Problem 1. 20%

Let $G[X, Y]$ be a simple bipartite graph, where $|X| = r$ and $|Y| = s$.

a) Show that $m \leq rs$.
b) Deduce that $m \leq n^2/4$.
c) Describe the simple bipartite graphs G for which equality holds in (b).

Problem 2. 20%

A *k-partite graph* is one whose vertex set can be partitioned into k subsets, or *parts*, in such a way that no edge has both ends in the same part. (Equivalently, one may think of the vertices as being colourable by k colours so that no edge joins two vertices of the same colour.) Let G be a simple k-partite graph with parts of sizes $a_1, a_2, \ldots, a_k$. Show that $m \leq \frac{1}{2} \sum_{i=1}^{k} a_i(n - a_i)$.

Problem 3. 10%

Show that if G is simple and $\delta \geq \nu/2$, then $\kappa' = \delta$.

Problem 4 20%

Prove the following facts about eigenvalues of graphs.

(1)

The sum of all eigenvalues of a graph is always 0.

(2)

If G has at least one edge, then it has a negative eigenvalue.

Problem 5. 30%

(1)

Show that every tree with exactly two vertices of degree one is a path.

(2)

Show that a simple connected graph that has exactly two vertices which are not cut vertices is a path.

(3)

Show that if G has no even cycles, then each block of G is either K_1 or K_2, or an odd cycle.