Identify each expression as a polynomial *or* not a polynomial. *For each polynomial, give the degree and identify it as a* monomial, binomial, trinomial, *or* none of these. ***See Example 4.***

37. $-5x^{11}$

38. $-4y^5$

39. $6x + 3x^4$

40. $-9y + 5y^3$

41. $-7z^5 - 2z^3 + 1$

42. $-9t^4 + 8t^3 - 7$

43. $15a^2b^3 + 12a^3b^8 - 13b^5 + 12b^6$

44. $-16x^5y^7 + 12x^3y^8 - 4xy^9 + 18x^{10}$

45. $\frac{3}{8}x^5 - \frac{1}{x^2} + 9$

46. $\frac{2}{3}t^6 + \frac{3}{t^5} + 1$

47. 5

48. 9

Add or subtract, as indicated. ***See Example 5.***

49. $(5x^2 - 4x + 7) + (-4x^2 + 3x - 5)$

50. $(3m^3 - 3m^2 + 4) + (-2m^3 - m^2 + 6)$

51. $2(12y^2 - 8y + 6) - 4(3y^2 - 4y + 2)$

52. $3(8p^2 - 5p) - 5(3p^2 - 2p + 4)$

53. $(6m^4 - 3m^2 + m) - (2m^3 + 5m^2 + 4m) + (m^2 - m)$

54. $-(8x^3 + x - 3) + (2x^3 + x^2) - (4x^2 + 3x - 1)$

Find each product. ***See Examples 6–8.***

55. $(4r - 1)(7r + 2)$

56. $(5m - 6)(3m + 4)$

57. $x^2\left(3x - \frac{2}{3}\right)\left(5x + \frac{1}{3}\right)$

58. $m^3\left(2m - \frac{1}{4}\right)\left(3m + \frac{1}{2}\right)$

59. $4x^2(3x^3 + 2x^2 - 5x + 1)$

60. $2b^3(b^2 - 4b + 3)$

61. $(2z - 1)(-z^2 + 3z - 4)$

62. $(3w + 2)(-w^2 + 4w - 3)$

63. $(m - n + k)(m + 2n - 3k)$

64. $(r - 3s + t)(2r - s + t)$

65. $(2x + 3)(2x - 3)(4x^2 - 9)$

66. $(3y - 5)(3y + 5)(9y^2 - 25)$

67. $(x + 1)(x + 1)(x - 1)(x - 1)$

68. $(t + 4)(t + 4)(t - 4)(t - 4)$

Find each product. ***See Examples 8 and 9.***

69. $(2m + 3)(2m - 3)$

70. $(8s - 3t)(8s + 3t)$

71. $(4x^2 - 5y)(4x^2 + 5y)$

72. $(2m^3 + n)(2m^3 - n)$

73. $(4m + 2n)^2$

74. $(a - 6b)^2$

75. $(5r - 3t^2)^2$

76. $(2z^4 - 3y)^2$

77. $[(2p - 3) + q]^2$

78. $[(4y - 1) + z]^2$

79. $[(3q + 5) - p][(3q + 5) + p]$

80. $[(9r - s) + 2][(9r - s) - 2]$

81. $[(3a + b) - 1]^2$

82. $[(2m + 7) - n]^2$

83. $(y + 2)^3$

84. $(z - 3)^3$

85. $(q - 2)^4$

86. $(r + 3)^4$

Perform the indicated operations. ***See Examples 5–9.***

87. $(p^3 - 4p^2 + p) - (3p^2 + 2p + 7)$

88. $(x^4 - 3x^2 + 2) - (-2x^4 + x^2 - 3)$

89. $(7m + 2n)(7m - 2n)$

90. $(3p + 5)^2$

91. $-3(4q^2 - 3q + 2) + 2(-q^2 + q - 4)$

92. $2(3r^2 + 4r + 2) - 3(-r^2 + 4r - 5)$

93. $p(4p - 6) + 2(3p - 8)$

94. $m(5m - 2) + 9(5 - m)$

95. $-y(y^2 - 4) + 6y^2(2y - 3)$

96. $-z^3(9 - z) + 4z(2 + 3z)$

Perform each division. ***See Examples 10 and 11.***

97. $\dfrac{-4x^7 - 14x^6 + 10x^4 - 14x^2}{-2x^2}$

98. $\dfrac{-8r^3s - 12r^2s^2 + 20rs^3}{-4rs}$

99. $\dfrac{4x^3 - 3x^2 + 1}{x - 2}$

100. $\dfrac{3x^3 - 2x + 5}{x - 3}$

101. $\dfrac{6m^3 + 7m^2 - 4m + 2}{3m + 2}$

102. $\dfrac{10x^3 + 11x^2 - 2x + 3}{5x + 3}$

103. $\dfrac{x^4 + 5x^2 + 5x + 27}{x^2 + 3}$

104. $\dfrac{k^4 - 4k^2 + 2k + 5}{k^2 + 1}$

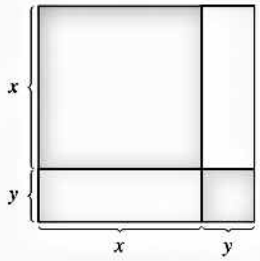
(Modeling) Solve each problem.

105. ***Geometric Modeling*** Consider the figure, which is a square divided into two squares and two rectangles.

(a) The length of each side of the largest square is $x + y$. Use the formula for the area of a square to write the area of the largest square as a power.

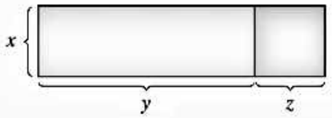
(b) Use the formulas for the area of a square and the area of a rectangle to write the area of the largest square as a trinomial that represents the sum of the areas of the four figures that make it up.

(c) Explain why the expressions in parts (a) and (b) must be equivalent.



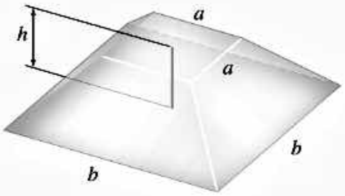
(d) What special product formula from this section does this exercise reinforce geometrically?

106. ***Geometric Modeling*** Use the figure to geometrically support the distributive property. Write a short paragraph explaining this process.



107. ***Volume of the Great Pyramid*** An amazing formula from ancient mathematics was used by the Egyptians to find the volume of the frustum of a square pyramid, as shown in the figure. Its volume is given by

$$V = \frac{1}{3}h(a^2 + ab + b^2),$$



where b is the length of the base, a is the length of the top, and h is the height. (*Source:* Freebury, H. A., *A History of Mathematics*, Macmillan Company, New York.)

(a) When the Great Pyramid in Egypt was partially completed to a height h of 200 ft, b was 756 ft, and a was 314 ft. Calculate its volume at this stage of construction to the nearest thousand feet.

EXAMPLE 4 Classifying Expressions as Polynomials

Identify each as a *polynomial* or *not a polynomial*. For each polynomial, give the degree and identify it as a *monomial, binomial, trinomial*, or *none of these*.

(a) $9x^7 - 4x^3 + 8x^2$ **(b)** $2t^4 - \frac{1}{t^2}$ **(c)** $-\frac{4}{5}x^3y^2$

SOLUTION

(a) $9x^7 - 4x^3 + 8x^2$ is a polynomial. The first term, $9x^7$, has greatest degree, so this a polynomial of degree 7. Because it has three terms, it is a trinomial.

(b) $2t^4 - \frac{1}{t^2}$ is not a polynomial because it has a variable in the denominator.

(c) $-\frac{4}{5}x^3y^2$ is a polynomial. Add the exponents $3 + 2 = 5$ to determine that it is of degree 5. Because there is one term, it is a monomial.

✔ Now Try Exercises 37, 39, and 45.

Addition and Subtraction Polynomials are added by adding coefficients of like terms. They are subtracted by subtracting coefficients of like terms.

EXAMPLE 5 Adding and Subtracting Polynomials

Add or subtract, as indicated.

(a) $(2y^4 - 3y^2 + y) + (4y^4 + 7y^2 + 6y)$

(b) $(-3m^3 - 8m^2 + 4) - (m^3 + 7m^2 - 3)$

(c) $(8m^4p^5 - 9m^3p^5) + (11m^4p^5 + 15m^3p^5)$

(d) $4(x^2 - 3x + 7) - 5(2x^2 - 8x - 4)$

SOLUTION

(a) $(2y^4 - 3y^2 + y) + (4y^4 + 7y^2 + 6y)$ y = 1y

$= (2 + 4)y^4 + (-3 + 7)y^2 + (1 + 6)y$ Add coefficients of like terms.

$= 6y^4 + 4y^2 + 7y$ Work inside the parentheses.

(b) $(-3m^3 - 8m^2 + 4) - (m^3 + 7m^2 - 3)$

$= (-3 - 1)m^3 + (-8 - 7)m^2 + [4 - (-3)]$ Subtract coefficients of like terms.

$= -4m^3 - 15m^2 + 7$ Simplify.

(c) $(8m^4p^5 - 9m^3p^5) + (11m^4p^5 + 15m^3p^5)$

$= 19m^4p^5 + 6m^3p^5$

(d) $4(x^2 - 3x + 7) - 5(2x^2 - 8x - 4)$

$= 4x^2 - 4(3x) + 4(7) - 5(2x^2) - 5(-8x) - 5(-4)$ Distributive property

$= 4x^2 - 12x + 28 - 10x^2 + 40x + 20$ Multiply.

$= -6x^2 + 28x + 48$ Add like terms.

✔ Now Try Exercises 49 and 51.

As shown in **Examples 5(a), (b), and (d),** polynomials in one variable are often written with their terms in **descending order** (or descending degree). The term of greatest degree is first, the one of next greatest degree is next, and so on.

Multiplication One way to find the product of two polynomials, such as $3x - 4$ and $2x^2 - 3x + 5$, is to distribute each term of $3x - 4$, multiplying by each term of $2x^2 - 3x + 5$.

$$\begin{aligned}
&(3x - 4)(2x^2 - 3x + 5) \\
&\quad = 3x(2x^2 - 3x + 5) - 4(2x^2 - 3x + 5) && \text{Distributive property} \\
&\quad = 3x(2x^2) + 3x(-3x) + 3x(5) - 4(2x^2) - 4(-3x) - 4(5) && \text{Distributive property again} \\
&\quad = 6x^3 - 9x^2 + 15x - 8x^2 + 12x - 20 && \text{Multiply.} \\
&\quad = 6x^3 - 17x^2 + 27x - 20 && \text{Combine like terms.}
\end{aligned}$$

Another method is to write such a product vertically, similar to the method used in arithmetic for multiplying whole numbers.

Place like terms in the same column.

$$\begin{array}{rrrrl}
 & 2x^2 & -\ 3x & +\ 5 & \\
 & & 3x & -\ 4 & \\
\hline
 & -8x^2 & +\ 12x & -\ 20 & \leftarrow -4(2x^2 - 3x + 5) \\
6x^3 & -\ 9x^2 & +\ 15x & & \leftarrow 3x(2x^2 - 3x + 5) \\
\hline
6x^3 & -\ 17x^2 & +\ 27x & -\ 20 & \text{Add in columns.}
\end{array}$$

EXAMPLE 6 Multiplying Polynomials

Multiply $(3p^2 - 4p + 1)(p^3 + 2p - 8)$.

SOLUTION

This process is an ordered method of applying the distributive property.

Write like terms in columns.

$$\begin{array}{rrrrrrl}
 & & & 3p^2 & -\ 4p & +\ 1 & \\
 & & p^3 & & +\ 2p & -\ 8 & \\
\hline
 & & & -24p^2 & +\ 32p & -\ 8 & \leftarrow -8(3p^2 - 4p + 1) \\
 & & 6p^3 & -\ 8p^2 & +\ 2p & & \leftarrow 2p(3p^2 - 4p + 1) \\
3p^5 & -\ 4p^4 & +\ p^3 & & & & \leftarrow p^3(3p^2 - 4p + 1) \\
\hline
3p^5 & -\ 4p^4 & +\ 7p^3 & -\ 32p^2 & +\ 34p & -\ 8 & \text{Add in columns.}
\end{array}$$

✔ **Now Try Exercise 63.**

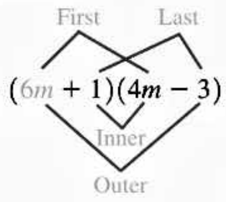
The **FOIL method** is a convenient way to find the product of two binomials. The memory aid **FOIL** (for **F**irst, **O**uter, **I**nner, **L**ast) gives the pairs of terms to be multiplied when distributing each term of the first binomial, multiplying by each term of the second binomial.

EXAMPLE 7 Using the FOIL Method to Multiply Two Binomials

Find each product.

(a) $(6m + 1)(4m - 3)$ **(b)** $(2x + 7)(2x - 7)$ **(c)** $r^2(3r + 2)(3r - 2)$

SOLUTION



$$\begin{aligned}
\textbf{(a)}\ (6m + 1)(4m - 3) &= \overset{\text{F}}{6m(4m)} + \overset{\text{O}}{6m(-3)} + \overset{\text{I}}{1(4m)} + \overset{\text{L}}{1(-3)} \\
&= 24m^2 - 18m + 4m - 3 && \text{Multiply.} \\
&= 24m^2 - 14m - 3 && \text{Combine like terms.}
\end{aligned}$$

(b) $(2x + 7)(2x - 7)$

$$= 4x^2 - 14x + 14x - 49 \qquad \text{FOIL method}$$

$$= 4x^2 - 49 \qquad \text{Combine like terms.}$$

(c) $r^2(3r + 2)(3r - 2)$

$$= r^2(9r^2 - 6r + 6r - 4) \qquad \text{FOIL method}$$

$$= r^2(9r^2 - 4) \qquad \text{Combine like terms.}$$

$$= 9r^4 - 4r^2 \qquad \text{Distributive property}$$

✔ **Now Try Exercises 55 and 57.**

In **Example 7(a),** the product of two binomials is a trinomial, while in **Examples 7(b) and (c)**, the product of two binomials is a binomial. The product of two binomials of the forms $x + y$ and $x - y$ is a special product form called a **difference of squares.** The squares of binomials, $(x + y)^2$ and $(x - y)^2$, are also special product forms called **perfect square trinomials.**

Special Products

Product of the Sum and Difference of Two Terms

$$(x + y)(x - y) = \underbrace{x^2 - y^2}_{\text{Difference of squares}}$$

Square of a Binomial

$$(x + y)^2 = x^2 + 2xy + y^2$$

$$(x - y)^2 = \underbrace{x^2 - 2xy + y^2}_{\text{Perfect square trinomials}}$$

EXAMPLE 8 Using the Special Products

Find each product.

(a) $(3p + 11)(3p - 11)$

(b) $(5m^3 - 3)(5m^3 + 3)$

(c) $(9k - 11r^3)(9k + 11r^3)$

(d) $(2m + 5)^2$

(e) $(3x - 7y^4)^2$

SOLUTION

(a) $(3p + 11)(3p - 11)$

$$= (3p)^2 - 11^2 \qquad (x + y)(x - y) = x^2 - y^2$$

$$= 9p^2 - 121 \qquad \text{Power rule 2}$$

$(3p)^2 = 3^2p^2$, not $3p^2$

(b) $(5m^3 - 3)(5m^3 + 3)$

$$= (5m^3)^2 - 3^2 \qquad (x - y)(x + y) = x^2 - y^2$$

$$= 25m^6 - 9 \qquad \text{Power rules 2 and 1}$$

(c) $(9k - 11r^3)(9k + 11r^3)$

$$= (9k)^2 - (11r^3)^2$$

$$= 81k^2 - 121r^6$$

Be careful applying the power rules.

(d) $(2m + 5)^2$

$$\begin{aligned} &= (2m)^2 + 2(2m)(5) + 5^2 && (x + y)^2 = x^2 + 2xy + y^2 \\ &= 4m^2 + 20m + 25 && \text{Power rule 2; Multiply.} \end{aligned}$$

(e) $(3x - 7y^4)^2$

$$\begin{aligned} &= (3x)^2 - 2(3x)(7y^4) + (7y^4)^2 && (x - y)^2 = x^2 - 2xy + y^2 \\ &= 9x^2 - 42xy^4 + 49y^8 && \text{Power rule 2; Multiply.} \end{aligned}$$

✓ **Now Try Exercises 69, 71, 73, and 75.**

CAUTION See **Examples 8(d) and (e).** ***The square of a binomial has three terms.*** Do **not** give $x^2 + y^2$ as the result of expanding $(x + y)^2$, or $x^2 - y^2$ as the result of expanding $(x - y)^2$.

$$\begin{aligned} (x + y)^2 &= x^2 + 2xy + y^2 \\ (x - y)^2 &= x^2 - 2xy + y^2 \end{aligned} \qquad \text{Remember to include the middle term.}$$

EXAMPLE 9 Multiplying More Complicated Binomials

Find each product.

(a) $[(3p - 2) + 5q][(3p - 2) - 5q]$ **(b)** $(x + y)^3$ **(c)** $(2a + b)^4$

SOLUTION

(a) $[(3p - 2) + 5q][(3p - 2) - 5q]$

$$\begin{aligned} &= (3p - 2)^2 - (5q)^2 && \text{Product of the sum and difference of two terms} \\ &= 9p^2 - 12p + 4 - 25q^2 && \text{Square both quantities.} \end{aligned}$$

(b) $(x + y)^3$ This does *not* equal $x^3 + y^3$.

$$\begin{aligned} &= (x + y)^2(x + y) && a^3 = a^2 \cdot a \\ &= (x^2 + 2xy + y^2)(x + y) && \text{Square } x + y. \\ &= x^3 + x^2y + 2x^2y + 2xy^2 + xy^2 + y^3 && \text{Multiply.} \\ &= x^3 + 3x^2y + 3xy^2 + y^3 && \text{Combine like terms.} \end{aligned}$$

(c) $(2a + b)^4$

$$\begin{aligned} &= (2a + b)^2(2a + b)^2 && a^4 = a^2 \cdot a^2 \\ &= (4a^2 + 4ab + b^2)(4a^2 + 4ab + b^2) && \text{Square each } 2a + b. \\ &= 16a^4 + 16a^3b + 4a^2b^2 + 16a^3b + 16a^2b^2 && \text{Distributive property} \\ &\quad + 4ab^3 + 4a^2b^2 + 4ab^3 + b^4 \\ &= 16a^4 + 32a^3b + 24a^2b^2 + 8ab^3 + b^4 && \text{Combine like terms.} \end{aligned}$$

✓ **Now Try Exercises 79, 83, and 85.**

Division The quotient of two polynomials can be found with an algorithm (that is, a step-by-step procedure) for long division similar to that used for dividing whole numbers. ***Both polynomials must be written in descending order to use this algorithm.***

EXAMPLE 10 Dividing Polynomials

Divide $4m^3 - 8m^2 + 5m + 6$ by $2m - 1$.

SOLUTION

$4m^3$ divided by $2m$ is $2m^2$.

$-6m^2$ divided by $2m$ is $-3m$.

$2m$ divided by $2m$ is 1.

$$\begin{array}{rrl}
 & 2m^2 - 3m + 1 & \\
2m - 1\big) & \overline{4m^3 - 8m^2 + 5m + 6} & \\
 & \underline{4m^3 - 2m^2} \phantom{{}+5m+6} & \leftarrow 2m^2(2m-1) = 4m^3 - 2m^2 \\
 & -6m^2 + 5m \phantom{{}+6} & \leftarrow \text{Subtract. Bring down the next term.} \\
 & \underline{-6m^2 + 3m} \phantom{{}+6} & \leftarrow -3m(2m-1) = -6m^2 + 3m \\
 & 2m + 6 & \leftarrow \text{Subtract. Bring down the next term.} \\
 & \underline{2m - 1} & \leftarrow 1(2m-1) = 2m - 1 \\
 & 7 & \leftarrow \text{Subtract. The remainder is 7.}
\end{array}$$

To subtract, add the opposite.

Thus, $\dfrac{4m^3 - 8m^2 + 5m + 6}{2m - 1} = 2m^2 - 3m + 1 + \dfrac{7}{2m - 1}$.

Remember to add $\frac{\text{remainder}}{\text{divisor}}$.

✔ Now Try Exercise 101.

When a polynomial has a missing term, we allow for that term by inserting a term with a 0 coefficient for it. For example,

$3x^2 - 7$ is equivalent to $3x^2 + 0x - 7$,

and $2x^3 + x + 10$ is equivalent to $2x^3 + 0x^2 + x + 10$.

EXAMPLE 11 Dividing Polynomials with Missing Terms

Divide $3x^3 - 2x^2 - 150$ by $x^2 - 4$.

SOLUTION Both polynomials have missing first-degree terms. Insert each missing term with a 0 coefficient.

$$\begin{array}{rrl}
 & 3x - 2 & \\
x^2 + 0x - 4\big) & \overline{3x^3 - 2x^2 + 0x - 150} & \leftarrow \text{Missing term} \\
\text{Missing term} \uparrow & \underline{3x^3 + 0x^2 - 12x} \phantom{{}-150} & \\
 & -2x^2 + 12x - 150 & \\
 & \underline{-2x^2 + 0x + 8} & \\
 & 12x - 158 & \leftarrow \text{Remainder}
\end{array}$$

The division process ends when the remainder is 0 or the degree of the remainder is less than that of the divisor. Because $12x - 158$ has lesser degree than the divisor $x^2 - 4$, it is the remainder. Thus, the entire quotient is written as follows.

$$\frac{3x^3 - 2x^2 - 150}{x^2 - 4} = 3x - 2 + \frac{12x - 158}{x^2 - 4}$$

✔ Now Try Exercise 103.

CONCEPT PREVIEW *Work each problem.*

6. Match each polynomial in Column I with its factored form in Column II.

I	II
(a) $x^2 + 10xy + 25y^2$	**A.** $(x + 5y)(x - 5y)$
(b) $x^2 - 10xy + 25y^2$	**B.** $(x + 5y)^2$
(c) $x^2 - 25y^2$	**C.** $(x - 5y)^2$
(d) $25y^2 - x^2$	**D.** $(5y + x)(5y - x)$

7. Match each polynomial in Column I with its factored form in Column II.

I	II
(a) $8x^3 - 27$	**A.** $(3 - 2x)(9 + 6x + 4x^2)$
(b) $8x^3 + 27$	**B.** $(2x - 3)(4x^2 + 6x + 9)$
(c) $27 - 8x^3$	**C.** $(2x + 3)(4x^2 - 6x + 9)$

8. Which of the following is the correct factorization of $6x^2 + x - 12$?

A. $(3x + 4)(2x + 3)$ **B.** $(3x - 4)(2x - 3)$

C. $(3x + 4)(2x - 3)$ **D.** $(3x - 4)(2x + 3)$

9. Which of the following is the correct complete factorization of $x^4 - 1$?

A. $(x^2 - 1)(x^2 + 1)$ **B.** $(x^2 + 1)(x + 1)(x - 1)$

C. $(x^2 - 1)^2$ **D.** $(x - 1)^2(x + 1)^2$

10. Which of the following is the correct factorization of $x^3 + 8$?

A. $(x + 2)^3$ **B.** $(x + 2)(x^2 + 2x + 4)$

C. $(x + 2)(x^2 - 2x + 4)$ **D.** $(x + 2)(x^2 - 4x + 4)$

Factor out the greatest common factor from each polynomial. ***See Examples 1 and 2.***

11. $12m + 60$

12. $15r - 27$

13. $8k^3 + 24k$

14. $9z^4 + 81z$

15. $xy - 5xy^2$

16. $5h^2j + hj$

17. $-4p^3q^4 - 2p^2q^5$

18. $-3z^5w^2 - 18z^3w^4$

19. $4k^2m^3 + 8k^4m^3 - 12k^2m^4$

20. $28r^4s^2 + 7r^3s - 35r^4s^3$

21. $2(a + b) + 4m(a + b)$

22. $6x(a + b) - 4y(a + b)$

23. $(5r - 6)(r + 3) - (2r - 1)(r + 3)$

24. $(4z - 5)(3z - 2) - (3z - 9)(3z - 2)$

25. $2(m - 1) - 3(m - 1)^2 + 2(m - 1)^3$

26. $5(a + 3)^3 - 2(a + 3) + (a + 3)^2$

27. ***Concept Check*** When directed to completely factor the polynomial $4x^2y^5 - 8xy^3$, a student wrote $2xy^3(2xy^2 - 4)$. When the teacher did not give him full credit, he complained because when his answer is multiplied out, the result is the original polynomial. Give the correct answer.

28. ***Concept Check*** Kurt factored $16a^2 - 40a - 6a + 15$ by grouping and obtained $(8a - 3)(2a - 5)$. Callie factored the same polynomial and gave an answer of $(3 - 8a)(5 - 2a)$. Which answer is correct?

Factor each polynomial by grouping. ***See Example 2.***

29. $6st + 9t - 10s - 15$

30. $10ab - 6b + 35a - 21$

31. $2m^4 + 6 - am^4 - 3a$

32. $4x^6 + 36 - x^6y - 9y$

33. $p^2q^2 - 10 - 2q^2 + 5p^2$

34. $20z^2 - 8x + 5pz^2 - 2px$

Factor each trinomial, if possible. ***See Examples 3 and 4.***

35. $6a^2 - 11a + 4$

36. $8h^2 - 2h - 21$

37. $3m^2 + 14m + 8$

38. $9y^2 - 18y + 8$

39. $15p^2 + 24p + 8$

40. $9x^2 + 4x - 2$

41. $12a^3 + 10a^2 - 42a$

42. $36x^3 + 18x^2 - 4x$

43. $6k^2 + 5kp - 6p^2$

44. $14m^2 + 11mr - 15r^2$

45. $5a^2 - 7ab - 6b^2$

46. $12s^2 + 11st - 5t^2$

47. $12x^2 - xy - y^2$

48. $30a^2 + am - m^2$

49. $24a^4 + 10a^3b - 4a^2b^2$

50. $18x^5 + 15x^4z - 75x^3z^2$

51. $9m^2 - 12m + 4$

52. $16p^2 - 40p + 25$

53. $32a^2 + 48ab + 18b^2$

54. $20p^2 - 100pq + 125q^2$

55. $4x^2y^2 + 28xy + 49$

56. $9m^2n^2 + 12mn + 4$

57. $(a - 3b)^2 - 6(a - 3b) + 9$

58. $(2p + q)^2 - 10(2p + q) + 25$

Factor each polynomial. ***See Examples 5 and 6.***

59. $9a^2 - 16$

60. $16q^2 - 25$

61. $x^4 - 16$

62. $y^4 - 81$

63. $25s^4 - 9t^2$

64. $36z^2 - 81y^4$

65. $(a + b)^2 - 16$

66. $(p - 2q)^2 - 100$

67. $p^4 - 625$

68. $m^4 - 1296$

69. $x^2 - 8x + 16 - y^2$

70. $m^2 + 10m + 25 - n^4$

71. $y^2 - x^2 + 12x - 36$

72. $9m^2 - n^2 - 2n - 1$

73. $8 - a^3$

74. $27 - r^3$

75. $125x^3 - 27$

76. $8m^3 - 27n^3$

77. $27y^9 + 125z^6$

78. $27z^9 + 64y^{12}$

79. $(r + 6)^3 - 216$

80. $(b + 3)^3 - 27$

81. $27 - (m + 2n)^3$

82. $125 - (4a - b)^3$

Factor each polynomial. ***See Example 7.***

83. $7(3k - 1)^2 + 26(3k - 1) - 8$

84. $6(4z - 3)^2 + 7(4z - 3) - 3$

85. $9(a - 4)^2 + 30(a - 4) + 25$

86. $4(5x + 7)^2 + 12(5x + 7) + 9$

87. $(a + 1)^3 + 27$

88. $(x - 4)^3 + 64$

89. $(3x + 4)^3 - 1$

90. $(5x - 2)^3 - 8$

91. $m^4 - 3m^2 - 10$

92. $a^4 - 2a^2 - 48$

93. $12t^4 - t^2 - 35$

94. $10m^4 + 43m^2 - 9$

Factor by any method. ***See Examples 1–7.***

95. $4b^2 + 4bc + c^2 - 16$

96. $(2y - 1)^2 - 4(2y - 1) + 4$

97. $x^2 + xy - 5x - 5y$

98. $8r^2 - 3rs + 10s^2$

99. $p^4(m - 2n) + q(m - 2n)$

100. $36a^2 + 60a + 25$

101. $4z^2 + 28z + 49$

102. $6p^4 + 7p^2 - 3$

103. $1000x^3 + 343y^3$

104. $b^2 + 8b + 16 - a^2$

105. $125m^6 - 216$

106. $q^2 + 6q + 9 - p^2$

107. $64 + (3x + 2)^3$

108. $216p^3 + 125q^3$

109. $(x + y)^3 - (x - y)^3$

110. $100r^2 - 169s^2$

111. $144z^2 + 121$

112. $(3a + 5)^2 - 18(3a + 5) + 81$

113. $(x + y)^2 - (x - y)^2$

114. $4z^4 - 7z^2 - 15$

R.4 Factoring Polynomials

- Factoring Out the Greatest Common Factor
- Factoring by Grouping
- Factoring Trinomials
- Factoring Binomials
- Factoring by Substitution

The process of finding polynomials whose product equals a given polynomial is called **factoring.** Unless otherwise specified, we consider only integer coefficients when factoring polynomials. For example, because

$$4x + 12 = 4(x + 3),$$

both 4 and $x + 3$ are **factors** of $4x + 12$, and $4(x + 3)$ is a **factored form** of $4x + 12$.

A polynomial with variable terms that cannot be written as a product of two polynomials of lesser degree is a **prime polynomial.** A polynomial is **factored completely** when it is written as a product of prime polynomials.

Factoring Out the Greatest Common Factor To factor a polynomial such as $6x^2y^3 + 9xy^4 + 18y^5$, we look for a monomial that is the **greatest common factor (GCF)** of the three terms.

$$\begin{aligned} &6x^2y^3 + 9xy^4 + 18y^5 \\ &\quad = 3y^3(2x^2) + 3y^3(3xy) + 3y^3(6y^2) \qquad \text{GCF} = 3y^3 \\ &\quad = 3y^3(2x^2 + 3xy + 6y^2) \qquad \text{Distributive property} \end{aligned}$$

EXAMPLE 1 Factoring Out the Greatest Common Factor

Factor out the greatest common factor from each polynomial.

(a) $9y^5 + y^2$ **(b)** $6x^2t + 8xt - 12t$

(c) $14(m + 1)^3 - 28(m + 1)^2 - 7(m + 1)$

SOLUTION

(a) $9y^5 + y^2$

$$\begin{aligned} &= y^2(9y^3) + y^2(1) \qquad \text{GCF} = y^2 \\ &= y^2(9y^3 + 1) \end{aligned}$$

Remember to include the 1.

CHECK Multiply out the factored form: $y^2(9y^3 + 1) = \overbrace{9y^5 + y^2}^{\text{Original polynomial}}$. ✓

(b) $6x^2t + 8xt - 12t$

$$= 2t(3x^2 + 4x - 6) \qquad \text{GCF} = 2t$$

CHECK $2t(3x^2 + 4x - 6) = 6x^2t + 8xt - 12t$ ✓

(c) $14(m + 1)^3 - 28(m + 1)^2 - 7(m + 1)$

$$\begin{aligned} &= 7(m + 1)[2(m + 1)^2 - 4(m + 1) - 1] \qquad \text{GCF} = 7(m + 1) \\ &= 7(m + 1)[2(m^2 + 2m + 1) - 4m - 4 - 1] \qquad \text{Square } m + 1\text{; distributive property} \end{aligned}$$

Remember the middle term.

$$\begin{aligned} &= 7(m + 1)(2m^2 + 4m + 2 - 4m - 4 - 1) \qquad \text{Distributive property} \\ &= 7(m + 1)(2m^2 - 3) \qquad \text{Combine like terms.} \end{aligned}$$

✔ **Now Try Exercises 13, 19, and 25.**

CAUTION In **Example 1(a),** the 1 is essential in the answer because

$$y^2(9y^3) \neq 9y^5 + y^2.$$

Factoring can always be checked by multiplying.

Factoring by Grouping When a polynomial has more than three terms, it can sometimes be factored using **factoring by grouping.** Consider this example.

$$\begin{aligned} & ax + ay + 6x + 6y \\ &= \overbrace{(ax + ay)}^{\text{Terms with common factor } a} + \overbrace{(6x + 6y)}^{\text{Terms with common factor 6}} && \text{Group the terms so that each group has a common factor.} \\ &= a(x + y) + 6(x + y) && \text{Factor each group.} \\ &= (x + y)(a + 6) && \text{Factor out } x + y. \end{aligned}$$

It is not always obvious which terms should be grouped. In cases like the one above, group in pairs. Experience and repeated trials are the most reliable tools.

EXAMPLE 2 Factoring by Grouping

Factor each polynomial by grouping.

(a) $mp^2 + 7m + 3p^2 + 21$ **(b)** $2y^2 + az - 2z - ay^2$

(c) $4x^3 + 2x^2 - 2x - 1$

SOLUTION

(a) $mp^2 + 7m + 3p^2 + 21$

$$\begin{aligned} &= (mp^2 + 7m) + (3p^2 + 21) && \text{Group the terms.} \\ &= m(p^2 + 7) + 3(p^2 + 7) && \text{Factor each group.} \\ &= (p^2 + 7)(m + 3) && p^2 + 7 \text{ is a common factor.} \end{aligned}$$

CHECK $(p^2 + 7)(m + 3)$

$$\begin{aligned} &= mp^2 + 3p^2 + 7m + 21 && \text{FOIL method} \\ &= mp^2 + 7m + 3p^2 + 21 \checkmark && \text{Commutative property} \end{aligned}$$

(b) $2y^2 + az - 2z - ay^2$

$$\begin{aligned} &= 2y^2 - 2z - ay^2 + az && \text{Rearrange the terms.} \\ &= (2y^2 - 2z) + (-ay^2 + az) && \text{Group the terms.} \\ &= 2(y^2 - z) - a(y^2 - z) && \text{Factor out 2 and } -a \text{ so that } y^2 - z \text{ is a common factor.} \\ &= (y^2 - z)(2 - a) && \text{Factor out } y^2 - z. \end{aligned}$$

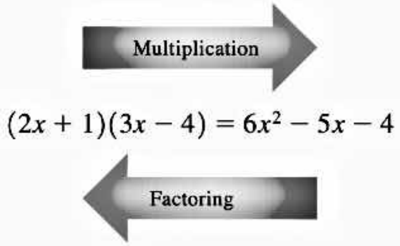
Be careful with signs here.

(c) $4x^3 + 2x^2 - 2x - 1$

$$\begin{aligned} &= (4x^3 + 2x^2) + (-2x - 1) && \text{Group the terms.} \\ &= 2x^2(2x + 1) - 1(2x + 1) && \text{Factor each group.} \\ &= (2x + 1)(2x^2 - 1) && \text{Factor out } 2x + 1. \end{aligned}$$

✔ **Now Try Exercises 29 and 31.**

Factoring Trinomials As shown in the diagram below, factoring is the opposite of multiplication.



$$(2x + 1)(3x - 4) = 6x^2 - 5x - 4$$

One strategy when factoring trinomials uses the FOIL method in reverse. This strategy requires trial-and-error to find the correct arrangement of coefficients of the binomial factors.

EXAMPLE 3 Factoring Trinomials

Factor each trinomial, if possible.

(a) $4y^2 - 11y + 6$

(b) $6p^2 - 7p - 5$

(c) $2x^2 + 13x - 18$

(d) $16y^3 + 24y^2 - 16y$

SOLUTION

(a) To factor this polynomial, we must find values for integers *a*, *b*, *c*, and *d* in such a way that

$$\begin{aligned}&4y^2 - 11y + 6\\&= (ay + b)(cy + d).\end{aligned}$$ FOIL method

Using the FOIL method, we see that $ac = 4$ and $bd = 6$. The positive factors of 4 are 4 and 1 or 2 and 2. Because the middle term has a negative coefficient, we consider only negative factors of 6. The possibilities are -2 and -3 or -1 and -6.

Now we try various arrangements of these factors until we find one that gives the correct coefficient of *y*.

$(2y - 1)(2y - 6)$	$(2y - 2)(2y - 3)$	$(y - 2)(4y - 3)$
$= 4y^2 - 14y + 6$	$= 4y^2 - 10y + 6$	$= 4y^2 - 11y + 6$
Incorrect	Incorrect	Correct

Therefore, $4y^2 - 11y + 6$ factors as $(y - 2)(4y - 3)$.

CHECK $(y - 2)(4y - 3)$

$= 4y^2 - 3y - 8y + 6$ FOIL method

$= 4y^2 - 11y + 6$ ✓ Original polynomial

(b) Again, we try various possibilities to factor $6p^2 - 7p - 5$. The positive factors of 6 could be 2 and 3 or 1 and 6. As factors of -5 we have only -1 and 5 or -5 and 1.

$(2p - 5)(3p + 1)$	$(3p - 5)(2p + 1)$
$= 6p^2 - 13p - 5$ Incorrect	$= 6p^2 - 7p - 5$ Correct

Thus, $6p^2 - 7p - 5$ factors as $(3p - 5)(2p + 1)$.

(c) If we try to factor $2x^2 + 13x - 18$, we find that none of the pairs of factors gives the correct coefficient of x. Additional trials are also unsuccessful.

$(2x + 9)(x - 2)$	$(2x - 3)(x + 6)$	$(2x - 1)(x + 18)$
$= 2x^2 + 5x - 18$	$= 2x^2 + 9x - 18$	$= 2x^2 + 35x - 18$
Incorrect	Incorrect	Incorrect

This trinomial cannot be factored with integer coefficients and is prime.

(d)

$$
\begin{aligned}
&16y^3 + 24y^2 - 16y \\
&= 8y(2y^2 + 3y - 2) && \text{Factor out the GCF, } 8y. \\
&= 8y(2y - 1)(y + 2) && \text{Factor the trinomial.}
\end{aligned}
$$

Remember to include the common factor in the final form.

✔ **Now Try Exercises 35, 37, 39, and 41.**

NOTE In **Example 3,** we chose positive factors of the positive first term (instead of two negative factors). This makes the work easier.

Each of the special patterns for multiplication can be used in reverse to obtain a pattern for factoring. Perfect square trinomials can be factored as follows.

Factoring Perfect Square Trinomials

$$x^2 + 2xy + y^2 = (x + y)^2$$

$$x^2 - 2xy + y^2 = (x - y)^2$$

EXAMPLE 4 Factoring Perfect Square Trinomials

Factor each trinomial.

(a) $16p^2 - 40pq + 25q^2$ **(b)** $36x^2y^2 + 84xy + 49$

SOLUTION

(a) Because $16p^2 = (4p)^2$ and $25q^2 = (5q)^2$, we use the second pattern shown in the box, with $4p$ replacing x and $5q$ replacing y.

$$
\begin{aligned}
&16p^2 - 40pq + 25q^2 \\
&= (4p)^2 - 2(4p)(5q) + (5q)^2 \\
&= (4p - 5q)^2
\end{aligned}
$$

Make sure that the middle term of the trinomial being factored, $-40pq$ here, is twice the product of the two terms in the binomial $4p - 5q$.

$$-40pq = 2(4p)(-5q)$$

Thus, $16p^2 - 40pq + 25q^2$ factors as $(4p - 5q)^2$.

CHECK $(4p - 5q)^2 = 16p^2 - 40pq + 25q^2$ ✓ Multiply.

(b) $36x^2y^2 + 84xy + 49$ factors as $(6xy + 7)^2$. $2(6xy)(7) = 84xy$

CHECK Square $6xy + 7$: $(6xy + 7)^2 = 36x^2y^2 + 84xy + 49$. ✓

✔ **Now Try Exercises 51 and 55.**

Factoring Binomials Check first to see whether the terms of a binomial have a common factor. If so, factor it out. The binomial may also fit one of the following patterns.

Factoring Binomials

Difference of Squares	$x^2 - y^2 = (x + y)(x - y)$
Difference of Cubes	$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$
Sum of Cubes	$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

CAUTION ***There is no factoring pattern for a sum of squares in the real number system.*** In particular, for real numbers x and y,

$$x^2 + y^2 \textbf{ does not factor as } (x + y)^2.$$

EXAMPLE 5 Factoring Differences of Squares

Factor each polynomial.

(a) $4m^2 - 9$

(b) $256k^4 - 625m^4$

(c) $(a + 2b)^2 - 4c^2$

(d) $x^2 - 6x + 9 - y^4$

(e) $y^2 - x^2 + 6x - 9$

SOLUTION

(a) $4m^2 - 9$

$= (2m)^2 - 3^2$ Write as a difference of squares.

$= (2m + 3)(2m - 3)$ Factor.

Check by multiplying.

(b) $256k^4 - 625m^4$

$= (16k^2)^2 - (25m^2)^2$ Write as a difference of squares.

Don't stop here.

$= (16k^2 + 25m^2)(16k^2 - 25m^2)$ Factor.

$= (16k^2 + 25m^2)(4k + 5m)(4k - 5m)$ Factor $16k^2 - 25m^2$.

CHECK $(16k^2 + 25m^2)(4k + 5m)(4k - 5m)$

$= (16k^2 + 25m^2)(16k^2 - 25m^2)$ Multiply the last two factors.

$= 256k^4 - 625m^4$ ✓ Original polynomial

(c) $(a + 2b)^2 - 4c^2$

$= (a + 2b)^2 - (2c)^2$ Write as a difference of squares.

$= [(a + 2b) + 2c][(a + 2b) - 2c]$ Factor.

$= (a + 2b + 2c)(a + 2b - 2c)$

Check by multiplying.

(d) $x^2 - 6x + 9 - y^4$

$= (x^2 - 6x + 9) - y^4$ Group terms.

$= (x - 3)^2 - y^4$ Factor the trinomial.

$= (x - 3)^2 - (y^2)^2$ Write as a difference of squares.

$= [(x - 3) + y^2][(x - 3) - y^2]$ Factor.

$= (x - 3 + y^2)(x - 3 - y^2)$

(e) $y^2 - x^2 + 6x - 9$

Be careful with signs. This is a perfect square trinomial.

$= y^2 - (x^2 - 6x + 9)$ Factor out the negative sign, and group the last three terms.

$= y^2 - (x - 3)^2$ Write as a difference of squares.

$= [y - (x - 3)][y + (x - 3)]$ Factor.

$= (y - x + 3)(y + x - 3)$ Distributive property

✔ Now Try Exercises 59, 61, 65, and 69.

CAUTION When factoring as in **Example 5(e),** be careful with signs. Inserting an open parenthesis following the minus sign requires changing the signs of all of the following terms.

EXAMPLE 6 Factoring Sums or Differences of Cubes

Factor each polynomial.

(a) $x^3 + 27$ **(b)** $m^3 - 64n^3$ **(c)** $8q^6 + 125p^9$

SOLUTION

(a) $x^3 + 27$

$= x^3 + 3^3$ Write as a sum of cubes.

$= (x + 3)(x^2 - 3x + 3^2)$ Factor.

$= (x + 3)(x^2 - 3x + 9)$ Apply the exponent.

(b) $m^3 - 64n^3$

$= m^3 - (4n)^3$ Write as a difference of cubes.

$= (m - 4n)[m^2 + m(4n) + (4n)^2]$ Factor.

$= (m - 4n)(m^2 + 4mn + 16n^2)$ Multiply; $(4n)^2 = 4^2n^2$.

(c) $8q^6 + 125p^9$

$= (2q^2)^3 + (5p^3)^3$ Write as a sum of cubes.

$= (2q^2 + 5p^3)[(2q^2)^2 - 2q^2(5p^3) + (5p^3)^2]$ Factor.

$= (2q^2 + 5p^3)(4q^4 - 10q^2p^3 + 25p^6)$ Simplify.

✔ Now Try Exercises 73, 75, and 77.

8. $\frac{4}{x-y}-\frac{9}{x-y}$ **9.** $\frac{2x}{5}+\frac{x}{4}$ **10.** $\frac{7}{x^2}-\frac{8}{y}$

Find the domain of each rational expression. ***See Example 1.***

11. $\frac{x+3}{x-6}$ **12.** $\frac{2x-4}{x+7}$ **13.** $\frac{3x+7}{(4x+2)(x-1)}$

14. $\frac{9x+12}{(2x+3)(x-5)}$ **15.** $\frac{12}{x^2+5x+6}$ **16.** $\frac{3}{x^2-5x-6}$

17. $\frac{x^2-1}{x+1}$ **18.** $\frac{x^2-25}{x-5}$ **19.** $\frac{x^3-1}{x-1}$

20. ***Concept Check*** Use specific values for x and y to show that in general, $\frac{1}{x}+\frac{1}{y}$ is not equivalent to $\frac{1}{x+y}$.

Write each rational expression in lowest terms. ***See Example 2.***

21. $\frac{8x^2+16x}{4x^2}$ **22.** $\frac{36y^2+72y}{9y^2}$ **23.** $\frac{3(3-t)}{(t+5)(t-3)}$

24. $\frac{-8(4-y)}{(y+2)(y-4)}$ **25.** $\frac{8k+16}{9k+18}$ **26.** $\frac{20r+10}{30r+15}$

27. $\frac{m^2-4m+4}{m^2+m-6}$ **28.** $\frac{r^2-r-6}{r^2+r-12}$ **29.** $\frac{8m^2+6m-9}{16m^2-9}$

30. $\frac{6y^2+11y+4}{3y^2+7y+4}$ **31.** $\frac{x^3+64}{x+4}$ **32.** $\frac{y^3-27}{y-3}$

Multiply or divide, as indicated. ***See Example 3.***

33. $\frac{15p^3}{9p^2}\div\frac{6p}{10p^2}$ **34.** $\frac{8r^3}{6r}\div\frac{5r^2}{9r^3}$ **35.** $\frac{2k+8}{6}\div\frac{3k+12}{2}$

36. $\frac{5m+25}{10}\div\frac{6m+30}{12}$ **37.** $\frac{x^2+x}{5}\cdot\frac{25}{xy+y}$ **38.** $\frac{y^3+y^2}{7}\cdot\frac{49}{y^4+y^3}$

39. $\frac{4a+12}{2a-10}\div\frac{a^2-9}{a^2-a-20}$ **40.** $\frac{6r-18}{9r^2+6r-24}\div\frac{4r-12}{12r-16}$

41. $\frac{p^2-p-12}{p^2-2p-15}\cdot\frac{p^2-9p+20}{p^2-8p+16}$ **42.** $\frac{x^2+2x-15}{x^2+11x+30}\cdot\frac{x^2+2x-24}{x^2-8x+15}$

43. $\frac{m^2+3m+2}{m^2+5m+4}\div\frac{m^2+5m+6}{m^2+10m+24}$ **44.** $\frac{y^2+y-2}{y^2+3y-4}\div\frac{y^2+3y+2}{y^2+4y+3}$

45. $\frac{x^3+y^3}{x^3-y^3}\cdot\frac{x^2-y^2}{x^2+2xy+y^2}$ **46.** $\frac{x^2-y^2}{(x-y)^2}\cdot\frac{x^2-xy+y^2}{x^2-2xy+y^2}\div\frac{x^3+y^3}{(x-y)^4}$

47. $\frac{xz-xw+2yz-2yw}{z^2-w^2}\cdot\frac{4z+4w+xz+wx}{16-x^2}$

48. $\frac{ac+ad+bc+bd}{a^2-b^2}\cdot\frac{a^3-b^3}{2a^2+2ab+2b^2}$

49. ***Concept Check*** Which of the following rational expressions is equivalent to -1? In choices A, B, and D, $x \neq -4$, and in choice C, $x \neq 4$. (*Hint:* There may be more than one answer.)

A. $\frac{x-4}{x+4}$ **B.** $\frac{-x-4}{x+4}$ **C.** $\frac{x-4}{4-x}$ **D.** $\frac{x-4}{-x-4}$

50. Explain how to find the least common denominator of several fractions.

Add or subtract, as indicated. ***See Example 4.***

51. $\frac{3}{2k}+\frac{5}{3k}$

52. $\frac{8}{5p}+\frac{3}{4p}$

53. $\frac{1}{6m}+\frac{2}{5m}+\frac{4}{m}$

54. $\frac{8}{3p}+\frac{5}{4p}+\frac{9}{2p}$

55. $\frac{1}{a}-\frac{b}{a^2}$

56. $\frac{3}{z}+\frac{x}{z^2}$

57. $\frac{5}{12x^2y}-\frac{11}{6xy}$

58. $\frac{7}{18a^3b^2}-\frac{2}{9ab}$

59. $\frac{17y+3}{9y+7}-\frac{-10y-18}{9y+7}$

60. $\frac{7x+8}{3x+2}-\frac{x+4}{3x+2}$

61. $\frac{1}{x+z}+\frac{1}{x-z}$

62. $\frac{m+1}{m-1}+\frac{m-1}{m+1}$

63. $\frac{3}{a-2}-\frac{1}{2-a}$

64. $\frac{4}{p-q}-\frac{2}{q-p}$

65. $\frac{x+y}{2x-y}-\frac{2x}{y-2x}$

66. $\frac{m-4}{3m-4}-\frac{5m}{4-3m}$

67. $\frac{4}{x+1}+\frac{1}{x^2-x+1}-\frac{12}{x^3+1}$

68. $\frac{5}{x+2}+\frac{2}{x^2-2x+4}-\frac{60}{x^3+8}$

69. $\frac{3x}{x^2+x-12}-\frac{x}{x^2-16}$

70. $\frac{p}{2p^2-9p-5}-\frac{2p}{6p^2-p-2}$

Simplify each complex fraction. ***See Example 5.***

71. $\frac{1+\frac{1}{x}}{1-\frac{1}{x}}$

72. $\frac{2-\frac{2}{y}}{2+\frac{2}{y}}$

73. $\frac{\frac{1}{x+1}-\frac{1}{x}}{\frac{1}{x}}$

74. $\frac{\frac{1}{y+3}-\frac{1}{y}}{\frac{1}{y}}$

75. $\frac{1+\frac{1}{1-b}}{1-\frac{1}{1+b}}$

76. $\frac{2+\frac{2}{1+x}}{2-\frac{2}{1-x}}$

77. $\frac{\frac{1}{a^3+b^3}}{\frac{1}{a^2+2ab+b^2}}$

78. $\frac{\frac{1}{x^3-y^3}}{\frac{1}{x^2-y^2}}$

79. $\frac{m-\frac{1}{m^2-4}}{\frac{1}{m+2}}$

80. $\frac{y+\frac{1}{y^2-9}}{\frac{1}{y+3}}$

81. $\frac{\frac{3}{p^2-16}+p}{\frac{1}{p-4}}$

82. $\frac{\frac{6}{x^2-25}+x}{\frac{1}{x-5}}$

83. $\frac{\frac{y+3}{y}-\frac{4}{y-1}}{\frac{y}{y-1}+\frac{1}{y}}$

84. $\frac{\frac{x+4}{x}-\frac{3}{x-2}}{\frac{x}{x-2}+\frac{1}{x}}$

85. $\frac{\frac{1}{x+h}-\frac{1}{x}}{h}$

86. $\frac{\frac{-2}{x+h}-\frac{-2}{x}}{h}$

87. $\frac{\frac{1}{(x+h)^2+9}-\frac{1}{x^2+9}}{h}$

88. $\frac{\frac{2}{(x+h)^2+16}-\frac{2}{x^2+16}}{h}$

R.5 Rational Expressions

- Rational Expressions
- Lowest Terms of a Rational Expression
- Multiplication and Division
- Addition and Subtraction
- Complex Fractions

Rational Expressions The quotient of two polynomials P and Q, with $Q \neq 0$, is a **rational expression.**

$$\frac{x+6}{x+2}, \quad \frac{(x+6)(x+4)}{(x+2)(x+4)}, \quad \frac{2p^2+7p-4}{5p^2+20p} \qquad \text{Rational expressions}$$

The **domain** of a rational expression is the set of real numbers for which the expression is defined. Because the denominator of a fraction cannot be 0, the domain consists of all real numbers except those that make the denominator 0. We find these numbers by setting the denominator equal to 0 and solving the resulting equation. For example, in the rational expression

$$\frac{x+6}{x+2},$$

the solution to the equation $x + 2 = 0$ is excluded from the domain. The solution is -2, so the domain is the set of all real numbers x not equal to -2.

$$\{x \mid x \neq -2\} \qquad \text{Set-builder notation}$$

If the denominator of a rational expression contains a product, we determine the domain with the **zero-factor property,** which states that $ab = 0$ if and only if $a = 0$ or $b = 0$.

EXAMPLE 1 Finding the Domain

Find the domain of the rational expression.

$$\frac{(x+6)(x+4)}{(x+2)(x+4)}$$

SOLUTION

$$\begin{aligned} (x+2)(x+4) &= 0 && \text{Set the denominator equal to zero.} \\ x+2=0 \quad \text{or} \quad x+4 &= 0 && \text{Zero-factor property} \\ x=-2 \quad \text{or} \quad x &= -4 && \text{Solve each equation.} \end{aligned}$$

The domain is the set of real numbers *not equal to* -2 or -4, written

$$\{x \mid x \neq -2, -4\}.$$

✔ **Now Try Exercises 11 and 13.**

Lowest Terms of a Rational Expression A rational expression is written in **lowest terms** when the greatest common factor of its numerator and its denominator is 1. We use the following **fundamental principle of fractions** to write a rational expression in lowest terms by dividing out common factors.

Fundamental Principle of Fractions

$$\frac{ac}{bc} = \frac{a}{b} \quad (b \neq 0, c \neq 0)$$

EXAMPLE 2 Writing Rational Expressions in Lowest Terms

Write each rational expression in lowest terms.

(a) $\dfrac{2x^2 + 7x - 4}{5x^2 + 20x}$ **(b)** $\dfrac{6 - 3x}{x^2 - 4}$

SOLUTION

(a) $\dfrac{2x^2 + 7x - 4}{5x^2 + 20x}$

$$\begin{aligned} &= \frac{(2x-1)(x+4)}{5x(x+4)} && \text{Factor.} \\ &= \frac{2x-1}{5x} && \text{Divide out the common factor.} \end{aligned}$$

To determine the domain, we find values of x that make the *original* denominator $5x^2 + 20x$ equal to 0, and exclude them.

$$\begin{aligned} 5x^2 + 20x &= 0 && \text{Set the denominator equal to 0.} \\ 5x(x+4) &= 0 && \text{Factor.} \\ 5x = 0 \quad \text{or} \quad x + 4 &= 0 && \text{Zero-factor property} \\ x = 0 \quad \text{or} \quad x &= -4 && \text{Solve each equation.} \end{aligned}$$

The domain is $\{x \mid x \neq 0, -4\}$. ***From now on, we will assume such restrictions when writing rational expressions in lowest terms.***

(b) $\dfrac{6 - 3x}{x^2 - 4}$

$$\begin{aligned} &= \frac{3(2-x)}{(x+2)(x-2)} && \text{Factor.} \\ &= \frac{3(2-x)(-1)}{(x+2)(x-2)(-1)} && \begin{array}{l} 2 - x \text{ and } x - 2 \text{ are opposites.} \\ \text{Multiply numerator and denominator by } -1. \end{array} \\ &= \frac{3(2-x)(-1)}{(x+2)(2-x)} && (x-2)(-1) = -x + 2 = 2 - x \\ &= \frac{-3}{x+2} && \text{Divide out the common factor.} \end{aligned}$$

Be careful with signs.

Working in an alternative way would lead to the equivalent result $\frac{3}{-x-2}$.

✔ Now Try Exercises 23 and 27.

LOOKING AHEAD TO CALCULUS

A standard problem in calculus is investigating what value an expression such as $\frac{x^2-1}{x-1}$ approaches as x approaches 1. We cannot do this by simply substituting 1 for x in the expression since the result is the indeterminate form $\frac{0}{0}$. When we factor the numerator and write the expression in lowest terms, it becomes $x + 1$. Then, by substituting 1 for x, we obtain $1 + 1 = 2$, which is the **limit** of $\frac{x^2-1}{x-1}$ as x approaches 1.

CAUTION The fundamental principle requires a pair of common *factors*, one in the numerator and one in the denominator. ***Only after a rational expression has been factored can any common factors be divided out.*** For example,

$$\frac{2x+4}{6} = \frac{2(x+2)}{2 \cdot 3} = \frac{x+2}{3}. \quad \text{Factor first, and then divide.}$$

Multiplication and Division

We now multiply and divide fractions.

Multiplication and Division

For fractions $\frac{a}{b}$ and $\frac{c}{d}$ $(b \neq 0, d \neq 0)$, the following hold.

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \quad \text{and} \quad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} \quad (c \neq 0)$$

That is, to find the product of two fractions, multiply their numerators to find the numerator of the product. Then multiply their denominators to find the denominator of the product.

To divide two fractions, multiply the* dividend *(the first fraction) by the reciprocal of the* divisor *(the second fraction).

EXAMPLE 3 Multiplying or Dividing Rational Expressions

Multiply or divide, as indicated.

(a) $\frac{2y^2}{9} \cdot \frac{27}{8y^5}$

(b) $\frac{3m^2 - 2m - 8}{3m^2 + 14m + 8} \cdot \frac{3m + 2}{3m + 4}$

(c) $\frac{3p^2 + 11p - 4}{24p^3 - 8p^2} \div \frac{9p + 36}{24p^4 - 36p^3}$

(d) $\frac{x^3 - y^3}{x^2 - y^2} \cdot \frac{2x + 2y + xz + yz}{2x^2 + 2y^2 + zx^2 + zy^2}$

SOLUTION

(a) $\frac{2y^2}{9} \cdot \frac{27}{8y^5}$

$= \frac{2y^2 \cdot 27}{9 \cdot 8y^5}$ Multiply fractions.

$= \frac{2 \cdot 9 \cdot 3 \cdot y^2}{9 \cdot 2 \cdot 4 \cdot y^2 \cdot y^3}$ Factor.

$= \frac{3}{4y^3}$ Lowest terms

Although we usually factor first and then multiply the fractions (see parts (b)–(d)), we did the opposite here. Either order is acceptable.

(b) $\frac{3m^2 - 2m - 8}{3m^2 + 14m + 8} \cdot \frac{3m + 2}{3m + 4}$

$= \frac{(m - 2)(3m + 4)}{(m + 4)(3m + 2)} \cdot \frac{3m + 2}{3m + 4}$ Factor.

$= \frac{(m - 2)(3m + 4)(3m + 2)}{(m + 4)(3m + 2)(3m + 4)}$ Multiply fractions.

$= \frac{m - 2}{m + 4}$ Lowest terms

(c) $$\frac{3p^2+11p-4}{24p^3-8p^2} \div \frac{9p+36}{24p^4-36p^3}$$

$$= \frac{(p+4)(3p-1)}{8p^2(3p-1)} \div \frac{9(p+4)}{12p^3(2p-3)} \qquad \text{Factor.}$$

$$= \frac{(p+4)(3p-1)}{8p^2(3p-1)} \cdot \frac{12p^3(2p-3)}{9(p+4)} \qquad \text{Multiply by the reciprocal of the divisor.}$$

$$= \frac{12p^3(2p-3)}{9 \cdot 8p^2} \qquad \text{Divide out common factors. Multiply fractions.}$$

$$= \frac{3 \cdot 4 \cdot p^2 \cdot p(2p-3)}{3 \cdot 3 \cdot 4 \cdot 2 \cdot p^2} \qquad \text{Factor.}$$

$$= \frac{p(2p-3)}{6} \qquad \text{Lowest terms}$$

(d) $$\frac{x^3-y^3}{x^2-y^2} \cdot \frac{2x+2y+xz+yz}{2x^2+2y^2+zx^2+zy^2}$$

$$= \frac{(x-y)(x^2+xy+y^2)}{(x+y)(x-y)} \cdot \frac{2(x+y)+z(x+y)}{2(x^2+y^2)+z(x^2+y^2)} \qquad \text{Factor. Group terms and factor.}$$

$$= \frac{(x-y)(x^2+xy+y^2)}{(x+y)(x-y)} \cdot \frac{(x+y)(2+z)}{(x^2+y^2)(2+z)} \qquad \text{Factor by grouping.}$$

$$= \frac{x^2+xy+y^2}{x^2+y^2} \qquad \text{Divide out common factors. Multiply fractions.}$$

✔ **Now Try Exercises 33, 43, and 47.**

Addition and Subtraction We add and subtract rational expressions in the same way that we add and subtract fractions.

Addition and Subtraction

For fractions $\frac{a}{b}$ and $\frac{c}{d}$ $(b \neq 0, d \neq 0)$, the following hold.

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$$

That is, to add (or subtract) two fractions in practice, find their least common denominator (LCD) and change each fraction to one with the LCD as denominator. The sum (or difference) of their numerators is the numerator of their sum (or difference), and the LCD is the denominator of their sum (or difference).

Finding the Least Common Denominator (LCD)

Step 1 Write each denominator as a product of prime factors.

Step 2 Form a product of all the different prime factors. Each factor should have as exponent the *greatest* exponent that appears on that factor.

EXAMPLE 4 Adding or Subtracting Rational Expressions

Add or subtract, as indicated.

(a) $\frac{5}{9x^2} + \frac{1}{6x}$ **(b)** $\frac{y}{y-2} + \frac{8}{2-y}$

(c) $\frac{3}{(x-1)(x+2)} - \frac{1}{(x+3)(x-4)}$

SOLUTION

(a) $\frac{5}{9x^2} + \frac{1}{6x}$

Step 1 Write each denominator as a product of prime factors.

$$9x^2 = 3^2 \cdot x^2$$
$$6x = 2^1 \cdot 3^1 \cdot x^1$$

Step 2 For the LCD, form the product of all the prime factors, with each factor having the greatest exponent that appears on it.

Greatest exponent on 3 is 2. Greatest exponent on x is 2.

$$\text{LCD} = 2^1 \cdot 3^2 \cdot x^2$$
$$= 18x^2$$

Write the given expressions with this denominator, and then add.

$$\frac{5}{9x^2} + \frac{1}{6x}$$
$$= \frac{5 \cdot 2}{9x^2 \cdot 2} + \frac{1 \cdot 3x}{6x \cdot 3x} \quad \text{LCD} = 18x^2$$
$$= \frac{10}{18x^2} + \frac{3x}{18x^2} \quad \text{Multiply.}$$
$$= \frac{10 + 3x}{18x^2} \quad \text{Add the numerators.}$$

Always check to see that the answer is in lowest terms.

(b) $\frac{y}{y-2} + \frac{8}{2-y}$ We arbitrarily choose $y - 2$ as the LCD.

$$= \frac{y}{y-2} + \frac{8(-1)}{(2-y)(-1)} \quad \text{Multiply the second expression by } -1 \text{ in both the numerator and the denominator.}$$
$$= \frac{y}{y-2} + \frac{-8}{y-2} \quad \text{Simplify.}$$
$$= \frac{y-8}{y-2} \quad \text{Add the numerators.}$$

We could use $2 - y$ as the common denominator instead of $y - 2$.

$$\frac{y(-1)}{(y-2)(-1)} + \frac{8}{2-y}$$

Multiply the first expression by −1 in both the numerator and the denominator.

$$= \frac{-y}{2-y} + \frac{8}{2-y}$$

Simplify.

$$= \frac{8-y}{2-y}$$

This equivalent expression results.

(c) $$\frac{3}{(x-1)(x+2)} - \frac{1}{(x+3)(x-4)}$$

The LCD is $(x-1)(x+2)(x+3)(x-4)$.

$$= \frac{3(x+3)(x-4)}{(x-1)(x+2)(x+3)(x-4)} - \frac{1(x-1)(x+2)}{(x+3)(x-4)(x-1)(x+2)}$$

$$= \frac{3(x^2-x-12)-(x^2+x-2)}{(x-1)(x+2)(x+3)(x-4)}$$

Multiply in the numerators, and then subtract them.

Be careful with signs.

$$= \frac{3x^2-3x-36-x^2-x+2}{(x-1)(x+2)(x+3)(x-4)}$$

Distributive property

$$= \frac{2x^2-4x-34}{(x-1)(x+2)(x+3)(x-4)}$$

Combine like terms in the numerator.

✔ **Now Try Exercises 57, 63, and 69.**

CAUTION ***When subtracting fractions where the second fraction has more than one term in the numerator, as in Example 4(c), be sure to distribute the negative sign to each term.*** Use parentheses as in the second step to avoid an error.

Complex Fractions

The quotient of two rational expressions is a **complex fraction**. There are two methods for simplifying a complex fraction.

EXAMPLE 5 Simplifying Complex Fractions

Simplify each complex fraction. In part (b), use two methods.

(a) $$\frac{6-\frac{5}{k}}{1+\frac{5}{k}}$$

(b) $$\frac{\frac{a}{a+1}+\frac{1}{a}}{\frac{1}{a}+\frac{1}{a+1}}$$

SOLUTION

(a) Method 1 for simplifying uses the identity property for multiplication. We multiply both numerator and denominator by the LCD of all the fractions, k.

$$\frac{6-\frac{5}{k}}{1+\frac{5}{k}} = \frac{k\left(6-\frac{5}{k}\right)}{k\left(1+\frac{5}{k}\right)} = \frac{6k-k\left(\frac{5}{k}\right)}{k+k\left(\frac{5}{k}\right)} = \frac{6k-5}{k+5}$$

Distribute k to *all* terms within the parentheses.

EXAMPLE 5 Applying the Simple Interest Formula

A woman borrowed \$5240 for new furniture. She will pay it off in 11 months at an annual simple interest rate of 4.5%. How much interest will she pay?

SOLUTION Use the simple interest formula $I = Prt$.

$$I = 5240(0.045)\left(\frac{11}{12}\right) = \$216.15 \qquad P = 5240,\ r = 0.045,\ \text{and } t = \tfrac{11}{12} \text{ (year)}$$

She will pay \$216.15 interest on her purchase.

✓ Now Try Exercise 59.

1.1 Exercises

CONCEPT PREVIEW *Fill in the blank to correctly complete each sentence.*

1. A(n) ________ is a statement that two expressions are equal.

2. To ________ an equation means to find all numbers that make the equation a true statement.

3. A linear equation is a(n) ________ because the greatest degree of the variable is 1.

4. A(n) ________ is an equation satisfied by every number that is a meaningful replacement for the variable.

5. A(n) ________ is an equation that has no solution.

CONCEPT PREVIEW *Decide whether each statement is* true *or* false.

6. The solution set of $2x + 5 = x - 3$ is $\{-8\}$.

7. The equation $5(x - 8) = 5x - 40$ is an example of an identity.

8. The equation $5x = 4x$ is an example of a contradiction.

9. Solving the literal equation $A = \frac{1}{2}bh$ for the variable h gives $h = \frac{A}{2b}$.

10. CONCEPT PREVIEW Which one is *not* a linear equation?

A. $5x + 7(x - 1) = -3x$

B. $9x^2 - 4x + 3 = 0$

C. $7x + 8x = 13x$

D. $0.04x - 0.08x = 0.40$

Solve each equation. ***See Examples 1 and 2.***

11. $5x + 4 = 3x - 4$

12. $9x + 11 = 7x + 1$

13. $6(3x - 1) = 8 - (10x - 14)$

14. $4(-2x + 1) = 6 - (2x - 4)$

15. $\frac{5}{6}x - 2x + \frac{4}{3} = \frac{5}{3}$

16. $\frac{7}{4} + \frac{1}{5}x - \frac{3}{2} = \frac{4}{5}x$

17. $3x + 5 - 5(x + 1) = 6x + 7$

18. $5(x + 3) + 4x - 3 = -(2x - 4) + 2$

19. $2[x - (4 + 2x) + 3] = 2x + 2$

20. $4[2x - (3 - x) + 5] = -6x - 28$

21. $\frac{1}{14}(3x - 2) = \frac{x + 10}{10}$

22. $\frac{1}{15}(2x + 5) = \frac{x + 2}{9}$

23. $0.2x - 0.5 = 0.1x + 7$

24. $0.01x + 3.1 = 2.03x - 2.96$

25. $-4(2x - 6) + 8x = 5x + 24 + x$

26. $-8(3x + 4) + 6x = 4(x - 8) + 4x$

27. $0.5x + \frac{4}{3}x = x + 10$

28. $\frac{2}{3}x + 0.25x = x + 2$

29. $0.08x + 0.06(x + 12) = 7.72$

30. $0.04(x - 12) + 0.06x = 1.52$

Determine whether each equation is an identity, *a* conditional equation, *or a* contradiction. *Give the solution set.* ***See Example 3.***

31. $4(2x + 7) = 2x + 22 + 3(2x + 2)$

32. $\frac{1}{2}(6x + 20) = x + 4 + 2(x + 3)$

33. $2(x - 8) = 3x - 16$

34. $-8(x + 5) = -8x - 5(x + 8)$

35. $4(x + 7) = 2(x + 12) + 2(x + 1)$

36. $-6(2x + 1) - 3(x - 4) = -15x + 1$

37. $0.3(x + 2) - 0.5(x + 2) = -0.2x - 0.4$

38. $-0.6(x - 5) + 0.8(x - 6) = 0.2x - 1.8$

Solve each formula for the specified variable. Assume that the denominator is not 0 if variables appear in the denominator. ***See Examples 4(a) and (b).***

39. $V = lwh$, for l (volume of a rectangular box)

40. $I = Prt$, for P (simple interest)

41. $P = a + b + c$, for c (perimeter of a triangle)

42. $P = 2l + 2w$, for w (perimeter of a rectangle)

43. $\mathcal{A} = \frac{1}{2}h(B + b)$, for B (area of a trapezoid)

44. $\mathcal{A} = \frac{1}{2}h(B + b)$, for h (area of a trapezoid)

45. $S = 2\pi rh + 2\pi r^2$, for h (surface area of a right circular cylinder)

46. $s = \frac{1}{2}gt^2$, for g (distance traveled by a falling object)

47. $S = 2lw + 2wh + 2hl$, for h (surface area of a rectangular box)

48. $z = \frac{x - \mu}{\sigma}$, for x (standardized value)

Solve each equation for x. ***See Example 4(c).***

49. $2(x - a) + b = 3x + a$

50. $5x - (2a + c) = 4(x + c)$

51. $ax + b = 3(x - a)$

52. $4a - ax = 3b + bx$

53. $\frac{x}{a - 1} = ax + 3$

54. $\frac{x - 1}{2a} = 2x - a$

55. $a^2x + 3x = 2a^2$

56. $ax + b^2 = bx - a^2$

57. $3x = (2x - 1)(m + 4)$

58. $-x = (5x + 3)(3k + 1)$

Simple Interest *Work each problem.* ***See Example 5.***

59. Elmer borrowed \$3150 from his brother Julio to pay for books and tuition. He agreed to repay Julio in 6 months with simple annual interest at 4%.

(a) How much will the interest amount to?

(b) What amount must Elmer pay Julio at the end of the 6 months?

60. Levada borrows \$30,900 from her bank to open a florist shop. She agrees to repay the money in 18 months with simple annual interest of 5.5%.

(a) How much must she pay the bank in 18 months?

(b) How much of the amount in part (a) is interest?

Celsius and Fahrenheit Temperatures In the metric system of weights and measures, temperature is measured in degrees Celsius (°C) instead of degrees Fahrenheit (°F). To convert between the two systems, we use the equations

$$C = \frac{5}{9}(F - 32) \quad \text{and} \quad F = \frac{9}{5}C + 32.$$

In each exercise, convert to the other system. Round answers to the nearest tenth of a degree if necessary.

61. 20°C

62. 200°C

63. 50°F

64. 77°F

65. 100°F

66. 350°F

Work each problem. Round to the nearest tenth of a degree, if necessary.

67. ***Temperature of Venus*** Venus is the hottest planet, with a surface temperature of 867°F. What is this temperature in Celsius? (*Source: World Almanac and Book of Facts.*)

68. ***Temperature at Soviet Antarctica Station*** A record low temperature of -89.4°C was recorded at the Soviet Antarctica Station of Vostok on July 21, 1983. Find the corresponding Fahrenheit temperature. (*Source: World Almanac and Book of Facts.*)

69. ***Temperature in South Carolina*** A record high temperature of 113°F was recorded for the state of South Carolina on June 29, 2012. What is the corresponding Celsius temperature? (*Source:* U.S. National Oceanic and Atmospheric Administration.)

70. ***Temperature in Haiti*** The average annual temperature in Port-au-Prince, Haiti, is approximately 28.1°C. What is the corresponding Fahrenheit temperature? (*Source:* www.haiti.climatemps.com)

1.2

Applications and Modeling with Linear Equations

- **Solving Applied Problems**
- **Geometry Problems**
- **Motion Problems**
- **Mixture Problems**
- **Modeling with Linear Equations**

Solving Applied Problems One of the main reasons for learning mathematics is to be able use it to solve application problems. While there is no one method that enables us to solve all types of applied problems, the following six steps provide a useful guide.

Solving an Applied Problem

Step 1 **Read** the problem carefully until you understand what is given and what is to be found.

Step 2 **Assign a variable** to represent the unknown value, using diagrams or tables as needed. Write down what the variable represents. If necessary, express any other unknown values in terms of the variable.

Step 3 **Write an equation** using the variable expression(s).

Step 4 **Solve** the equation.

Step 5 **State the answer** to the problem. Does it seem reasonable?

Step 6 **Check** the answer in the words of the original problem.

A linear equation is a **first-degree equation** because the greatest degree of the variable is 1.

$$3x + \sqrt{2} = 0, \quad \frac{3}{4}x = 12, \quad 0.5(x+3) = 2x - 6 \qquad \text{Linear equations}$$

$$\sqrt{x} + 2 = 5, \quad \frac{1}{x} = -8, \quad x^2 + 3x + 0.2 = 0 \qquad \text{Nonlinear equations}$$

EXAMPLE 1 Solving a Linear Equation

Solve $3(2x-4) = 7 - (x+5)$.

SOLUTION

$$\begin{aligned}
3(2x-4) &= 7-(x+5) && \\
6x - 12 &= 7 - x - 5 && \text{Distributive property} \\
6x - 12 &= 2 - x && \text{Combine like terms.} \\
6x - 12 + x &= 2 - x + x && \text{Add } x \text{ to each side.} \\
7x - 12 &= 2 && \text{Combine like terms.} \\
7x - 12 + 12 &= 2 + 12 && \text{Add 12 to each side.} \\
7x &= 14 && \text{Combine like terms.} \\
\frac{7x}{7} &= \frac{14}{7} && \text{Divide each side by 7.} \\
x &= 2 &&
\end{aligned}$$

Be careful with signs.

CHECK

A check of the solution is recommended.

$$\begin{aligned}
3(2x-4) &= 7-(x+5) && \text{Original equation} \\
3(2\cdot 2-4) &\stackrel{?}{=} 7-(2+5) && \text{Let } x = 2. \\
3(4-4) &\stackrel{?}{=} 7-(7) && \text{Work inside the parentheses.} \\
0 &= 0 \ \checkmark && \text{True}
\end{aligned}$$

Replacing x with 2 results in a true statement, so 2 is a solution of the given equation. The solution set is $\{2\}$.

✔ Now Try Exercise 13.

EXAMPLE 2 Solving a Linear Equation with Fractions

Solve $\frac{2x+4}{3} + \frac{1}{2}x = \frac{1}{4}x - \frac{7}{3}$.

SOLUTION

$$\begin{aligned}
\frac{2x+4}{3} + \frac{1}{2}x &= \frac{1}{4}x - \frac{7}{3} && \\
12\left(\frac{2x+4}{3} + \frac{1}{2}x\right) &= 12\left(\frac{1}{4}x - \frac{7}{3}\right) && \text{Multiply by 12, the LCD of the fractions.} \\
12\left(\frac{2x+4}{3}\right) + 12\left(\frac{1}{2}x\right) &= 12\left(\frac{1}{4}x\right) - 12\left(\frac{7}{3}\right) && \text{Distributive property} \\
4(2x+4) + 6x &= 3x - 28 && \text{Multiply.} \\
8x + 16 + 6x &= 3x - 28 && \text{Distributive property} \\
14x + 16 &= 3x - 28 && \text{Combine like terms.} \\
11x &= -44 && \text{Subtract } 3x. \text{ Subtract 16.} \\
x &= -4 && \text{Divide each side by 11.}
\end{aligned}$$

Distribute to *all* terms within the parentheses.

CHECK

$$\begin{aligned} \frac{2x+4}{3} + \frac{1}{2}x &= \frac{1}{4}x - \frac{7}{3} && \text{Original equation} \\ \frac{2(-4)+4}{3} + \frac{1}{2}(-4) &\stackrel{?}{=} \frac{1}{4}(-4) - \frac{7}{3} && \text{Let } x = -4. \\ \frac{-4}{3} + (-2) &\stackrel{?}{=} -1 - \frac{7}{3} && \text{Simplify on each side.} \\ -\frac{10}{3} &= -\frac{10}{3} \checkmark && \text{True} \end{aligned}$$

The solution set is $\{-4\}$. ✔ **Now Try Exercise 21.**

Identities, Conditional Equations, and Contradictions An equation satisfied by every number that is a meaningful replacement for the variable is an **identity.**

$$3(x+1) = 3x + 3 \quad \text{Identity}$$

An equation that is satisfied by some numbers but not others is a **conditional equation.**

$$2x = 4 \quad \text{Conditional equation}$$

The equations in **Examples 1 and 2** are conditional equations. An equation that has no solution is a **contradiction.**

$$x = x + 1 \quad \text{Contradiction}$$

EXAMPLE 3 Identifying Types of Equations

Determine whether each equation is an *identity*, a *conditional equation*, or a *contradiction.* Give the solution set.

(a) $-2(x+4) + 3x = x - 8$ **(b)** $5x - 4 = 11$ **(c)** $3(3x-1) = 9x + 7$

SOLUTION

(a) $-2(x+4) + 3x = x - 8$

$$\begin{aligned} -2x - 8 + 3x &= x - 8 && \text{Distributive property} \\ x - 8 &= x - 8 && \text{Combine like terms.} \\ 0 &= 0 && \text{Subtract } x. \text{ Add 8.} \end{aligned}$$

When a *true* statement such as $0 = 0$ results, the equation is an identity, and the solution set is **{all real numbers}.**

(b) $5x - 4 = 11$

$$\begin{aligned} 5x &= 15 && \text{Add 4 to each side.} \\ x &= 3 && \text{Divide each side by 5.} \end{aligned}$$

This is a conditional equation, and its solution set is $\{3\}$.

(c) $3(3x-1) = 9x + 7$

$$\begin{aligned} 9x - 3 &= 9x + 7 && \text{Distributive property} \\ -3 &= 7 && \text{Subtract } 9x. \end{aligned}$$

When a *false* statement such as $-3 = 7$ results, the equation is a contradiction, and the solution set is the **empty set,** or **null set,** symbolized $\varnothing$.

✔ **Now Try Exercises 31, 33, and 35.**

Identifying Types of Linear Equations

1. If solving a linear equation leads to a true statement such as $0 = 0$, the equation is an **identity.** Its solution set is **{all real numbers}. (See Example 3(a).)**
2. If solving a linear equation leads to a single solution such as $x = 3$, the equation is **conditional.** Its solution set consists of a single element. **(See Example 3(b).)**
3. If solving a linear equation leads to a false statement such as $-3 = 7$, the equation is a **contradiction.** Its solution set is $\varnothing$. **(See Example 3(c).)**

Solving for a Specified Variable (Literal Equations) A formula is an example of a **literal equation** (an equation involving letters).

EXAMPLE 4 Solving for a Specified Variable

Solve each formula or equation for the specified variable.

(a) $I = Prt$, for t **(b)** $A - P = Prt$, for P

(c) $3(2x - 5a) + 4b = 4x - 2$, for x

SOLUTION

(a) This is the formula for **simple interest** I on a principal amount of P dollars at an annual interest rate r for t years. To solve for t, we treat t as if it were the only variable, and the other variables as if they were constants.

$$I = Prt \quad \text{Goal: Isolate } t \text{ on one side.}$$

$$\frac{I}{Pr} = \frac{Prt}{Pr} \qquad \text{Divide each side by } Pr.$$

$$\frac{I}{Pr} = t, \quad \text{or} \quad t = \frac{I}{Pr}$$

(b) The formula $A = P(1 + rt)$, which can also be written $A - P = Prt$, gives the **future value,** or **maturity value,** A of P dollars invested for t years at annual simple interest rate r.

$$A - P = Prt \quad \text{Goal: Isolate } P\text{, the specified variable.}$$

$$A = P + Prt \qquad \text{Transform so that all terms involving } P \text{ are on one side.}$$

Pay close attention to this step.

$$A = P(1 + rt) \qquad \text{Factor out } P.$$

$$\frac{A}{1 + rt} = P, \quad \text{or} \quad P = \frac{A}{1 + rt} \qquad \text{Divide by } 1 + rt.$$

(c)

$$3(2x - 5a) + 4b = 4x - 2 \qquad \text{Solve for } x.$$

$$6x - 15a + 4b = 4x - 2 \qquad \text{Distributive property}$$

$$6x - 4x = 15a - 4b - 2 \qquad \text{Isolate the } x\text{-terms on one side.}$$

$$2x = 15a - 4b - 2 \qquad \text{Combine like terms.}$$

$$x = \frac{15a - 4b - 2}{2} \qquad \text{Divide each side by 2.}$$

✓ **Now Try Exercises 39, 47, and 49.**