gaussian Elimination? (Hint: computers represent everything in terms of numbers located at an index.)

Problem 9 (6 points)

A system of linear equaitons.

$$\begin{array}{rrrcr} 2x & -2y & -6z & = & 12 \\ -6x & +8y & +4z & = & -24 \\ x & -2y & & = & 6 \end{array}$$

9a (2 points)

Write the above system as an augmented matrix.

9b (3 points)

Solve the system by applying GE to that matrix.

9c (1 points)

How many solutions are there?

Problem 10 (4 points)

Consider the general 2×2 linear system.

$$\begin{array}{rrl} ax & +by & = u \\ cx & +dy & = v \end{array}$$

Assume that a unique solution exists. Verify that

$$x = \frac{du - bv}{ad - bc} \qquad y = \frac{av - cu}{ad - bc}.$$

2 points for verifying x, 2 points for verifying y.

Problem 11 (9 points)

Compute the reduced row echelon form for each matrix below. State which elementary row operation you do at each step.

11a (3 points)

$$\begin{bmatrix} 2 & -1 & 3 \\ 1 & -1 & 2 \\ 3 & -3 & 7 \end{bmatrix}$$

11b (3 points)

$$\begin{bmatrix} 1 & -3 & 1 & 1 \\ 2 & -5 & 4 & -1 \\ -2 & 4 & 3 & -5 \\ 1 & 1 & 1 & -3 \end{bmatrix}$$

11c (3 points)

$$\begin{bmatrix} 0 & 0 & 0 & 1 & -1 \\ 1 & 2 & -1 & 2 & 1 \\ 2 & 4 & -2 & 6 & 0 \\ 3 & 6 & -6 & 0 & 0 \end{bmatrix}$$

Problem 12 (8 points)

Consider an augmented matrix containing a variable a.

$$\left[\begin{array}{ccc|c} 1+a & 2 & 0 & 2a \\ 0 & 4 & 0 & 4a \\ 3+3a & 6 & a & 1+6a \end{array}\right]$$

12a (2 points)

Find any row echelon for in terms of a.

12b (2 points)

What values of a give ∞ solutions?

12c (2 points)

What values of a give 0 solutions?

12d (2 points)

What values of a give 1 solution?

Problem 13 (6 points)

Let $[M|b]$ be the augmented matrix of Problem 9. Let

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

be a solution vector. Double check your answer to Problem 9 by computing the matrix multiplication Mv. (It should be be equal to b.)

Problem 14 (4 points)

Compute these matrix products.

14a (2 points)

$$\begin{bmatrix} 1 & -2 & 3 \\ 3 & -6 & 12 \end{bmatrix} \begin{bmatrix} 1/3 & 8 \\ 1/3 & 4 \\ 1/3 & 1 \end{bmatrix} = ?$$

14b (2 points)

$$\begin{bmatrix} -6 & 3 & 10 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 3 \\ -2 & -2 \end{bmatrix} = ?$$

Problem 15 (4 points)

Let

$$A = \begin{bmatrix} -6 & 2 \\ 0 & 2 \\ 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 & 7 \\ -4 & 0 & 3 \end{bmatrix}$$

15a (2 points)

Copmute $C = A^\top$.

15b (2 points)

Copmute $D = \frac{1}{2}C + \frac{1}{2}B$.