Answer **ALL** *the two questions.*

Question 1 [20 marks]

A person **Q** is involved in a game of betting on an event **E**, where

$$\mathbf{P}(\mathbf{E}) = \mathbf{p} \quad \text{and} \quad \mathbf{P}(\text{not } \mathbf{E}) = 1 - \mathbf{p}.$$

Here

$$0 < \mathbf{p} < 1$$

is a given (fixed) number. Denote

$$\text{not } \mathbf{E} \quad \text{by} \quad \neg\mathbf{E}.$$

The payoff of the game (per round) is

$$1 : 1.$$

[That is, in each round of the game, for every 1 unit **Q** wagers on **E**, if **E** occurs, **Q** collects 1 unit as the payoff, and keeps the 1 unit wagered; and if $\neg$**E** comes out, **Q** loses the 1 unit wagered.]

Q's thought is to play safe after each lost, and to be aggressive after each win (to a certain degree). With this in mind, **Q** applies the following betting strategy (described in $*_1 - *_7$).

$*_1$ **Q** plays exactly 10 "rounds of the game" (or simply, "*round*").

$*_2$ **Q** always bets on **E** in each and every round.

$*_3$ **Q** starts the first round with 1 unit (betting on **E**).

$*_4$ After each and every lost, say on the n-th round, **Q** bets 1 unit on the next round [that is, on the $(n+1)$-th round, provided that $n + 1 \leq 10$].

$*_5$ In case **Q** encounters a win, say, in the m-th round (where $1 \leq m \leq 9$), **Q** bets 2 units in the $(m+1)$-th round. In addition, if **Q** also wins the $(m+1)$-th round, and $m + 1 \leq 9$, **Q** goes on to bet 4 units in the $(m+2)$-th round. { In case **Q** also wins the $(m+2)$-th round [together with the m-th round and $m+1)$-th round], then we say that **Q** has a *triple consecutive wins* .}

$*_6$ After a *triple consecutive wins*, **Q** returns to bet 1 unit on the following round, provided the number of the following round does not exceed 10.

$*_7$ The rules in $*_1 - *_6$ repeat themselves until the last round (that is, the 10-th round).

For example:

The outcome	:	**E**	**E**	**E**	**E**	**E**	$\neg$**E**	$\neg$**E**	**E**	**E**	$\neg$**E**	
The amount wagered	:	**1**	**2**	**4**	**1**	**2**	**4**	**1**	**1**	**2**	**4**	(**ended**)

– *Q.* 1 *continues on the next page* –

Q. 1 *continues....*

In the following, " *strategy* " always refers to $*_1 - *_7$ of the above.

(i) What is the maximum (combined) gain for $\mathbf{Q}$ by applying the strategy? Justify your answer.

(ii) What is the maximum (combined) lost for $\mathbf{Q}$ by applying the strategy? Justify your answer.

(iii) By applying the strategy, what is the probability of having at **least** one *triple consecutive wins*? You answer should be in terms of $\mathbf{p}$, and in its simplest form. Justify your answer.

(iv) Compute the expectation of $\mathbf{Q}$ by applying this strategy. You answer should be in terms of $\mathbf{p}$, and in its simplest form. Justify your answer.

– *Q.* 2 *starts on the next page* –

Question 2 [20 marks]

Consider the number $C(n\,,\,r)$ given by

$$C(n\,,\,r) \;=\; \frac{n\,!}{(n\,-\,r\,)\,!\cdot r\,!}\quad, \tag{2.0}$$

where n and r are integers satisfying $n \geq 1$ and $n \geq r \geq 0$ ($0! = 1$). We have the following identity (the Pascal Triangle relation).

$$C(n\,+\,1\,,\,r\,) \;=\; C(n\,,\,r\,) \,+\, C(n\,,\,r\,-\,1\,) \quad \text{for} \quad n \geq r \geq 1\,. \tag{2.1}$$

You are NOT required to prove (2.1) . See for example a portion of the Pascal Triangle Diagram in the PDF file found in the file "**e**–Test" [LumiNUS (GET1018)] with the file name Pascal-Triangle.pdf. By ignoring the number "1" on top , we start with

"1 1",

and called it the first row . Likewise , the second row contains the numbers

"1 2 1",

and so on .

Given an integer $m \geq 4$, consider the number

$$C\,(\,2m\,,\;m\,)\,,$$

which can be found in the $(\,2\,m\,)$ - th row of the Pascal Triangle Diagram. For example , take $m = 7$, then

$$C\,(\,2m\,,\;m\,) \;=\; C\,(\,14\,,\;7\,) \;=\; 3,432$$

($\uparrow$ found in the 14 - th row of the Pascal Triangle Diagram) .

Via the Pascal Triangle relation (2.1) , we have

$$C\,(\,2m\,,\;m\,) \;=\; C\,(\,2m\,-\,1\,,\;m\,) \,+\, C\,(\,2m\,-\,1\,,\;m\,-\,1\,)\,. \tag{2.2}$$

You are NOT required to prove (2.2) . The numbers

$$C\,(\,2m\,-\,1\,,\;m\,) \quad \text{and} \quad C\,(\,2m\,-\,1\,,\;m\,-\,1\,)$$

can be found in the $(\,2\,m\,-\,1\,)$ - th row of the Pascal Triangle Diagram. Likewise , we apply the Pascal Triangle relation to the numbers $C\,(\,2m\,-\,1\,,\;m\,)$ and $C\,(\,2m\,-\,1\,,\;m\,-\,1\,)$, obtaining

– *Q. 2 continues on the next page* –

Q. 2 continues....

$$C\,(\,2m\,-\,1\,,\;m\,)\;=\;C\,(\,2m\,-\,2\,,\;m\,)\;+\;C\,(\,2m\,-\,2\,,\;m\,-\,1\,) \tag{2.3}$$

and

$$C\,(\,2m\,-\,1\,,\;m\,-\,1\,)\;=\;C\,(\,2m\,-\,2\,,\;m\,-\,1\,)\;+\;C\,(\,2m\,-\,2\,,\;m\,-\,2\,)\,. \tag{2.4}$$

[Again, you are NOT required to prove (2.3) and (2.4).] The numbers on the right hand sides of (2.3) and (2.4) can be found in the $(\,2m\,-\,2\,)$-th row of the Pascal Triangle Diagram.

We continues to move upward of the Pascal Triangle by using the Pascal Triangle relation (2.1), as in (2.2), (2.3) and (2.4), from $(\,2m\,-\,2\,)$-th row to $(\,2m\,-\,3\,)$-th row, and so on, until we arrive at the m-th row of the Pascal Triangle Diagram (recall that $m\;\geq\;4$). It follows that

$$C\,(\,2m\,,\;m\,)$$

$$=\;a_o\cdot C\,(\,m\,,\;0\,)\;+\;a_1\cdot C\,(\,m\,,\;1\,)\;+\cdots\;+\;a_r\cdot C\,(\,m\,,\;r\,)\;+\;\cdots\;+\;a_m\cdot C\,(\,m\,,\;m\,)\,.$$

Here $a_o\,,\;a_1\,,\;\cdots,\;a_r\,,\;\cdots,\;a_m$ are coefficients to be determined.

(i) By using the procedure described in the above [(2.2), (2.3) and (2.4), and so on], determine all the coefficients $a_o\,,\;a_1\,,\;\cdots,\;a_r\,,\;\cdots,\;a_m$ in terms of

$$C\,(\,m\,,\;0\,),\;\;C\,(\,m\,,\;1\,),\cdots,\;C\,(\,m\,,\;r\,),\cdots,\;C\,(\,m\,,\;m\,)\,.$$

Your answers should be in their simplest forms. Justify your answer.

Note that if you use another method [not from (2.2), (2.3) or (2.4), and so on], you may not be awarded full credit to this part.

(ii) Using the result in **(i)**, find

$$[\,C\,(\,m\,,\;0\,)\,]^2\;+\;[\,C\,(\,m\,,\;1\,)\,]^2\;+\;\cdots\;+\;[\,C\,(\,m\,,\;r\,)\,]^2\;+\;\cdots\;+\;[\,C\,(\,m\,,\;m\,)\,]^2\,.$$

Your answer should be of the form

$$C\,(\,X\,,\;Y\,)\qquad [\text{ refer to } (2.0)\,],$$

where X and Y are integers to be determined by you (expressed in their simplest forms). Justify your answer.

Note that if you use another method [not directly from **(i)**], you may not be awarded full credit to this part.

– **END OF THE e - Test** –