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#1 Evaluate $\int \frac{dx}{(x^2-6x+11)^2}$

$= \int \frac{1}{[(x-3)^2+2]^2} dx$

make $u = x-3 \Rightarrow du = dx$

$= \int \frac{1}{(u^2+2)^2} du$

Apply integral substitution $u = \sqrt{2}\, v \quad du = \sqrt{2}\, dv$

$= \int \frac{\sqrt{2}\, dv}{(2v^2+2)^2}$

$= \sqrt{2} \int \frac{1}{4(v^2+1)^2} dv$

Apply trig substitution $v = \tan w$

$= \frac{\sqrt{2}}{4} \int \frac{1}{\sec^2(w) \cdot \cos^2(w)} dw$

$= \frac{\sqrt{2}}{4} \int \frac{1}{\left(\frac{1}{\cos^2 w}\right) \cos^2 w}$

$= \frac{\sqrt{2}}{4} \cos^2(w)\, dw$

$= \frac{\sqrt{2}}{4} \cdot \frac{1}{2} \int 1 + \cos 2w \, dw$

$= \frac{\sqrt{2}}{4} \cdot \frac{1}{2} \left(w + \frac{1}{2} \sin(2w)\right)$

$= \frac{1}{4\sqrt{2}} \tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right) + \frac{(x-3)}{4(x^2-6x+11)} + C$

#3 Find $\lim_{x \to 0^+} (1+\sin 4x)^{\cot x}$

$a^x = e^{\ln(a^x)} = e^{x \ln(a)}$

so $\lim_{x \to 0^+} \left(e^{\cot(x)\ln(1+\sin(4x))}\right)$

Apply limit chain rule

$= e^4$

#4 Solve $\frac{du}{dt} = \frac{2t + \sec^2 t}{2u}$, $u(0) = -5$

$2u\, du = (2t + \sec^2 t)\, dt$

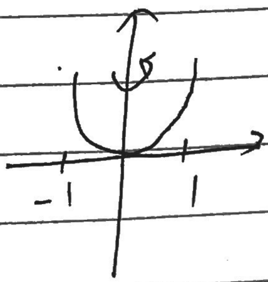
$\int 2u\, du = \int (2t + \sec^2 t)\, dt$

$2\frac{u^2}{2} + u(0) = \frac{2t^2}{2} + \tan(t)$

$u^2 - 5 = t + \tan t$

$u^2 - t - \tan t = 5$

#5 Find the area of the surface obtained by rotating the curve $y = x^2$, $0 \le x \le 1$, about the y-axis



$$y = x^2$$

$$\frac{dy}{du} = 2x \qquad \left(\frac{dy}{du}\right)^2 = 4x^2$$

$$S.A = \int 2\pi x \, dx$$

$$= \int_0^1 2\pi x \sqrt{1 + \left(\frac{dy}{du}\right)^2} \, dx$$

$$= 2\pi \int_0^1 x \sqrt{1 + 4x^2} \, dx$$

$\cancel{= 2\pi}$ make $u = 1 + 4x^2$

$$du = 8x \, dx \Rightarrow \frac{du}{8} = x \, dx$$

$$= \frac{2\pi}{8} \int_1^5 \sqrt{u} \, du$$

$$= \frac{2\pi}{8} \left[\frac{2}{3} u^{3/2} \right] \Big|_1^5$$

$$= \frac{2\pi}{8} \left[\frac{10\sqrt{5}}{3} - \frac{2}{3} \right]$$

$$= \frac{1}{4}\pi \left[\frac{10\sqrt{5}}{3} - \frac{2}{3} \right]$$

#6 Find the arc length fun for the curve $y = x^2 - \frac{\ln x}{8}$ taking $P(1,1)$ as the starting point.

Arc length Fuction:

$$y = x^2 - \frac{\ln x}{8} \quad P(1,1)$$

$$\frac{dy}{dx} = 2x - \frac{1}{8x}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2x - \frac{1}{8x}\right)^2$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\left(2x-\frac{1}{8x}\right)^2}$$

Arc length : $\int_{x_0}^{x} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$

$$= \int_1^x \sqrt{1+\left(2x-\frac{1}{8x}\right)^2}\, dx$$

$$= \int_1^x \frac{16x^2+1}{8x}\, dx$$

$$= \frac{1}{8}\cdot 16\left(\frac{x^2}{2} - \frac{1}{2}\right) + \ln|x|$$

$$= x^2 - 1 + \ln|x|$$

#7 Find $\int_0^2 \frac{dx}{(x-1)^{2/3}}$ (Hint: use improper integral)

$$= \int_0^1 \frac{1}{(x-1)^{2/3}}\,dx + \int_1^2 \frac{1}{(x-1)^{2/3}}\,dx$$

$$\Rightarrow \int \frac{1}{(x-1)^{2/3}}\,dx \qquad \text{make } u = x-1 \quad du = dx$$

$$= \int \frac{1}{u^{2/3}}\,du = \int u^{-2/3}\,du = \frac{u^{-2/3+1}}{-\frac{2}{3}+1}$$

$$\Rightarrow = \left[\frac{u^{-2/3+1}}{-2/3+1}\right]_{-1}^{0} + \left[\frac{u^{-2/3+1}}{-2/3+1}\right]_{0}^{1}$$

$$= (3u^{1/3})\Big]_{-1}^{0} + \left[3u^{1/3}\right]_{0}^{1}$$

$$= 3 + 3 = 6$$

#8 Test for convergence: $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2+4}}$

$a_n = \frac{1}{\sqrt{n^2+4}}$

$a_{n+1} = \frac{1}{\sqrt{(n+1)^2+4}}$

Ratio test complete

$\lim_{n\to\infty} \left[\frac{a_{n+1}}{a_n} \right]$

$= \lim_{n\to\infty} \frac{\sqrt{n^2+4}}{\sqrt{(n+1)^2+4}} = 1 \to$ inconclusive

$\lim_{n\to\infty} \sqrt[n]{a_n}$

$= \lim_{n\to\infty} \left[\frac{1}{\sqrt{n^2+4}} \right]^{1/n} = 1 \to$ inconclusive

comparision test: $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2}} = \sum_{n=3}^{\infty} \frac{1}{n}$

since it diverges, our series also diverges by deduction from comparsion test.

#9 Find the Maclaurin Series for $f(x) = x^2 e^x$

Maclaurin Series $f(x) = x^2 e^x$

Maclaurin series of e^t is:

$$\sum_{n=1}^{\infty} \frac{1}{n!} t^n$$

So, multiplying by x^2 and subsitituting $t = x$

$$x^2 e^x = x^2 \cdot \sum_{n=0}^{\infty} \frac{1}{n!} (x)^n$$

$$= \sum_{n=0}^{\infty} \frac{(1)^n}{n!} \cdot x^{n+2}$$

#10 Find the radius and interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(-1)^2}{4^n \cdot n} (x-5)^n$$

General from power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{4^{n+1} \cdot n+1} \cdot (x-5)^{n+1} \cdot \frac{4^n \cdot n}{(-1)^n \cdot (x-5)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(-1)(x-5) \cdot n}{4 \cdot (n+1)} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(5-x)}{4} \frac{n}{(1+\frac{1}{n})n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{5-x}{4} \cdot \frac{1}{1+\frac{1}{n}} \right|$$

$$= \left| \frac{5-x}{4} \right|$$

to convege : $\left| \frac{5-x}{4} \right| < 1 \Rightarrow 9 > x > -9$

$\boxed{-9 < x < 9}$ Interval is $[-9, 9]$ as converges on both point.

#2 use integration by parts and table #67 to evaluate $\int x \sin^2 x \cos x \, dx$

$\int x \sin^2 x \cos x \, dx$

$= \int x \cdot (1-\cos^2 x) \cos x \, dx$

$= \int x \cos x \, dx - \int x \cos^3 x \, dx$

By integration by parts:

$u = x \quad v' = \cos(x) \, dx$

$\int x \cos x \, dx = x \sin x - \int \sin(x) dx$

$= x \sin x + \cos x$

similarly:

$\int x \cos^3 x \, dx$

$u = \cancel{x} \, x, \quad v' = \cos^3 x \, dx$

$\int x \cos^3 x \, dx = x \left(\sin(x) - \frac{1}{3} \sin^3(x) \right) - \int \sin(x) - \frac{1}{3} \sin^3(x) \, dx$

$= x \left(\sin(x) - \frac{1}{3} \sin^3(x) \right) - \left(-\frac{2}{3} \cos(x) - \frac{1}{9} \cos^3(x) \right)$

$= x \sin(x) - \frac{x}{3} \sin^3(x) + \frac{2}{3} \cos(x) + \frac{1}{9} \cancel{x} \cos^3(x)$

Final answers:

$= x \sin x + \cos x - \left[x \sin x - \frac{x}{3} \sin^3 x + \frac{2}{3} \cos(x) + \frac{1}{9} \cos^3 x \right]$

$= \frac{x}{3} \sin^3 x + \frac{1}{3} \cos x - \frac{1}{9} \cos^3 x$