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Not all Infinite Numbers are the Same

The concept of infinite numbers emerged over a long period. Aristotle observed that although it is possible to subdivide the universe in infinite ways, the concept of infinity was unthinkable (Wildberger 1). The Greek mathematicians conceptualized infinity in terms of counting infinities, thus limiting their scope on the understanding of the entire concept. In 1600, Galileo observed that there could be an infinite~~ly~~ number of small gaps. However, it would not be possible to express whether one was smaller or larger ~~to~~ than another. In 1655, John Wallis introduced the symbol ∞ representing infinity (Wildberger 2 ?). In 1874, George Cantor developed a deeper conceptualization of infinity. According to Cantor, it would be possible to add or even subtract infinities. In addition, some infinities were large than others. Cantor studied functions that had a Fourier series convergence (Wildberger 2). Cantor proved that the trigonometric series $\sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx = 0$ converges to zero at all points except where there exists a finite k^{th}derived. For the finite k, then $a_n = b_n = 0$, n = 1, 2, 3, 4, ….

This is not listed in the references, there is only one paper by Wildberger

? Cantor showed that if the Fourier series converges to 0 at all but finitely many points then the coefficients are 0.

Many ancient cultures have ideas about infinite, but most of them defined the infinite as a philosophical concept instead of mathematics concepts. The symbol ∞ did not exist until 17th century when John Wallis introduced it in 1655 (Cajori 44). John Wallis wanted to divide a region into infinitesimal strips, and each strip is $\frac{1}{\infty}$. That is why mathematicians need the infinite numbers. They need an arbitrary large number for some certain situation. Given a large number L from ℝ, there must be a larger number L+1 exists (Cajori 46). To move forward for a single step, if mathematicians take L+1 as the new large number, there also be a large number L+2 exist. In this way, no one could express the largest number. To express 'large number', the notation ∞ and definition of infinite were invented.

here use either "some situations" or "certain situations" or "some specific situations"

Say either "there must be a larger number L+1" OR "a larger number L+1 exists", but not both, as "there must ... exist"

Do the same for this

However, a new question will be elicited: whether all infinite numbers are same. It is hard to image and convince other people without using mathematical proof.

Many infinite numbers are related to limits. Expression (1) $\lim_{n\to\infty}(n)$ and expression (2) $\lim_{n\to\infty}(n)^2$ are two limits and their values are all infinite, ∞ (Calori 259). If we just compare their result, the two positive infinites, ∞, seem to be same. However, expression (2) could be rewritten as

$$\lim_{n\to\infty}(n)^2=\lim_{n\to\infty}(n\times n)=\lim_{n\to\infty}(n)\times\lim_{n\to\infty}(n)=\infty$$

Obviously, $\lim_{n\to\infty}(n)\times\lim_{n\to\infty}(n)>\lim_{n\to\infty}(n)$ when we know $\lim_{n\to\infty}(n)>1$. Thus, it is enough to say $\lim_{n\to\infty}(n)<\lim_{n\to\infty}(n)^2$ although $\lim_{n\to\infty}(n)^2=\infty$ and $\lim_{n\to\infty}(n)=\infty$.

Another way to think of this is that if $f(n)\to\infty$ and $g(n)\to\infty$, $f(n)\to\infty$ faster if $\frac{f(n)}{g(n)}\to\infty$ (so $\frac{n^2}{n}\to\infty$, $\frac{e^n}{n^k}\to\infty$ for any fixed k, etc.

So, infinite numbers may not have to be same to each other. Some are relatively larger, and some are relatively smaller, even though all of them are denoted by the same notation ∞. In other words, infinite numbers, ∞, represent a tendency of increasing and never decrease. Just like $\lim_{n\to\infty}(n)$, as n increases toward positive direction, the value of the expression will increase. Similarly, in $\lim_{n\to\infty}(n)^2$, the $(n)^2$ will also increase as n increase toward being large. However, the difference from $\lim_{n\to\infty}(n)$ is the value of $\lim_{n\to\infty}(n)^2$ increase faster than the value of $\lim_{n\to\infty}(n)$. Although mathematicians cannot

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$x+\sin 2x\geq x-1$

so

$x+\sin 2x\to\infty$

but as $x\to\infty$

$\frac{d}{dx}(x+\sin 2x)$

$=1+2\cos 2x$

is <0 for some x

express their value by an exact number, they are not same number. The infinite numbers are numbers that is larger than a certain boundary (likes a supremum). If a number inside the boundary, we could express it in a normal way, such as $\frac{q}{r}$ for r and q are natural numbers. This isn't true, there are bounded real numbers like $\sqrt{2}$ that are not rational.

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In addition, we could consider the infinite in another way. ℕ is the set of natural numbers and ℝ is the set of real numbers. ℕ is a proper sub-set of ℝ, so the cardinal number of ℝ is larger than cardinal number of ℕ. However, if considering ℕ and ℝ respectively. Both ℕ and ℝ have infinite elements. If comparing the result, the cardinal number of ℝ should be equal to cardinal number of ℕ. But it does not. Thus, infinite, ∞, does not means a certain number, it represents some very large number. As such, not all infinite numbers are same.

The set $\{2n : n \in \mathbb{N}\}$ is a proper subset of $\mathbb{N}$ but they have the same cardinality. The ~~set~~ interval $(0,1)$ is a proper subset of $\mathbb{R}$ but they have the same cardinality – the map $x \to -\frac{1}{x} + \frac{1}{1-x}$ is a bijection from $(0,1)$ to $\mathbb{R}$

There exist different levels of infinity. We can deduce the existence of these levels by making comparisons of infinite sets, an idea developed by Georg Cantor (Wildberger 3). The actual method involves comparing sets of integers and natural numbers. It is possible to develop a method of comparing a specific integer to a natural number by making use of the negative and positive integers. The following table (Table 1) shows the one-to-one comparisons of the set of integers and natural numbers.

Table 1.

Integers	0	1	-1	2	-2	3	-3	4
Natural	1	2	3	4	5	6	7	8

The natural numbers represent subsets of integers. Since this is one-to-one mapping, the two sets are equal or have the same size. As such, the set of natural numbers and

(BUT there is no bijection from $\mathbb{N}$ to $\mathbb{R}$)

integers is a countably infinite set. When we look at real numbers such as $\frac{1}{3}$, they give a recurring set of decimals when divided, that is 1.333…. This is the same with π. It is impossible to develop a correspondence like the one above between integers and natural numbers for the real numbers.

Not true. π is irrational. All repeating decimals are rational.

In a real-world analogy, if we ask an average person to describe the wealth of Bill Gates and Warren Buffett, he or she may just say billions of dollars instead of $86B and $75.6B. Why? Because whatever $86B or $75.6B, both numbers are too far from an average person, he or she just thinks there is no difference for her or him. Thus, he or she just termed $86B and $75.6B as billions of dollars, because $86B and $75.6B are higher than his or her boundary. However, $86B and $75.6B are not same. Similarly, in mathematical number system, some numbers are too large for us to describe and we denote them as infinite, but they are not same.

Not really. They don't necesserily satisfy the same axioms.

In summary, infinite numbers are same as normal numbers. The small difference is that we just denote their value by the symbol ∞. In my understanding, the symbol ∞ just means their value are above our expression ability and does not mean their value are same. As Cantor asserts, infinities are not equal; some infinities are larger than others are. While the integers and the natural numbers give a countably indefinite set, the real numbers give an uncountable indefinite set as shown by Cantor. The proof by diagonal clearly shows that infinities are not equal. Nonetheless, it is worth noting that it is not possible to express infinities in specific values.

Some can be expressed in "specific values" relative to ~~For~~ each other. For example, $\mathbb{N}$ is a specific set with a specific "cardinality", that is less than the cardinality of $\{S : S \text{ is a subset of } \mathbb{N}\}$ so those are two distinct specific values of infinity. (The second is sometimes written $2^{\mathbb{N}}$.)

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Reference

(1) Cajori, Florian. A History of Mathematical Notations. 1993. Web.

(2) Clawson, Calvin C. *The Mathematical Traveler: Exploring the Grand History of Numbers*. New York: Plenum Press, 1994. Internet resource.

(3) Wildberger, N. *Numbers, Infinities, and Infinitesimals*. School of Mathematics, University of New South Wales, October 17, 2006. Web.

(It turns out that $2^{\mathbb{N}}$ has the same cardinality as $\mathbb{R}$.) [A question is whether there is any set with cardinality greater than that of $\mathbb{N}$ and less than that of $\mathbb{R}$. This is called the continuum hypothesis. (Look it up in Wikipedia) It turns out that it cannot be proved ~~by using the st~~ or disproved by using standard set theory.

→ Sometimes the cardinality of $\mathbb{N}$ is called ~~ℵ~~ "Aleph$_0$"

$\aleph_0$ (Aleph is a Hebrew letter)