Final

1. $\int \frac{1}{(x^2-6x+11)^2}\,dx$

$= \int \frac{1}{((x-3)^2+2)^2}\,dx$ $\qquad$ let $u=x-3$ $\quad du=dx$

$= \int \frac{1}{(u^2+2)^2}\,du$ $\qquad$ $u=\sqrt{2}\tan\theta$ $\quad du=\sqrt{2}\sec^2\theta$

$= \int \frac{\sqrt{2}\sec^2\theta}{(2\tan^2\theta+2)^2}\,d\theta$

$= \frac{\sqrt{2}}{4}\int \cos^2\theta\,d\theta$

$= \frac{\sqrt{2}}{4}\int \frac{1}{2}\cos(2\theta)+\frac{1}{2}\,d\theta$

$= \frac{\sqrt{2}}{8}\left(\frac{1}{2}\sin(2\theta)+\frac{\theta}{2}\right)+C$

$= \frac{\sqrt{2}}{8}\left(\sin\theta\cos\theta+\frac{\theta}{2}\right)+C$

$= \frac{\sqrt{2}}{8}\left(\left(\frac{u/\sqrt{2}}{\sqrt{1+u^2/2}}\right)\left(\frac{1}{\sqrt{1+u^2/2}}\right)+\frac{\arctan(u/\sqrt{2})}{2}\right)+C$

$= \frac{\sqrt{2}}{8}\left(\frac{(x-3)/\sqrt{2}}{1+(x-3)^2/2}+\frac{\arctan((x-3)/\sqrt{2})}{2}\right)+C$

$= \frac{x-3}{8+4(x-3)^2}+\frac{\arctan((x-3)/\sqrt{2})}{2^{5/2}}+C$

2. $\int x \sin^2 x \cos x \, dx$

$f = x \quad g' = \cos x \sin^2 x \quad \int f g' = fg - \int f' g$

$f' = 1 \quad g = \frac{\sin^3 x}{3}$

$= \frac{x \sin^3 x}{3} - \int \frac{\sin^3 x}{3} dx$

$= \frac{x \sin^3 x}{3} - \frac{1}{3} \int \sin^3 x \, dx$ by table 67 $\int \sin^3 u = -\frac{1}{3}(2 + \sin^2 u) \cos u + C$

$= \frac{x \sin^3 x}{3} - \frac{1}{9}(2 + \sin^2 x) \cos x + C$

3. $\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x}$

$a^x = e^{\ln(a^x)} = e^{x \cdot \ln a}$

$(1 + \sin 4x)^{\cot x} = e^{\cot x \ln(1 + \sin 4x)}$

$= \lim_{x \to 0^+} e^{\cot x \ln(1 + \sin 4x)}$

$g(x) = \cot x \ln(1 + \sin 4x), \ f(u) e^u$

$= \lim_{x \to 0^+} \cot x \ln(1 + \sin 4x)$

$\underline{LH} \lim_{x \to 0^+} \left(\frac{\frac{4 \cos 4x}{1 + \sin 4x}}{\sec^2 x} \right) = \lim_{x \to 0^+} \frac{4 \cos 4x}{\sec^2 x (1 + \sin 4x)} = \frac{4 \cos(4 \cdot 0)}{\sec^2(0)(1 + \sin 4 \cdot 0)}$

$= 4$

$\lim_{u \to 4^+} e^u = e^4$

4. $\frac{du}{dt} = \frac{2t+\sec^2 t}{2u}$ $u(0) = -5$

$2u\,du = (2t+\sec^2 t)\,dt$

$\int 2u\,du = \int (2t+\sec^2 t)\,dt$

$u^2 = t^2 + \tan t + C$

when $t=0$

$u^2 = (-5)^2 = 25 = 0+0+C \Rightarrow C = 25$

$\therefore u^2 = t^2 + \tan t + 25$

$u = -\sqrt{t^2+\tan t+25}$ $(-5<0)$

5. $y=x^2$ $0 \le x \le 1$

$S = \int_0^1 2\pi x \sqrt{1+(\frac{dy}{dx})^2}\,dx$ $\frac{dy}{dx} = 2x$

$= \int_0^1 2\pi x \sqrt{1+4x^2}\,dx$ let $u = 1+4x^2$ $du = 8x\,dx$

$= 2\pi \cdot \frac{1}{8} \int_1^5 \sqrt{u}\,du$ when $x=0$, $u=1$ | $x=1$, $u=5$

$= \frac{\pi}{4} \cdot [\frac{2u^{\frac{3}{2}}}{3}]_1^5$

$= \frac{\pi \cdot 2}{4 \cdot 3}(5^{\frac{3}{2}}-1)$

$= \frac{\pi}{6}(5\sqrt{5}-1)$

Arc length function

6. $y = x^2 - \frac{\ln x}{8}$ taking $P(1,1)$ as the starting point

$\frac{dy}{dx} = 2x - \frac{1}{8x}$

Arclength $= \int_{x_0}^{x} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dt$

$= \int_1^x \sqrt{1+(2t-\frac{1}{8t})^2}\, dt$

$= \int_1^x \sqrt{1+4t^2+\frac{1}{64t^2}-\frac{1}{2}}$

$= \int_1^x \sqrt{\frac{1}{2}+4t^2+\frac{1}{64t^2}}\, dt$

$= \int_1^x \sqrt{(2t+\frac{1}{8t})^2}\, dt$

$= \int_1^x 2t + \frac{1}{8t}\, dt$

$= t^2 + \frac{\ln t}{8}\Big|_1^x$

$= x^2 + \frac{\ln x}{8} - 1 - \frac{\ln 1}{8} = x^2 + \frac{\ln x}{8} - 1$

7. $\int_0^2 \frac{dx}{(x-1)^{2/3}}$

$= \int_0^1 \frac{1}{(x-1)^{2/3}}\, dx + \int_1^2 \frac{1}{(x-1)^{2/3}}\, dx$

$= \lim_{a\to 1^-} \int_0^a \frac{1}{(x-1)^{2/3}}\, dx + \lim_{b\to 1^+} \int_b^2 \frac{1}{(x-1)^{2/3}}\, dx$

$= \lim_{a\to 1^-} 3(x-1)^{1/3}\Big|_0^a + \lim_{b\to 1^+} 3(x-1)^{1/3}\Big|_b^2$

$= 3+3 = 6$

8. $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2+4}}$

$a_n = \frac{1}{\sqrt{n^2+4}}$ $b_n = \frac{1}{n}$ $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n^2+4}}}{\frac{1}{n}}$

$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+4}} = 1 > 0$

Since $\sum_{n=3}^{\infty} b_n$ diverges,

$\sum_{n=3}^{\infty} \frac{1}{\sqrt{n^2+4}}$ diverges.

9. Mclaurin Series for $f(x) = x^2 e^x$

The Mclaurin Series for e^x is

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$f(x) = x^2 e^x \rightarrow$ Mclaurin Series

$$\sum_{n=0}^{\infty} \frac{1}{n!} x^{n+2} = x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \cdots$$

10. $a_n = \frac{(-1)^n}{4^n \cdot n} (x-5)^n$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} (-1) \frac{(x-5)^{n+1} / 4^{n+1} (n+1)}{(x-5)^n / 4^n \cdot n}$$

$$= \lim_{n\to\infty} \frac{(x-5) n}{4n+4} = (x-5) \lim_{n\to\infty} \frac{n}{4n+4} = \frac{1}{4}(x-5) < 1$$

$|x-5| < 4 \rightarrow$ the radius of convergence is 4

$|x-5| < 4 \Rightarrow -4 < x-5 < 4 \Rightarrow 1 < x < 9$

① $x=1$, $\sum a_n = \sum \frac{4^n}{4^n \cdot n} = \sum \frac{1}{n}$ diverges $x \neq 1$

② $x=9$, $\sum a_n = \sum \frac{(-1)^n}{n}$ converges by Alternating series,
since $\lim_{n\to\infty} \frac{1}{n} = 0$ and $\frac{1}{n}$ decreases

So the interval of convergence is $(1, 9]$.