Helpful Formulas

$$\Gamma(a) = \int_0^\infty u^{a-1} \cdot e^{-u}\, du \quad \text{for} \quad a > 0. \tag{1}$$

$$\text{Also } \Gamma(1/2) = \sqrt{\pi}$$

Notice that for integer $a \geq 0$, the identity $\Gamma(a+1) = a!$ holds.
As a result, we can use

$$\int_0^\infty u^k \cdot e^{-ru}\, du = \frac{k!}{r^{k+1}} \tag{2}$$

valid for integer $k \geq 0$ and $r > 0$, because $\Gamma(k+1) = k!$

$$\int_0^1 u^{a-1} \cdot (1-u)^{b-1}\, du = \frac{\Gamma(a) \cdot \Gamma(b)}{\Gamma(a+b)} \tag{3}$$

valid for $a > 0$ and $b > 0$.

Problem 1 [15 points = 5 + 5 + 5]

Independent random variables $\mathbf{X} = \{X_k : \ 1 \le k \le 8\}$ share the common density

$$f(x) = 6 \cdot \exp(-6x) \quad \text{for} \quad x > 0 \quad \text{and} \ \ f(x) = 0, \ \ \text{elsewhere.}$$

Consider two transformed variables,

$$U = X_1 + X_2 + X_3, \ \ W = \sum_{k=1}^{8} X_k \quad \text{and} \quad Q = \frac{W - U}{W} = 1 - \frac{U}{W}$$

1. Determine density function for Q and specify where it is positive.
2. Find expected value and variance of Q
3. Find expectation

$$\mathbf{E}\left[\frac{1-Q}{Q}\right]$$

Solution

Problem 2 [15 points = 5 + 4 + 6]

A continuous random variable Q has density defined for $0 < q < 1$ as

$$f^Q(q) = 6 \cdot q^5$$

and $f^Q(q) = 0$, elsewhere.
You are given that $X|Q = q$ is uniformly distributed over the interval $(0 < x < q)$.

1. Evaluate **marginal** density function for X and specify where it is positive
2. Find expected value of X
3. Derive the **conditional** density function for $(Q|X = x)$ and specify where it is positive.

Solution

Problem 3 [15 points = 4 + 6 + 5]

Consider independent random variables $\mathbf{Z} = \{Z_k : \ k \geq 1\}$ sharing the same standard normal distribution, $\mathbf{N}[0,1]$, with density defined for $-\infty < z < \infty$ as

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$$

An integer-valued random variable M is independent of all Z_k and

$$\mathbf{P}[M \geq m] = \left(\frac{3}{4}\right)^m \quad \text{for} \quad m \geq 0$$

Consider a random sum,

$$S = \sum_{k=1}^{3N+1} (Z_k)^2$$

1. Evaluate expectation of S
2. Derive variance of S
3. Find the second moment of S, that is: $\mathbf{E}[S^2]$

Hint If Z has standard normal distribution, then $Y = Z^2$ is $\chi^2[1]$-distributed with one degree of freedom, with density defined for $y > 0$ as follows:

$$f^Y(y) = \frac{1}{\sqrt{2} \cdot \Gamma(0.5)} \cdot y^{-0.5} \cdot \exp\left(-\frac{y}{2}\right)$$

Solution

Problem 4 [15 points = 4 + 4 + 7]

Consider two **continuous** random variables, (Y, W) such that

- Y is Gamma-distributed with mean and variance

$$\mathbf{E}[Y] = \mathbf{2} \quad \text{and} \quad \mathbf{Var}[Y] = \mathbf{1}$$

- $(W|Y = y)$ is uniformly distributed over the interval $(0 < w < y)$

1. Identify density of Y
2. Evaluate expectation of W
3. Derive variance of W

Solution

Problem 5 [20 points = 5 + 5 + 5 + 5]

A continuous random variable Q has cumulative distribution function

$$F^Q(q) = 4q^3 - 3q^4 \quad \text{for} \quad 0 \le q \le 1$$

and $F^Q(q) = 0$ for $q < 0$, while $F^Q(q) = 1$ for $q > 1$.
Given $Q = q$, a binary random variable N has probability mass function

$$\mathbf{P}[N = 1 | Q = q] = q \quad \text{and} \quad \mathbf{P}[N = 0 | Q = q] = 1 - q.$$

1. Find density function for Q and identify it.
2. Derive **marginal** distribution of N
3. Determine expectation and variance of N
4. Find the second moment, $\mathbf{E}[N^2]$

Solution

Problem 6 [20 points = 10 + 10]

Proceed with the assumptions from Problem 5. A continuous random variable Q has cumulative distribution function

$$F^Q(q) = 4q^3 - 3q^4 \quad \text{for} \quad (0 \leq q \leq 1)$$

and $F^Q(q) = 0$ for $q < 0$, while $F^Q(q) = 1$ for $q > 1$.
Given $Q = q$, an integer-valued random variable N has probability mass function:

$$\mathbf{P}\left[N = n | Q = q\right] = \frac{2!}{n! \cdot (2-n)!} \cdot q^n \cdot (1-q)^{2-n} \quad \text{for} \quad (0 \leq n \leq 2)$$

1. Evaluate **conditional** density of $(Q|N = n)$ for all values of N.
   $f^{Q|N}(q|0) =$

   $f^{Q|N}(q|1) =$

   $f^{Q|N}(q|2) =$

2. Determine **conditional** expectation of Q, given $N = n$ for values $n = 0$, $n = 1$, and $n = 2$.
   $\mathbf{E}\left[Q|N = 0\right] =$

   $\mathbf{E}\left[Q|N = 1\right] =$

   $\mathbf{E}\left[Q|N = 2\right] =$

Solution