1. A sphere of radius $r > 0$ is immersed in a fluid of density $\rho > 0$ and viscosity $\mu > 0$. When subject to the force q, the sphere attains a terminal velocity $v > 0$. We suppose there is a unit-free equation

$$v = f(r, g, \rho, \mu),$$

where $[q] = MLT^{-2}$, $[\rho] = ML^{-3}$, $[\mu] = ML^{-1}T^{-1}$.

   (a) Find a reduced form of $v = f(r, q, \rho, \mu)$.

   (b) Fix q, ρ, μ. Let v_* be the value of the terminal velocity of a sphere of radius $r = 1$. Write an expression for the value of the terminal velocity of a sphere of radius $r = 3$ in terms of v_*.

2. Consider the system:

$$\begin{cases} \frac{dx}{dt} = \frac{y}{2} + x - \left(x^2 + \frac{y^2}{2}\right) x \\ \frac{dy}{dt} = -x + y - \left(x^2 + \frac{y^2}{2}\right) y \end{cases}$$

   (a) Show that the origin is an unstable equilibrium.

   (b) Show there exists a periodic orbit.

   HINT: You can use an appropriate Lyapunov functional.

3. The displacement x at time t of a non-linear spring-mass system is modeled by the following problem, where m, σ, $h > 0$ and $c, x_0 \geq 0$ are parameters, and $[m] = M$, $[t] = T$ and $[x] = L$.

$$\begin{cases} m\frac{d^2x}{dt^2} = -c\frac{dx}{dt} - \sigma x^5 \quad \text{for } t > 0 \\ m\frac{dx}{dt}\big|_{t=0} = h, \\ x|_{t=0} = x_0. \end{cases}$$

   (a) Find a set of associated scales for t, x involving only m, σ, $h > 0$.

   (b) Find the scaled problem. How many independent dimensionless parameters does it contain?

(c) Which dimensionless parameter must be very small to obtain the approximate equation

$$\frac{d^2\overline{x}}{d\overline{t}^2} \approx -\gamma\overline{x}^5,$$

for some $\gamma > 0$?

4. A bead of mass $m > 0$, subject to gravitational acceleration $g > 0$, slides along a wire hoop of radius $r > 0$, which is rotating at angular velocity $\omega \geq 0$. If the sliding occurs with a high friction coefficient $\eta > 0$, then a model for the bead angle $\phi \in (-\pi, \pi]$ at time $t \geq 0$ is as follows:

$$\begin{cases} \eta\frac{d\phi}{dt} = -mg\sin\phi + mr\omega^2\sin\phi\cos\phi \\ \phi|_{t=0} = \phi_0 \end{cases}$$

(a) Let $\tau = t/a$ and find a scale a so that the scaled equation becomes

$$\frac{d\phi}{d\tau} = -\sin\phi + h\sin\phi\cos\phi.$$

Identify the parameter h.

(b) Construct a bifurcation diagram for the re-scaled model. Considering only $-\pi < \phi \leq \pi$ and $h \geq 0$, show that there are either two or four equilibria for any h.

(c) Forgiven m, r, g, η, show there is a critical value of the rotation rate ω, below which the only stable state is trivial, and above which the only stable states are suspended.