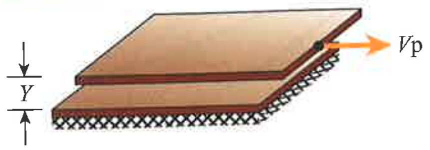
Example 3.5



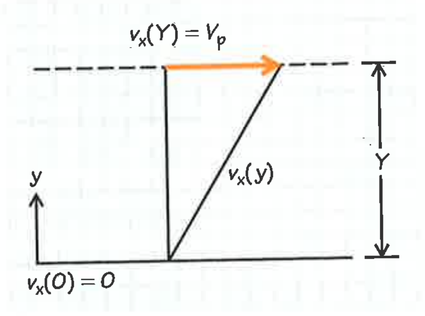
Consider a steady (i.e., time-invariant), linear velocity distribution for flow between two plates in which the lower plate is stationary and the upper plate moves with a known velocity V_p. The distance between the plates is Y. Determine the mass flow rate per unit flow depth (Z) for motor oil at 300 K with $V_p = 0.8$ m/s and $Y = 0.5$ mm.

Solution

Known Linear $v_x(y)$, V_p, Y

Find $\dot{m}/Z$

Sketch



Assumptions

i. One-dimensional flow
ii. Constant density

Analysis Since the velocity profile is linear in the y-direction [i.e., $v_x(y) = ay$] the slope a is simply

$$a = \frac{v_x(Y) - v_x(0)}{Y}$$

$$= \frac{V_p - 0}{Y} = \frac{V_p}{Y}$$

$$= \frac{0.8 \text{ m/s}}{0.0005 \text{ m}} = 1{,}600 \text{ s}^{-1}.$$

We now apply Eq. 3.9b in a straightforward fashion:

$$\dot{m} = \int_0^Y \rho v_x(y) Z dy$$

$$= \frac{\rho V_p Z}{Y} \int_0^Y y dy$$

$$= \frac{\rho V_p Z Y}{2}.$$

With the oil density (884.1 kg/m³) from Appendix G, we numerically evaluate this as

$$\frac{\dot{m}}{Z} = \frac{884.1(0.8)0.0005}{2} = 0.1768$$

$$[=](\text{kg/m}^3)(\text{m/s})\text{m} = \frac{\text{kg/s}}{\text{m}}.$$

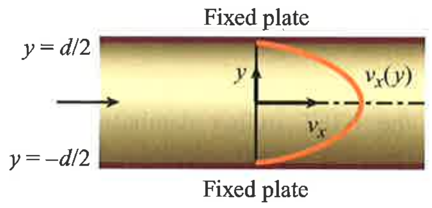
Comment This flow, known as Couette flow, has application to fluid-film bearings. If the radius of curvature of the bearing surface is large compared to the fluid-film thickness, then a one-dimensional Cartesian system, as employed in this example, can be used to model the flow.

Self Test 3.2

☑ **Consider the solution of Example 3.5 when applied to a 1-m depth ($Z = 1$ m). Does the flow rate of 0.1768 kg/s violate the steady-state assumption?**

(Answer: No, it does not. Although the flow is per unit time, it does not depend on time (i.e., is steady state). This flow will always be 0.1768 kg/s no matter when it is observed.)

Example 3.6



Consider a steady flow of water at 320 K and 2 atm between two fixed, parallel plates having a separation d of 1 mm. As illustrated in the sketch, the velocity profile $v_x(y)$ is parabolic, with a maximum value of a at the centerline ($y = 0$) and zero values at the surface of each plate ($y = \pm d/2$):

$$v_x(y) = a\left[1 - 4\left(\frac{y}{d}\right)^2\right] \quad [=]\text{m/s}.$$

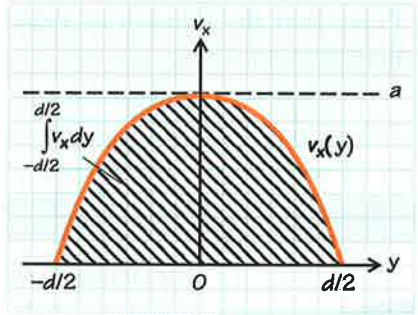
Determine the mass flow rate of the water per unit flow depth z when a is 0.2 m/s.

Solution

Known $v_x(y)$, a, d, P, T

Find $\dot{m}/z$

Sketch



Assumptions

i. One-dimensional flow
ii. Constant density

Analysis Recognizing the Cartesian geometry, we apply Eq. 3.9 as follows:

$$\dot{m} = \int_{-d/2}^{d/2} \rho v_x(y) Z dy.$$

Substituting the given velocity distribution, taking advantage of the symmetry of the integral, and removing constants from the integrand, we obtain

$$\dot{m} = 2\rho Z a \int_0^{d/2} \left[1 - 4\left(\frac{y}{d}\right)^2\right] dy.$$

Performing the integration yields

$$\dot{m} = 2\rho Z a \left[y - \frac{4y^3}{3d^2}\right]_0^{d/2},$$

and evaluating the limits results in

$$\dot{m} = 2\rho Z a \left[\frac{d}{2} - \frac{4d^3}{24d^2}\right] = \frac{2\rho Z a d}{3},$$

or

$$\frac{\dot{m}}{Z} = \frac{2\rho a d}{3}.$$

Using a value for the density from the NIST online database and substituting numerical values give

$$\frac{\dot{m}}{Z} = \frac{2(989.5)0.2(0.001)}{3} = 0.132$$

$$[=](\text{kg/m}^3)(\text{m/s})\text{m} = \frac{\text{kg/s}}{\text{m}}.$$

Comment Note the implicit use of the equation of state $\rho = \rho(T, P)$ in our use of the NIST database to obtain a value for the density.

Self Test 3.3 ☑ **Prove that the solution to Example 3.6 obeys the no-slip law at the wall and has a maximum at $y = 0$.**

Example 3.7

For laminar flow in circular tube or pipe, the velocity distribution obeys the following parabolic form:

$$v_x(r) = a\left(1 - \frac{r^2}{R^2}\right),$$

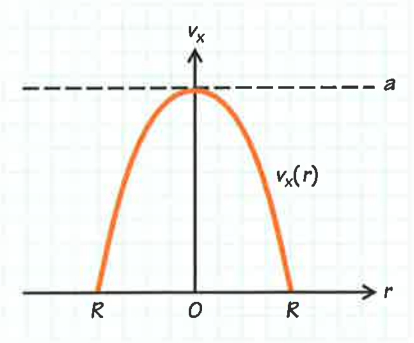
where r is the radial distance from the tube center and R is the inside radius of the tube or pipe. Determine an algebraic expression for the mass flow rate for a constant-density fluid.

Solution

Known Parabolic $v_x(r)$, ρ

Find $\dot{m}$

Sketch



Assumptions

i. Laminar flow
ii. Constant density

Analysis The straightforward integration of Eq. 3.9c is all that is required here. Starting with

$$\dot{m} = 2\pi\rho \int_0^R v_x(r) r dr,$$

we substitute $v_x(r)$ to obtain

$$\dot{m} = 2\pi\rho \int_0^R a\left(1 - \frac{r^2}{R^2}\right) r dr.$$

Integrating yields

$$\dot{m} = 2\pi\rho a \left[\frac{r^2}{2} - \frac{r^4}{4R^2}\right]_0^R.$$

Substitution of the limits generates our final result:

$$\dot{m} = \pi\rho a R^2/2.$$

Comment Rearranging our result to $\dot{m} = \rho(a/2)\pi R^2$, we notice that the flow rate is the product of the density, one-half of the centerline velocity (i.e., $a/2$), and the tube cross-sectional area (πR^2). From this we recognize that $a/2$ must be the area-weighted average velocity, as discussed in the next section.

Example 3.8

In steady turbulent flows through circular tubes, the following power law [3] approximates the velocity profile:

$$v_x(r) = a\left(1 - \frac{r}{R}\right)^{1/n},$$

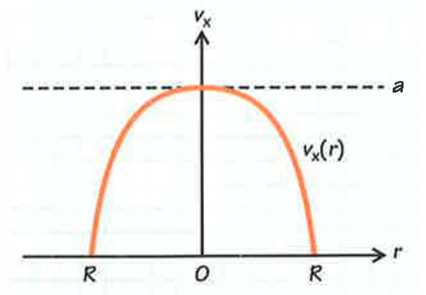
where n is an integer ranging between 6 and 10, depending on the flow conditions. Assuming a uniform density, find an expression for the mass flow rate for $n = 7$. Also draw a graph of $v_x(r)$ for this particular power law and compare it with the parabolic distribution from Example 3.7.

Solution

Known $v_x(r)$, n, uniform ρ

Find $\dot{m}$

Sketch



Assumptions

i. Steady flow
ii. Uniform density

Analysis To find an expression for $\dot{m}$, we directly apply Eq. 3.9c:

$$\dot{m} = \int_0^R \rho a \left(1 - \frac{r}{R}\right)^{1/n} 2\pi r dr.$$

We can put this in a standard form by defining

$$X = r/R,$$

so

$$dX = dr/R.$$

Substituting these into Eq. 3.9c and removing constants from the integrand yields

$$\dot{m} = 2\rho a \pi R^2 \int_0^1 (1 - X)^{1/n} X dX.$$

The solution to this integral, which can be found in standard integral tables (e.g., Ref. [4]), is

$$\int_0^1 (1 - X)^{1/n} X dX = \left[\frac{1}{\left(\frac{1}{n} + 2\right)} (1 - X)^{\frac{1}{n}+2} - \frac{1}{\left(\frac{1}{n} + 1\right)} (1 - X)^{\frac{1}{n}+1} \right]_0^1.$$

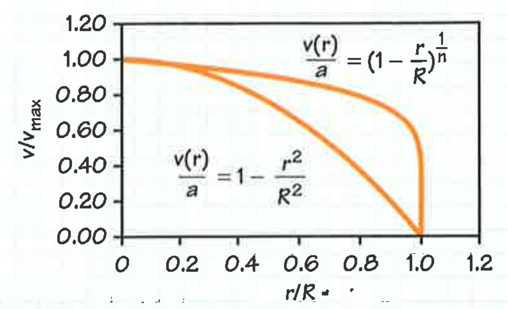
Evaluating the limits and substituting the result back into our expression for $\dot{m}$ yields

$$\dot{m} = \left(\frac{2n^2}{(n + 1)(2n + 1)}\right) \rho a \pi R^2.$$

For our particular case with $n = 7$,

$$\dot{m} = \frac{49}{60} \rho a \pi R^2.$$

The second part of the problem is easily solved using spreadsheet software. Graphical results are as follows:



Comments From the graph, we see that the velocity profile for turbulent flow in a tube is much flatter than the parabolic profile for laminar flows. The steep velocity gradient at the tube wall ($r/R = 1$) has implications for frictional effects and pressure losses in tube and pipe flows. These concepts are developed in Chapter 10.

Self Test 3.4

☑ **Consider the expression for the mass flow rate derived in Example 3.8 and compare it with that of Example 3.7. Which one would have the greatest average velocity?**

(Answer: For the same value of $a = v_{max}$, the turbulent flow has the greater average velocity.)

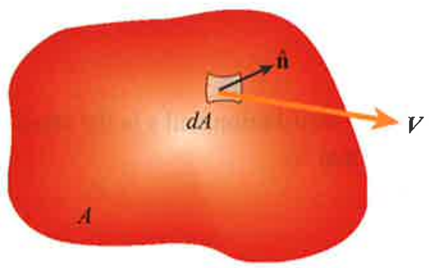


FIGURE 3.6
Arbitrary flow through arbitrary area defined by the variation of velocity vector V and area unit normal vector n̂ over the area A.

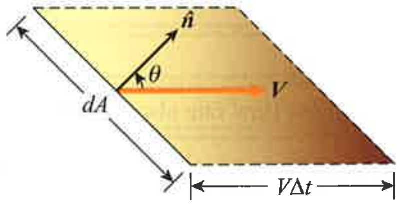


FIGURE 3.7
Two-dimensional representation of flow of unit depth with velocity V through the differential area dA. The volume of fluid passing through dA in a time interval Δt is the volume of the parallelpiped V Δt dA cos θ.

Generalized Definition

A flow rate can be defined for an arbitrary velocity distribution associated with an arbitrary flow area. Consider the velocity vector $\boldsymbol{V}$ associated with flow through the surface A. Surface A may have any shape. The particular shape is defined by specifying the direction of the unit normal at each location on the surface, as suggested in Fig. 3.6. The volume of fluid passing through the differential surface element in a time interval Δt is equal to the product of the projected area of dA in the direction of $\boldsymbol{V}$ (i.e., $dA \cos\theta$) and the distance traveled by a fluid element in time Δt (i.e., $V\,\Delta t$), so

$$\Delta \mathcal{V} = dA \cos\theta\, V\, \Delta t.$$

The flow rate through dA is then

$$\frac{\Delta \mathcal{V}}{\Delta t} = V\, dA \cos\theta,$$

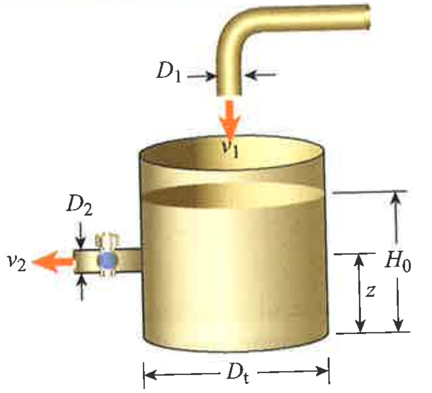
or

$$\frac{\Delta \mathcal{V}}{\Delta t} = (\boldsymbol{V} \cdot \hat{\boldsymbol{n}}) dA.$$

These relationships can be more easily visualized by considering the two-dimensional differential area shown in Fig. 3.7, rather than the three-dimensional situation presented in Fig. 3.6. In Fig. 3.7, we see that the volume of fluid passing through dA in the time Δt is a parallelpiped of unit depth (perpendicular to the page) as indicated by the dashed line. To obtain the total occurs in the filling of an air-compressor storage tank (Fig. 3.10c). In this case, the control volume remains fixed, whereas the density of the air in the tank continually increases as air is pumped in. A common example in which both the volume and the density vary with time is the inflation of a rubber balloon (Fig. 3.10b).

We illustrate these concepts now with an example.

Example 3.12



Consider the tank and water supply system as shown in the sketch. The diameter of the supply pipe D_1 is 20 mm, and the average incoming velocity v_1 is 0.595 m/s. A shut-off valve is located at $z = 0.1$ m, and the exit pipe diameter D_2 is 10 mm. The tank diameter D_t is 0.3 m. The water density is 997 kg/m^3.

A. Determine the time to fill the tank to a depth of 1 m ($\equiv H_0$) assuming the tank is initially empty and the shut-off valve is closed. Neglect the volume associated with the short pipe connecting the tank to the shut-off valve.

B. At the instant the water level reaches $H_0 = 1$ m the shut-off valve is opened. The instantaneous average velocity of the outflow depends on the water depth above z, that is, $H(t) - z$, and is given by

$$v_2 = 0.85[g(H(t) - z)]^{1/2},$$

where g is the gravitational acceleration. Determine whether the tank continues to fill or begins to empty immediately after the valve is opened.

C. Determine the steady-state value of the water depth.

D. Determine the time required to achieve steady state after the valve is opened.

Solution

Known $D_1, D_2, D_t, z, H_0, v_1$, expression for v_2

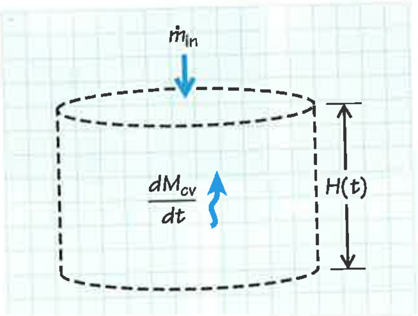
Find t_0, H_{ss}, t_{ss}

Sketch *See the sketches that follow for each part of the problem*

Assumptions

i. Incompressible flow
ii. Outlet velocity instantaneously adjusts to changes in H

Analysis (Part A) For this part, we select an expanding control volume that contains all of the water in the tank as shown in the sketch.



Conservation of mass for the unsteady filling process is expressed by Eq. 3.19a, where $\dot{m}_{out} = 0$:

$$\dot{m}_{in} = \dot{m}_1 = \frac{dM_{cv}}{dt}.$$

The inlet mass flow rate is expressed as (Eq. 3.15)

$$\dot{m}_1 = \rho v_1 A_1,$$

and the instantaneous mass within the control volume is simply the product of the density and the instantaneous volume of water within the tank, that is,

$$M_{cv} = \rho \mathcal{V}(t) = \rho A_t H(t),$$

where $A_t (= \pi D_t^2/4)$ is the cross-sectional area of the tank. Substituting these expressions for $\dot{m}_1$ and M_{cv} into Eq. 3.19a yields

$$\rho A_t \frac{dH(t)}{dt} = \rho v_1 A_1.$$

This first-order, ordinary differential equation is easily integrated from the initial condition $H(t = 0) = 0$ to $H(t_0) = H_0$ as follows:

$$\int_0^{H_0} dH(t) = \int_0^{t_0} v_1 \frac{A_1}{A_t} dt$$

$$H_0 = v_1 \frac{A_1}{A_t} t_0.$$

Solving for the unknown t_0 yields

$$t_0 = \frac{H_0 A_t}{v_1 A_1}.$$

Since the ratio A_t/A_1 is the ratio D_t^2/D_1^2, we evaluate this as

$$t_0 = \frac{H_0 D_t^2}{v_1 D_1^2}$$

$$= \frac{1.0(0.3)^2}{0.595(0.020)^2}\ \text{s}$$

$$= 378\ \text{s or } 6.30\ \text{min}.$$

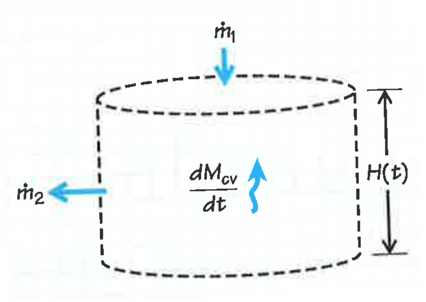
Analysis (Part B) To determine whether the water level continues to increase or begins to fall when the valve is opened, we need to know whether the mass in the tank is increasing or decreasing since

$$\frac{dM_{cv}}{dt} = \rho A_t \frac{dH(t_0)}{dt}.$$

To determine this, we apply the conservation of mass principle,

$$\dot{m}_1 - \dot{m}_2 = \frac{dM_{cv}}{dt},$$

for the control volume sketched here:



The instantaneous outlet mass flow rate $\dot m_2$ is expressed as

$$\dot m_2 = \rho v_2 A_2$$
$$= \rho\left(0.85[g(H(t_0) - z)]^{1/2}\right)\frac{\pi D_2^2}{4}.$$

We evaluate $\dot m_2(t_0)$ and $\dot m_1(t_0)$ as follows:

$$\dot m_2 = 997\left[0.85[9.81(1.0 - 0.1)]^{1/2}\right]\frac{\pi(0.01)^2}{4} = 0.198$$
$$[=]\,\frac{\text{kg}}{\text{m}^3}\left[\left(\frac{\text{m}}{\text{s}^2}\right)\text{m}\right]^{1/2}\text{m}^2 = \text{kg/s}$$

and

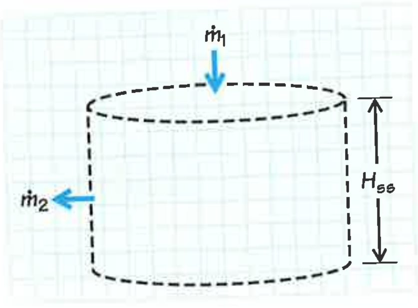
$$\dot m_1 = \rho v_1 \pi D_1^2/4$$
$$= 997(0.595)\pi(0.020)^2/4 \text{ kg/s}$$
$$= 0.186 \text{ kg/s}.$$

Thus,

$$\frac{dM_{cv}}{dt} = 0.186 - 0.198 \text{ kg/s}$$
$$= -0.012 \text{ kg/s},$$

where the negative sign indicates that the water level must start to fall when the valve is opened.

Analysis (Part C) For steady state, dM_{cv}/dt is zero. The appropriate sketch and mathematical expression for mass conservation are



and

$$\dot m_1 = \dot m_2.$$

The inlet mass flow rate is as previously calculated (0.186 kg/s), whereas $\dot m_2$ is expressed in terms of the steady-state water level H_{ss}. Thus,

$$\dot m_1 = \dot m_2 = \rho\left(0.85[g(H_{ss} - z)]^{1/2}\right)\frac{\pi D_2^2}{4}.$$

Solving for H_{ss} yields

$$H_{ss} = \frac{1}{g}\left(\frac{4\dot m_1}{0.85\rho\pi D_2^2}\right)^2 + z$$
$$= \frac{1}{9.81}\left(\frac{4(0.186)}{0.85(997)\pi(0.010)^2}\right)^2 + 0.1$$
$$= 0.90$$
$$[=]\,\frac{1}{(\text{m/s}^2)}\left(\frac{\text{kg/s}}{(\text{kg/m}^3)\text{m}^2}\right)^2 = \text{m}.$$

Analysis (Part D) From the time that the valve is opened until steady state is achieved, conservation of mass is expressed as in part B, except that H is a function of t, rather than being a fixed value, that is,

$$\dot m_1 - \rho\left(0.85[g(H(t) - z)]^{1/2}\right)A_2 = \frac{dM_{cv}}{dt}.$$

From the geometry, we also know that

$$\frac{dM_{cv}}{dt} = \rho A_t \frac{dH(t)}{dt}.$$

Combining these two equations and solving for $dH(t)/dt$ yield

$$\frac{dH(t)}{dt} = \frac{\dot m_1}{\rho A_t} - \frac{0.85}{A_t}[g(H(t) - z)]^{1/2} A_2.$$

Integration of this ordinary differential equation with the limits $H(t = t_0) = H_0$ and $H(t = t_{ss}) = H_{ss}$ enables us to find the desired time interval $\Delta t \equiv t_{ss} - t_0$. Separating the H and t variables yields

$$\frac{dH}{\frac{\dot m_1}{\rho A_t} - \frac{0.85}{A_t}[g(H - z)]^{1/2}A_2} = dt.$$

This can be expressed more compactly by defining

$$a \equiv \frac{-g(0.85)^2 A_2^2 z}{A_t^2} = \frac{-g(0.85)^2 z D_2^4}{D_t^4},$$
$$b \equiv \frac{g(0.85)^2 A_2^2}{A_t^2} = \frac{g(0.85)^2 D_2^4}{D_t^4},$$

and

$$c \equiv \frac{\dot m_1}{\rho A_t}.$$

Thus,

$$\frac{dH}{c - (a + bH)^{1/2}} = dt.$$

Integrating between our limits yields

$$\int_{H_0}^{H_{ss}} \frac{dH}{c - (a + bH)^{1/2}} = \int_{t_0}^{t_{ss}} dt = \Delta t.$$

Using the substitution $u \equiv c - (a + bH)^{1/2}$ makes the evaluation of this integral relatively straightforward. The final result is

$$\Delta t = \frac{-2c}{b} \ln\left[\frac{c - (a + bH_{ss})^{1/2}}{c - (a + bH_0)^{1/2}}\right] + \frac{2}{b}\left[(a + bH_0)^{1/2} - (a + bH_{ss})^{1/2}\right].$$

Substituting $H_0 = 1.0$ m and $H_{ss} = 0.90$ m, along with the numerical values

$$a = \frac{-9.81(0.85)^2\, 0.1(0.01)^4}{(0.3)^4} = -8.750 \times 10^{-7},$$

$$b = \frac{9.81(0.85)^2\,(0.01)^4}{(0.3)^4} = 8.750 \times 10^{-6},$$

and

$$c = \frac{0.186(4)}{997(0.3)^2\pi} = 2.639 \times 10^{-3},$$

yields to two significant digits

$$\Delta t = 2{,}000 \text{ s} \quad \text{or } 33 \text{ min}.$$

Comments This example illustrates the use of both steady and unsteady expressions of mass conservation for a control volume. Note that, in all parts of the problem, we chose a control surface that contained only the water in the tank; thus, our control volume expanded or contracted with time. An alternative, but more complex, choice would have been to choose a larger fixed volume containing some air above the water. Our original choice is clearly superior because of its simplicity. We also note that formulation of the unsteady problem generated an ordinary differential equation, a common result for this class of problems.

Self Test 3.8

☑ **A constant flow rate of 3 kg/s of water is entering a partially full bathtub measuring 6 ft by 2 ft. The drain plug is removed and at one instant the water level in the tub is decreasing at 0.5 in/min. Determine the exit mass flow rate at this instant.**

(Answer: 3.236 kg/s).

Many different control volumes can be selected to analyze this unsteady mid-air refueling process. Photograph courtesy of NASA.

To conclude this section, we present the most general integral form of conservation of mass:

$$\underbrace{-\int_{cs} \rho(\mathbf{V}_{\mathrm{rel}} \cdot \hat{\mathbf{n}})\,dA}_{\substack{\text{Net mass}\\ \text{flow across}\\ \text{the control}\\ \text{surface}}} = \underbrace{\frac{d}{dt}\left[\int_{\mathcal{V}} \rho\, d\mathcal{V}\right]}_{\substack{\text{Rate of}\\ \text{increase of}\\ \text{mass within the}\\ \text{control volume}}}, \tag{3.21}$$

where $\mathbf{V}_{\mathrm{rel}}$ is the local velocity relative to the control surface, that is, the velocity seen by an observer fixed to the control surface. Use of a relative velocity is required when the control surface is moving with respect to a fixed reference frame. A control system boundary may be moving because the entire control volume is in motion, or the control volume is being deformed, or a combination of these. The physical meaning here is the same as our more simple statements (Eqs. 3.19a and 3.19b); however, both the time rate of change of mass within the control volume and the net mass flow into the control volume are expressed in the most general way possible. For many engineering analyses, Eqs. 3.19a and 3.19b are the more useful starting points.

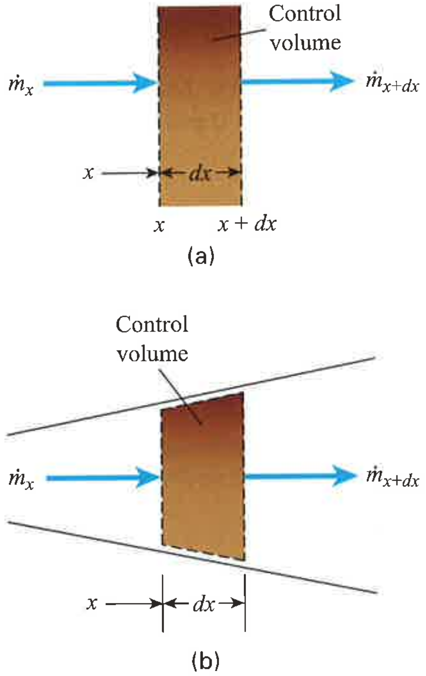


FIGURE 3.11
Control volumes for (a) one-dimensional unbounded flow and (b) one-dimensional bounded flow.

3.3f Differential Control Volumes

Steady-State, Steady Flow

We begin our analysis of differential control volumes with the one-dimensional case. We then extend this simple analysis to three-dimensional flows.

One-Dimensional Analysis Consider a differential control volume having a length dx (Fig. 3.11). For the situation depicted in Fig. 3.11a, the y- and z-coordinates of the flow domain extend indefinitely, and flow properties vary only in the x-direction [e.g., $v_x = v_x(x \text{ only})$]. Since we assume the flow is steady

$$\dot{m}_x = \dot{m}_{x+dx} = \text{constant}, \tag{3.22a}$$

or

$$(\rho v_x A)_x = (\rho v_x A)_{x+dx} = \text{constant}, \tag{3.22b}$$

where A is an arbitrary, but fixed, area perpendicular to the flow. Since A does not vary with x (Fig. 3.11a), we can rewrite Eq. 3.22b as

$$(\rho v_x)_x = (\rho v_x)_{x+dx}. \tag{3.23}$$

The quantity ρv_x, the mass flow rate per unit area, has a special meaning in fluid dynamics and is frequently known as the **mass flux**,

$$\dot{m}'' \equiv \rho v_x. \tag{3.24}$$

The mass flux has units of kg/s·m². For our special one-dimensional flow, then

$$\dot{m}'' = \text{constant}. \tag{3.25}$$

The usual expression of mass conservation for our differential control volume involves taking the spatial derivative, $d(\;)/dx$, of Eqs. 3.24 or 3.25; thus,

$$\frac{d\dot{m}''}{dx} = \frac{d(\rho v_x)}{dx} = 0. \tag{3.26}$$

Expanding this equation yields

$$\rho\frac{dv_x}{dx} + v_x\frac{d\rho}{dx} = 0. \tag{3.27}$$

Differential expressions of conservation of mass, such as Eq. 3.27, are frequently referred to as **continuity equations** or simply **continuity**.

Figure 3.11b depicts another one-dimensional flow. In this case, the flow area A is allowed to vary with x [i.e., $A = A(x)$]. Mass conservation as expressed by Eqs. 3.22a and 3.22b still holds. The mass flux, however, does

Examples 11.9–11.12 in Chapter 11 illustrate the use of the 1-D approximation to analyze compressible flows.

Example 12.10 in Chapter 12 relates the compressible-flow nozzle analysis to the performance of a jet engine.