where $V(=[v_x^2+v_y^2]^{1/2})$ is the magnitude of $\boldsymbol{V}$. Because ρ, v_x, and V are all uniform,

$$x\text{-momentum flow} = \hat{\imath}\rho v_x V\frac{\pi D^2}{4}.$$

Similarly, the y-component is

$$y\text{-momentum flow} = \hat{\jmath}\rho v_y V\frac{\pi D^2}{4}.$$

In both components, we recognize that $\rho V\pi D^2/4$ is $\dot{m}$, the mass flow rate through the area A, and from the geometry, that

$$v_x = V\cos\theta,$$
$$v_y = V\sin\theta.$$

Our final result is, thus,

$$x\text{-momentum flow} = \dot{m}v_x = \dot{m}V\cos\theta,$$
$$y\text{-momentum flow} = \dot{m}v_y = \dot{m}V\sin\theta,$$

with numerical values

$$x\text{-direction: } 2.05\cos 20^\circ\ \text{N} = 1.93\ \text{N},$$
$$y\text{-direction: } 2.05\sin 20^\circ\ \text{N} = 0.70\ \text{N}.$$

Comments From this example, we see that the momentum flux vector is simply $\dot{m}\boldsymbol{V}$ or $\dot{m}(\hat{\imath}v_x+\hat{\jmath}v_y+\hat{k}v_z)$ when the velocity and density are uniform over the region of interest.

Self Test 6.5

A 5-cm-diameter horizontal jet of water with velocity of 25 m/s strikes a curved deflector, which diverts the stream 90° upward. Determine the horizontal and vertical momentum flow through a control volume surrounding the deflector.

(Answer: x-momentum flow = −1227.2 N, y-momentum flow = 1227.2 N)

Example 6.9

As shown in Fig. 6.16c, water enters a long tube with a uniform velocity profile. At the tube exit, the velocity profile is parabolic and is expressed as

$$v_x(r) = v_{x,0}\left[1-\left(\frac{r}{R}\right)^2\right],$$

where $v_{x,0}$ is the centerline velocity and R is the tube radius. The flow rate through the tube is 8.3×10^{-3} kg/s and the water density is 997 kg/m^3. The tube diameter is 25 mm. Determine the momentum flow at the inlet and outlet of the tube.

Solution

Known $v_x(r)$, $\dot{m}$, ρ, D

Find Inlet and exit momentum flows

Sketch See Fig. 6.16c.

Assumptions

i. Constant ρ
ii. Steady state

Analysis At the inlet, the flow is essentially the same as in Example 6.8, except now the velocity vector and control surface normal vector are in opposite directions; thus,

$$\boldsymbol{V}\cdot\hat{\boldsymbol{n}} = v_{\text{inlet}}\cos 180^\circ = -v_{\text{inlet}}.$$

The inlet momentum flow is then

$$\int_A \boldsymbol{V}\rho(\boldsymbol{V}\cdot\hat{\boldsymbol{n}})dA = \hat{\imath}v_{\text{inlet}}(-\rho v_{\text{inlet}}A) = -\hat{\imath}\dot{m}v_{\text{inlet}}.$$

Since the inlet velocity is not given, we apply the definition of mass flow rate (Eq. 3.15) to obtain this quantity:

$$v_{\text{inlet}} = v_{\text{avg}} = \frac{\dot{m}}{\rho A_{\text{x-sec}}},$$

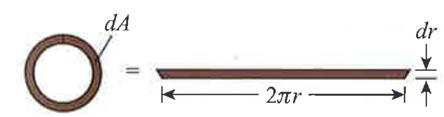
where we recognize that the uniform inlet velocity equals the average velocity. The value of v_{inlet} is

$$v_{\text{inlet}} = \frac{8.3\times10^{-3}}{997\pi(0.025)^2/4} = 0.017\ \text{m/s},$$

and the inlet momentum flow is

$$\int_A \boldsymbol{V}\rho(\boldsymbol{V}\cdot\hat{\boldsymbol{n}})dA = -\hat{\imath}(8.3\times10^{-3})0.017 = -\hat{\imath}\,1.41\times10^{-4}\ \text{N}.$$

At the outlet, the situation is more complex because the velocity is no longer uniform. Recognizing that $\boldsymbol{V}=\hat{\imath}v_x(r)$ and that $\boldsymbol{V}\cdot\hat{\boldsymbol{n}} = +v_x(r)$, we apply the definition of momentum flow (Eq. 6.50) as follows:



$$\int_A \boldsymbol{V}\rho(\boldsymbol{V}\cdot\hat{\boldsymbol{n}})dA = \int_0^R \hat{\imath}v_x(r)\rho v_x(r)2\pi r dr,$$

where $dA = 2\pi r dr$ for the axisymmetric geometry. Substituting the given expression for $v_x(r)$ and removing constant quantities from the integrand yield

$$\int_A \boldsymbol{V}\rho(\boldsymbol{V}\cdot\hat{\boldsymbol{n}})dA = \hat{\imath}2\pi\rho v_{x,0}^2\int_0^R\left[1-\left(\frac{r}{R}\right)^2\right]^2 r dr$$
$$= \hat{\imath}2\pi\rho v_{x,0}^2\int_0^R\left[1-2\left(\frac{r}{R}\right)^2+\left(\frac{r}{R}\right)^4\right] r dr$$
$$= \hat{\imath}2\pi\rho v_{x,0}^2\left[\frac{r^2}{2}-\frac{2}{R^2}\frac{r^4}{4}+\frac{1}{R^4}\frac{r^6}{6}\right]_0^R$$
$$= \hat{\imath}2\pi\rho v_{x,0}^2\left(\frac{R^2}{6}\right).$$

The only remaining task is to relate the centerline velocity $v_{x,0}$ to known quantities. We do this by applying mass conservation to the cylindrical control volume (Fig. 6.16c). With the assumption of steady state, Eq. 3.18a applies:

$$\dot{m}_{\text{inlet}} = \dot{m}_{\text{outlet}}$$

or

$$\rho v_{\text{inlet}} A = \rho v_{\text{avg,out}} A.$$

Thus, we find that

$$v_{\text{avg,out}} = v_{\text{inlet}}.$$

In Chapter 3, we determined that the centerline velocity $v_{x,0}$ is twice the average velocity for a parabolic velocity distribution. Thus,

$$v_{x,0} = 2v_{\text{avg,out}} = 2v_{\text{inlet}},$$

and the outlet momentum flux is easily evaluated as

$$\begin{aligned}\int \mathbf{V}\rho(\mathbf{V}\cdot\hat{\mathbf{n}})dA &= \hat{\imath}2\pi\rho(2v_{\text{avg,out}})^2R^2/6\\ &= \hat{\imath}\frac{4}{3}\rho v_{\text{avg, out}}^2\pi R^2\\ &= \hat{\imath}\frac{4}{3}\dot{m}v_{\text{avg,out}},\end{aligned}$$

recognizing that $\rho v_{\text{avg,out}}\pi R^2 = \dot{m}$. Numerically evaluating the exit momentum flow yields

$$\begin{aligned}\hat{\imath}\frac{4}{3}\dot{m}v_{\text{avg,out}} &= \hat{\imath}\frac{4}{3}8.3\times10^{-3}(0.017)\\ &= \hat{\imath}\frac{4}{3}(1.41\times10^{-4})\text{ N}\\ &= \hat{\imath}1.88\times10^{-4}\text{ N}.\end{aligned}$$

The reader should verify the units here.

Comments We see that the magnitude of the outlet momentum flux exceeds that of the inlet by the factor 4/3. This result requires that the contribution of the higher velocity in the central portion of the tube to the momentum flow more than compensates for the less-than-average velocity toward the tube walls. This observation also suggests that a correction factor (β) for nonuniform velocity distributions can be used to relate the actual momentum flow to that based on the average velocity, that is,

$$\int \mathbf{V}\rho(\mathbf{V}\cdot\hat{\mathbf{n}})dA \equiv \beta\dot{m}V_{\text{avg}}.$$

See Table 5.5 in Chapter 5.

This is similar to the kinetic energy correction factors defined in Chapter 5. We explore momentum flow correction factors later in this chapter.

Self Test 6.6

☑ **Redo Self Test 6.5 if the curved deflector turns the jet a full 180°, (i.e., opposite to the direction from which it came).**

(Answer: x-momentum flow = −2454.4 N, y-momentum flow = 0 N)

6.5 LINEAR MOMENTUM CONSERVATION FOR CONTROL VOLUMES

To maintain steady, level flight, the drag force is balanced by a net horizontal momentum flow from the aircraft's engines.

In this section, we restrict our analysis to nonaccelerating control volumes. This is equivalent to the requirement that we work with an inertial coordinate system, that is, a coordinate system that is fixed or moves at a constant velocity with respect to a stationary coordinate system. This restriction prevents us from analyzing a whole class of problems such as accelerating rockets; however, many interesting and challenging problems can still be treated with our restricted analysis. For the interested reader, the appendix to this chapter presents conservation of momentum expressions for noninertial (accelerating) coordinate systems.

6.5a Simplified General View

With the knowledge of how to express and calculate momentum flows, we are now able to write explicit momentum conservation statements for control volumes. In the following subsections, we develop the simplest statements and then add complexity. In all cases, we carefully define the restrictions that apply and state the momentum conservation principle with mathematical rigor. Before doing so, however, it is instructive to transform the generic statement of the control-volume conservation principles presented in Chapter 1 to a general statement of momentum conservation. Although lacking in detailed definition and rigor, this general statement is very useful for developing an understanding of the concept of momentum conservation applied to control volumes. Our subsequent developments will add the necessary rigor.

See Eq. 3.19 for mass conservation and Eq. 5.69 for energy conservation.

The transformation of the generic conservation principle to the specific case of momentum conservation parallels the development of the mass and energy conservation principles presented in Chapters 3 and 5, respectively. Applying the conservation of momentum principle to a control volume, however, is inherently more complex than applying either the conservation of mass or energy principles because forces and momentum flows are vectors. We begin with Eq. 1.2, the rate form of the generic conservation principle:

$$\dot{X}_{\text{in}} - \dot{X}_{\text{out}} + \dot{X}_{\text{generated}} = \dot{X}_{\text{stored}}.$$

Identifying the conserved quantity as momentum, we recognize that $\dot{X}_{\text{in}}$ and $\dot{X}_{\text{out}}$ are the respective momentum flows in and out of the control volume (see Fig. 6.17), that is,

$$\dot{X}_{\text{in}} \equiv (\dot{m}\mathbf{V})_{\text{in}}, \tag{6.51a}$$

$$\dot{X}_{\text{out}} \equiv (\dot{m}\mathbf{V})_{\text{out}}. \tag{6.51b}$$

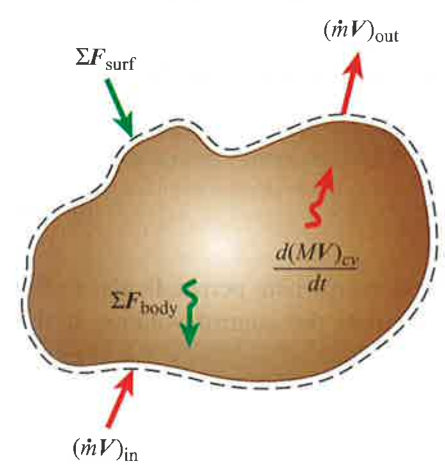


FIGURE 6.17
Control volume schematically showing momentum flowing in, momentum flowing out, and surface and body forces that act as sources (or sinks) of momentum on a time-rate basis. The rate at which momentum is stored in the control volume is represented with a squiggly arrow inside the control volume.

Forces acting on the control volume generate momentum on a time-rate basis; thus,

$$\dot{X}_{\text{generated}} \equiv \sum \mathbf{F}_{\text{cv}}. \tag{6.51c}$$

Depending on the direction of application, a force can also act as a sink for momentum. The time rate of change of momentum within the control volume is simply

$$\dot{X}_{\text{stored}} \equiv \frac{d(M\mathbf{V})_{\text{cv}}}{dt}. \tag{6.51d}$$

Using these definitions (Eqs. 6.51a–6.51d), we write the conservation of momentum principle for a control volume as

$$\underset{\substack{\text{Rate of}\\\text{momentum}\\\text{flowing into}\\\text{control volume}}}{(\dot{m}V)_{\text{in}}} - \underset{\substack{\text{Rate of}\\\text{momentum}\\\text{flowing out of}\\\text{control volume}}}{(\dot{m}V)_{\text{out}}} + \underset{\substack{\text{Rate at which}\\\text{momentum is generated}\\\text{by forces acting on}\\\text{control volume}}}{\sum F_{\text{cv}}} = \underset{\substack{\text{Time rate of}\\\text{change of}\\\text{momentum within}\\\text{control volume}}}{\frac{d(MV)_{\text{cv}}}{dt}}. \tag{6.52}$$

6.5b Integral Control Volumes with Steady Flow

We begin by considering steady flow for an integral control volume having a single inlet and a single outlet. For this simplified situation, conservation of momentum can be stated as follows:

In steady state, the vector momentum flow in minus the vector momentum flow out plus the vector sum of all forces acting on the control volume must equal zero.

Assuming that the velocity is uniform over the inlet and the outlet, this is expressed symbolically as

$$\underset{\substack{\text{Vector}\\\text{momentum flow}\\\text{into control}\\\text{volume}}}{\dot{m}V_{\text{in}}} - \underset{\substack{\text{Vector}\\\text{momentum flow}\\\text{out of control}\\\text{volume}}}{\dot{m}V_{\text{out}}} + \underset{\substack{\text{Vector sum of}\\\text{forces acting}\\\text{on control}\\\text{volume}}}{\sum F_{\text{cv}}} = 0 \quad (uniform\ V) \tag{6.53}$$

The horizontal reaction force exerted by the test stand on this jet engine equals the difference between the momentum flow out and the momentum flow in.

If the velocity is not uniformly distributed at the inlet or outlet, then

$$\dot{m}\beta_{\text{in}}V_{\text{avg,in}} - \dot{m}\beta_{\text{out}}V_{\text{avg,out}} + \sum F_{\text{cv}} = 0 \quad (distributed\ V), \tag{6.54}$$

where the **momentum flow correction factor** β is defined by the following:

$$\beta \equiv \frac{\int_A V\rho|(V\cdot\hat{n})|dA}{\dot{m}V_{\text{avg}}}. \tag{6.55a}$$

Here we assume that the velocity vector is everywhere perpendicular to the control surface where the flow enters or exits the control volume. If the density is constant, this expression simplifies to

$$\beta = \frac{1}{A}\int_A \frac{V^2}{V_{\text{avg}}^2}dA, \tag{6.55b}$$

where V is the magnitude of the local velocity, and V_{avg} is the magnitude of the average velocity over A (see Eq. 3.13). Values for β associated with common velocity distributions for circular flow areas are given in Table 6.1.

The forces acting on the control volume are the surface forces resulting from pressure or viscous stress, the gravitational body force, and other forces exposed where a control surface cuts through a solid, as discussed at the beginning of this chapter.

Table 6.1 Linear Momentum Correction Factors for Parabolic and Power-Law* Velocity Distributions over Circular Flow Areas

Velocity Distribution	Correction Factor (β)
Parabolic†	1.333
Power law	$\dfrac{(n+1)^2(2n+1)^2}{2n^2(2n+2)(n+2)}$
$n = 6$	1.027
$n = 7$	1.020
$n = 10$	1.011

* The power-law distribution is given by $v(r) = a(1 - r/R)^{1/n}$.
† See Example 6.9.

Equations 6.53 and 6.54 can be made more general by considering multiple inlets and outlets. For this situation, momentum conservation is expressed as

$$\sum_{j=1}^{N\text{ outlets}} \dot{m}_{\text{in},j}\beta_{\text{in},j}V_{\text{avg,in},j} - \sum_{k=1}^{M\text{ outlets}} \dot{m}_{\text{out},k}\beta_{\text{out},k}V_{\text{avg,out},k} + \sum F_{\text{cv}} = 0. \tag{6.56}$$

All of the previous expressions of momentum conservation are specific cases of the following general mathematical representation:

$$-\int_{CS} V\rho(V_{\text{rel}}\cdot\hat{n})dA + \int_{CS} dF_P + \int_{CS} dF_{\text{visc}} + \int_{CS} dF_{\text{other}} + \int_{CV} dF_{\text{grav}} = 0, \tag{6.57}$$

Appendix 6A presents the extension of momentum conservation to noninertial (accelerating) control volumes.

where $\int_{CS}$ indicates integration over the entire control surface. Note that the minus sign in the first term causes momentum flows *in* to be positive and momentum flows *out* to be negative. The appearance of the relative velocity V_{rel} handles the situation where the control volume moves with a constant velocity with respect to a fixed observer. If the control volume is fixed, $V_{\text{rel}} \equiv V$. This use of a relative velocity is already embodied in the simpler expressions (Eqs. 6.53 and 6.54) where the mass flow rate across the surface of interest, $\dot{m}$, takes this into account.

Example 6.10

Converging–diverging nozzles are treated in detail in Chapter 11—see Examples 11.10 and 11.11.

The Space Shuttle orbiter has three main engines located at the rear of the vehicle. These engines burn hydrogen with oxygen at high temperatures and pressures. The products of combustion, primarily steam and excess hydrogen, expand in a converging–diverging nozzle to produce a high-velocity jet at the nozzle exit. Consider a single Space Shuttle main engine (SSME) secured in a test stand and firing vertically downward as shown in Fig. 6.18. During a test at full power, the hydrogen and oxygen flow rates are 72.7 kg/s and 440.9 kg/s, respectively. The exit diameter of the nozzle is 2.3 m, and the temperature and pressure of the exhaust gases at the exit plane are 1166 K and 0.1915 atm, respectively. The apparent molecular weight of the exhaust gases is 13.78 kg/kmol. The engine weighs 31,140 N.

FIGURE 6.18
The Space Shuttle main engine secured in a test stand burns hydrogen with oxygen to create a high-velocity gas flow at the nozzle exit. Photograph courtesy of NASA.

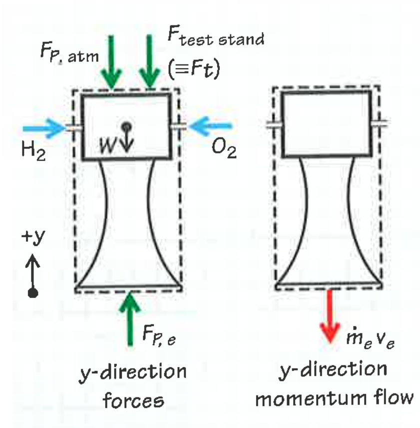
Determine the thrust exerted by the single SSME on the test stand for an ambient pressure of 1 atm.

Solution

Known $\dot{m}_{H_2}$, $\dot{m}_{O_2}$, P_{atm}, P_e, T_e, $\mathcal{M}_e$, W

Find F_t

Sketch



Assumptions

i. Steady state
ii. Uniform velocity at exit plane
iii. No y-direction forces or momentum flows associated with incoming H_2 and O_2.

Analysis We select a cylindrical control volume that surrounds the entire engine. The control surface cuts through the engine test stand mount to expose the desired thrust force. The surface also cuts through the H_2 and O_2 supply lines, which we assume are oriented in a radial direction as shown in the sketch. Atmospheric pressure acts over the entire control surface except at the exit plane where $P_e < P_{atm}$. All y-direction forces are shown in the sketch. Note that the area associated with the pressure at the top of the control volume is identical to that of the nozzle exit. Since the H_2 and O_2 enter radially, there are no y-directed forces or momentum flows associated with these streams. The only y-direction momentum flow is at the exit plane directed downward.

Using the sketch as a guide, we apply the conservation of momentum principle (Eq. 6.53) as follows:

$$0 - [\dot{m}_e(-v_e)] + F_{P,e} - F_{P,atm} - F_t - W = 0.$$

Note the minus sign attached to v_e indicating that v_e is directed in the negative y-direction. From mass conservation, the exit mass flow rate is

$$\dot{m}_e = \dot{m}_{H_2} + \dot{m}_{O_2}$$
$$= 72.7 + 440.9 \text{ kg/s} = 513.6 \text{ kg/s}.$$

With the assumption of a uniform exit velocity,

$$\dot{m}_e = \rho_e v_e A_e.$$

Combining this with the ideal-gas equation of state to find ρ_e, we obtain the exit velocity:

$$v_e = \frac{\dot{m}_e}{\left(\dfrac{P_e \mathcal{M}_e}{R_u T_e}\right)\dfrac{\pi D_e^2}{4}}$$
$$= \frac{513.6}{\left(\dfrac{0.1915(101{,}325)13.78}{8314.5(1166)}\right)\dfrac{\pi(2.3)^2}{4}}$$
$$= \frac{513.6 \text{ kg/s}}{0.0276 \text{ kg/m}^3\ 4.15 \text{ m}^2}$$
$$= 4480 \text{ m/s}.$$

The reader should verify the units in the calculation of ρ_e. The exit momentum flow is thus

$$\dot{m}_e v_e = 513.6\,(4480) = 2{,}300{,}000$$
$$[=] \frac{\text{kg}}{\text{s}}\frac{\text{m}}{\text{s}}\left[\frac{1 \text{ N}}{\text{kg}\cdot\text{m/s}^2}\right] = \text{N}.$$

The pressure forces are evaluated as

$$F_{P,e} - F_{P,atm} = (P_e - P_{atm})\pi D_e^2/4$$
$$= (0.1915 - 1)101{,}325\ \pi(2.3)^2/4$$
$$= -81{,}921\,(4.15) = -340{,}000$$
$$[=] \frac{\text{N}}{\text{m}^2}\text{m}^2 = \text{N}.$$

Solving our momentum conservation expression for the unknown test-stand force (i.e., the thrust) and evaluating yield

$$F_t = \dot{m}_e v_e + (F_{P,e} - F_{P,atm}) - W$$
$$= 2{,}300{,}000 - 340{,}000 - 31{,}140 \text{ N}$$
$$= 1{,}930{,}000 \text{ N}.$$

Comments First, we note that this force is very large (~434,000 lb$_f$). In comparison, automobiles weigh a few thousand pounds. Other big numbers are associated with this engine; for example, the turbopump that supplies the H_2 to the combustion chamber is rated at 75,000 hp. Clearly, huge rates of energy transfer are commonplace in the Space Shuttle propulsion system. Second, we note the importance of the choice of control volume and the simplifying assumptions invoked. Choosing a good control volume and sketching all of the forces and momentum flows are key to solving problems such as this.

Self Test 6.7

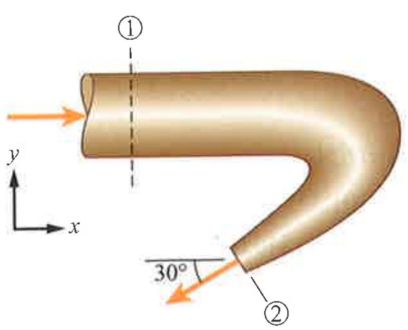
Using the momentum flows determined in Example 6.9, calculate the force required to hold the long tube illustrated in Fig. 6.16c. Neglect gravitational effects and ignore any pressure drop between the inlet and the exit.

(Answer: $F_x = 4.7 \times 10^{-5}$ N, $F_y = 0$ N)

In many applications, a fluid stream exits a control volume as a **free jet,** unconstrained by any solid boundaries. Examples of free jets are a flow from a pipe into the atmosphere and the jet of air one creates to blow out a candle. For jets of incompressible fluids, or jets with low Mach numbers ($Ma < 0.3$, say), the pressure at the exit plane where the jet enters the ambient fluid is essentially uniform and equal to the pressure in the quiescent ambient fluid (i.e., $P_{exit} \approx P_{amb}$).

In the following example, we take advantage of this **jet-exit boundary condition** to evaluate the pressure force where a jet exits a control volume. The previous example involved a high-speed compressible flow. In that case, the jet exit plane and the atmospheric pressures were not equal. Chapter 11 provides further insight into compressible flows.

Example 6.11



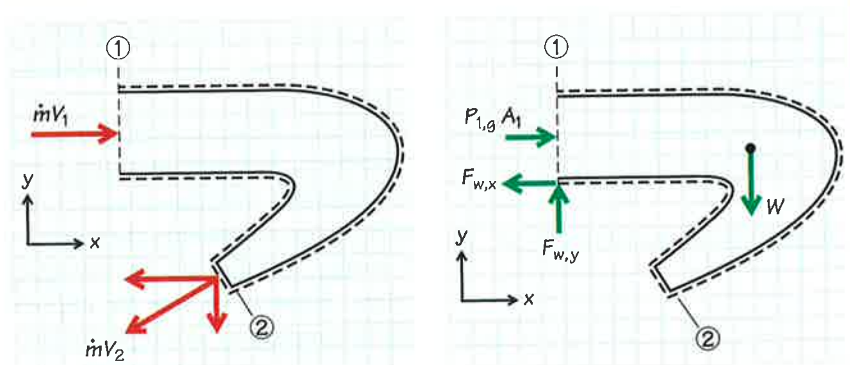
Water ($\rho = 997$ kg/m^3) exits into the atmosphere (100 kPa) from a backward-curving nozzle as shown in the sketch. At station 1, the water velocity is 2 m/s, the absolute pressure is 257 kPa, and the inside diameter of the pipe is 75 mm. The nozzle exit diameter is 25 mm. The weight of the pipe–nozzle assembly and the water it contains between stations 1 and 2 is 54 N. Determine the equivalent reaction forces, $F_{w,x}$ and $F_{w,y}$, in the pipe wall at station 1.

Solution

Known $\rho_w, P_{atm}, P_1, V_1, D_1, D_2, W_{tot}$

Find $F_{w,x}, F_{w,y}$

Sketch



Assumptions

i. Steady state
ii. Uniform entrance and exit velocities
iii. $P_2 = P_{atm}$
iv. $F_{w,x}$ acts to the left, $F_{w,y}$ acts upward

Analysis We choose a control volume that cuts through the pipe wall and the entering water at station 1, is then contiguous with the outer surface of the

pipe and nozzle, and finally cuts through the exiting jet. This choice results in the only net pressure force being the product of the gage pressure and cross-sectional area at station 1 because atmospheric pressure acts uniformly over the entire control volume. The forces acting on the control volume are shown in the sketch. The directions of $F_{w,x}$ and $F_{w,y}$ are chosen arbitrarily. If our analysis shows either to be negative, they must then act opposite to the direction chosen. The momentum flows are illustrated in the top sketch. Note that $\dot m \mathbf{V}_2$ can be resolved into x- and y-components as shown. With all forces and all momentum flows identified, we can apply linear momentum conservation for steady flow with uniform entrance and exit velocities expressed by Eq. 6.53:

$$\dot m \mathbf{V}_1 - \dot m \mathbf{V}_2 + \sum \mathbf{F}_{cv} = 0.$$

Resolving this into x- and y-components yields

$$\dot m v_{x,1} - \dot m v_{x,2} - F_{w,x} + P_{1g}A_1 = 0 \quad (x\text{-direction})$$

and

$$\dot m v_{y,1} - \dot m v_{y,2} + F_{w,y} - W_{tot} = 0 \quad (y\text{-direction}).$$

We note from the sketch that both $v_{x,2}$ and $v_{y,2}$ are negative quantities.

To solve these equations requires values for the velocity components. Using mass conservation, we first find the magnitude of $\mathbf{V}_2$ as follows:

$$\dot m = \rho_w |\mathbf{V}_1| A_1 = \rho_w |\mathbf{V}_2| A_2.$$

Thus,

$$V_2 = V_1 \frac{A_1}{A_2} = V_1 \frac{\pi D_1^2/4}{\pi D_2^2/4} = V_1 \frac{D_1^2}{D_2^2}$$

$$= 2\left(\frac{0.075}{0.025}\right)^2 \text{ m/s} = 18 \text{ m/s}.$$

From the geometry, the x- and y-velocity components are

$$v_{x,1} = 2 \text{ m/s},$$
$$v_{y,1} = 0,$$
$$v_{x,2} = -V_2 \cos 30° = -18(0.866) \text{ m/s} = -15.588 \text{ m/s},$$
$$v_{y,2} = -V_2 \sin 30° = -18(0.500) \text{ m/s} = 9.000 \text{ m/s}.$$

The mass flow rate is

$$\dot m = \rho_w V_1 \frac{\pi D_1^2}{4} = 997(2)\frac{\pi (0.075)^2}{4} = 8.809$$

$$[=] \frac{\text{kg}}{\text{m}^3}\frac{\text{m}}{\text{s}}\text{m}^2 = \text{kg/s}.$$

Rearranging the x-component expression of momentum conservation and substituting numerical values yield

$$F_{w,x} = \dot m (v_{x,1} - v_{x,2}) + (P_1 - P_{atm})A_1$$

$$= 8.809[2 - (-15.588)] + (257 \times 10^3 - 100 \times 10^3)\frac{\pi (0.075)^2}{4}$$

$$= 154.9 + 693.6 = 848.5$$

$$[=] \frac{\text{kg}}{\text{s}}\frac{\text{m}}{\text{s}}\left[\frac{1 \text{ N}}{\text{kg} \cdot \text{m/s}^2}\right] = \text{N}$$

and

$$\begin{aligned} F_{w,y} &= W - \dot{m}(v_{y,1} - v_{y,2}) \\ &= 54.0 - 8.809[0 - (-9.000)]\ \mathrm{N} \\ &= -25.3\ \mathrm{N}. \end{aligned}$$

The minus sign here indicates that $F_{w,y}$ must act *downward* rather than upward as originally assumed.

Comments We note how the choice of control volumes made dealing with the pressure forces easy, as the atmospheric pressure components canceled. Note also the importance of keeping track of the vector nature of momentum conservation and the use of positive and negative signs with the scalar velocity components.

The momentum within a control volume defined by the exterior surfaces of the Space Shuttle changes as a result of two factors: the mass decreases as the propellants are consumed and expelled, and the velocity increases as the system accelerates.

6.5c Integral Control Volumes with Unsteady Flow

If the linear momentum associated with the control volume as a whole varies with time, an unsteady term must be included in our statement of momentum conservation. Here we are concerned with the control-volume momentum itself, distinct from any momentum flow across the control surface. The control-volume momentum is most generally expressed as

$$\text{control-volume momentum} \equiv \int_{\mathrm{CV}} \boldsymbol{V}dM = \int_{\mathrm{CV}} \boldsymbol{V}\rho d\mathcal{V}. \tag{6.58}$$

The integrand of Eq. 6.58 is $\boldsymbol{V}dM$ $(= \boldsymbol{V}\rho d\mathcal{V})$, the momentum of a fluid element with mass dM. Allowing the velocity and/or the density of the fluid to vary from point to point within the control volume requires integration over the control volume to evaluate the total control-volume momentum. If both the velocity and density are the same at every point within the control volume, but still allowed to vary with time, the control-volume momentum is simply $M_{\mathrm{cv}}\boldsymbol{V}_{\mathrm{cv}}$.

With this understanding of the control-volume momentum, we state the general conservation of momentum principle for a nonaccelerating control volume:

$$\underbrace{\int_{\text{Inlets}} \boldsymbol{V}\rho(-1)(\boldsymbol{V}_{\mathrm{rel}}\cdot\hat{\boldsymbol{n}})dA}_{\substack{\text{Flow of momentum}\\ \text{into the control volume}}} - \underbrace{\int_{\text{Outlets}} \boldsymbol{V}\rho(\boldsymbol{V}_{\mathrm{rel}}\cdot\hat{\boldsymbol{n}})dA}_{\substack{\text{Flow of momentum}\\ \text{out of the control volume}}}$$

$$+ \underbrace{\sum \boldsymbol{F}}_{\substack{\text{The sum of all forces}\\ \text{(surface and body)}\\ \text{acting on the control}\\ \text{volume}}} = \underbrace{\frac{d}{dt}\int_{\mathrm{CV}} \boldsymbol{V}\rho d\mathcal{V}.}_{\substack{\text{The time rate of change}\\ \text{of the momentum within}\\ \text{the control volume}}} \tag{6.59a}$$

This particular presentation of momentum conservation preserves the physical interpretation that the momentum within the control volume can be changed in three ways: 1. by having a force act on the control volume, 2. by

an inflow of momentum, and 3. by an outflow of momentum. Note that the negative sign within the inlet momentum flow integral is required to make this a positive quantity since the entering fluid velocity $\boldsymbol{V}_{\mathrm{rel}}$ is directed opposite to the surface normal $\hat{\boldsymbol{n}}$ (i.e., $\boldsymbol{V}_{\mathrm{rel}}\cdot\hat{\boldsymbol{n}} < 0$). To provide a statement of conservation of momentum in a form consistent with our steady-flow analysis (Eq. 6.57), we combine the entering and exiting momentum flows in a single term and rearrange Eq. 6.59a to yield

$$\underbrace{-\int_{\mathrm{CS}} \boldsymbol{V}\rho(\boldsymbol{V}_{\mathrm{rel}}\cdot\hat{\boldsymbol{n}})dA}_{\substack{\text{Net flow of momentum}\\ \text{into the control volume}}} + \underbrace{\sum \boldsymbol{F}}_{\substack{\text{Sum of all forces}\\ \text{(surface and body)}\\ \text{acting on the}\\ \text{control volume}}} = \underbrace{\frac{d}{dt}\int_{\mathrm{CV}} \boldsymbol{V}\rho d\mathcal{V}.}_{\substack{\text{Time rate of change of}\\ \text{the momentum within}\\ \text{the control volume}}} \tag{6.59b}$$

The force term and the momentum flow term in Eqs. 6.59a and 6.59b are handled in the same manner as discussed in the previous section, the only difference being that these terms are now instantaneous expressions that can vary with time.

Recall that our development of Eq. 6.59 assumes a nonaccelerating coordinate system. Appendix 6A shows how to deal with noninertial coordinate systems.

The following example illustrates the application of Eq. 6.59.

Example 6.12

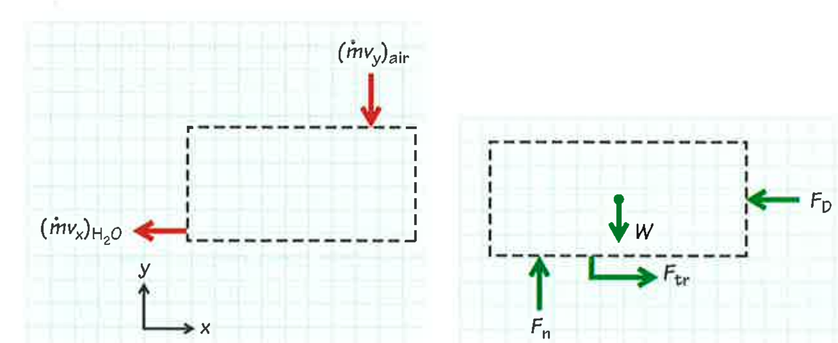
A water truck empties its load of water ($\rho_{\mathrm{H_2O}} = 996\ \mathrm{kg/m^3}$) through a 50-mm-diameter pipe at the rear as shown in the sketch. The truck is traveling forward at 10 miles/hr (4.47 m/s), and the velocity of the water jet with respect to the moving truck is 5 m/s rearward. Air enters the tank through a vent at the top of the truck, replacing the water that exits from the tank. The total drag force F_{D} opposing the motion of the truck is 95.9 N. Estimate the traction force required to keep the truck moving at a constant speed.

Solution

Known $\rho_{\mathrm{H_2O}}$, D, $v_{x,\mathrm{truck}}$, $V_{x,\mathrm{rel}}$, F_{D}

Find F_{tr}

Sketch



Assumptions

i. The water-jet velocity is uniform.
ii. F_D accounts for all horizontal forces acting on control volume other than F_{tr}.
iii. $\rho_{air} \ll \rho_{H_2O}$ so that $\dot{m}_{air} \ll \dot{m}_{H_2O}$.
iv. Momentum flows associated with the engine air and exhaust products are negligible as is the change in truck mass from fuel burning.

Analysis We choose a control volume that cuts between the tires and the road to expose the unknown traction force F_{tr}. With assumption ii, the only forces acting in the horizontal direction are F_{tr} and F_D. Two momentum flows are shown in the sketch: one associated with the incoming air (vertical) and one associated with the outgoing water (horizontal). We note that the momentum of the control volume (truck plus tank and its contents) is changing, so the problem is unsteady; thus, linear momentum conservation is expressed by Eq. 6.59b. The x-component of this equation can be written as

$$0 - (\dot{m}_{H_2O}v_{x,H_2O})_{out} + F_{tr} - F_D = \frac{d(M_{cv}v_{x,cv})}{dt},$$

where the zero indicates that there is no flow of x-momentum into the control volume. The velocities here are all with respect to a fixed observer, so that

$$v_{x,cv} = v_{x,truck} = 4.47 \text{ m/s}$$

and

$$\begin{aligned} v_{x,H_2O,out} &= v_{x,H_2O,rel} + v_{x,truck} \\ &= -5 + 4.47 \text{ m/s} = -0.53 \text{ m/s}. \end{aligned}$$

The flow rate $\dot{m}_{H_2O}$, however, is based on the velocity at which the water crosses the control surface, $v_{x,H_2O,rel}$; thus,

$$\begin{aligned} \dot{m}_{H_2O} &= \rho_{H_2O}v_{x,H_2O,rel}\pi D^2/4 \\ &= 996(5)\pi(0.05)^2/4 = 9.78 \\ [=] & \frac{\text{kg}}{\text{m}^3}\frac{\text{m}}{\text{s}}\text{m}^2 = \text{kg/s}. \end{aligned}$$

We focus now on the time rate of change of the momentum within the control volume. Since $v_{x,cv}$ is constant,

$$\frac{d(M_{cv}v_{x,cv})}{dt} = v_{x,cv}\frac{dM_{cv}}{dt}.$$

The time rate of change of the mass with the control volume can be related to the water outflow through Eq. 3.19a, an unsteady expression of the conservation of mass principle. For our control volume,

$$\dot{m}_{air,in} - \dot{m}_{H_2O,out} = \frac{dM_{cv}}{dt}.$$

Neglecting the air mass flow rate, we have

$$\frac{dM_{cv}}{dt} = -\dot{m}_{H_2O,out}.$$

Reassembling the momentum conservation expression yields

$$-\dot{m}_{H_2O,out}v_{x,H_2O,out} + F_{tr} - F_D = v_{x,cv}\frac{dM_{cv}}{dt} = v_{x,cv}(-\dot{m}_{H_2O,out}).$$

Solving for F_{tr} and substituting numerical values, we obtain

$$\begin{aligned} F_{tr} &= F_D + \dot{m}_{H_2O,out}v_{x,H_2O,out} + v_{x,cv}(-\dot{m}_{H_2O,out}) \\ &= F_D + \dot{m}_{H_2O,out}(v_{x,H_2O,out} - v_{x,cv}) \\ &= 95.9 + 9.78(-0.53 - 4.47) \\ &= 47.0 \\ [=] & \frac{\text{kg}}{\text{s}}\frac{\text{m}}{\text{s}}\left[\frac{1\text{ N}}{\text{kg}\cdot\text{m/s}^2}\right] = \text{N}. \end{aligned}$$

Comments In this example, we see the importance of interpreting velocities in the reference frames of a fixed observer and for an observer traveling with the control volume. We also see that keeping track of signs (±) in both mass and momentum conservation is essential to obtaining the proper solution. To further reinforce the concepts presented in this example, the reader is encouraged to solve this problem using the inertial reference frame of the moving control volume. (See Problem 6.87.)

6.5d Differential Control Volumes

See Chapters 3 and 5.

In our previous applications of mass and energy conservation to differential control volumes, we began with a simple one-dimensional steady flow and then added complexity. Since you should now be quite familiar with the general process used to derive such differential equations, we start with the three-dimensional (Cartesian), unsteady case rather than building up to it. We begin by applying the general conservation of momentum relation (Eq. 6.59) to the small control volume shown in Fig. 6.19. The pressure forces acting on this control volume are as shown previously in Fig. 6.7; the viscous forces are obtained by multiplying the viscous stresses shown in Fig. 6.4a by the area upon which they act. For simplicity, let us consider only the forces that act in the x-direction, all of which are shown in Fig. 6.20 except for the x-component of the body force. Summing these x-direction forces yields the following:

$$\begin{aligned} \sum F_{x\text{-dir}} = {} & \underbrace{[P_x - P_{x+\Delta x}]\Delta y\Delta z}_{\text{Net pressure force in the }x\text{-direction}} + \underbrace{[(\tau_{xx})_{x+\Delta x} - (\tau_{xx})_x]\Delta y\Delta z}_{\text{Net viscous normal force in }x\text{-direction}} \\ & + \underbrace{[(\tau_{yx})_{y+\Delta y} - (\tau_{yx})_y]\Delta x\Delta z + [(\tau_{zx})_{z+\Delta z} - (\tau_{zx})_z]\Delta x\Delta y}_{\text{Net viscous shear forces in the }x\text{-direction}} \\ & + \underbrace{\rho g_x\Delta x\Delta y\Delta z.}_{x\text{-component of gravitational body force}} \end{aligned} \quad (6.60)$$

Similar expressions can be written for the y- and z-directions, a task presented as a homework exercise.

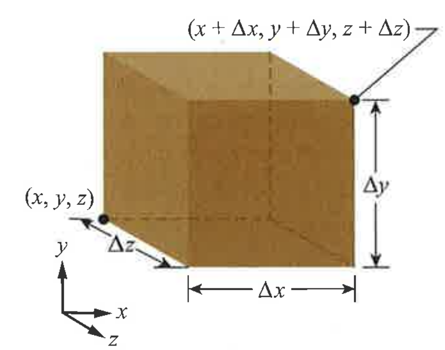


FIGURE 6.19
We apply momentum conservation to a stationary control volume of dimensions Δx, Δy, and Δz. Viscous and pressure forces act on all six faces, and momentum flows through all six faces. Gravity is the single body force acting on the control volume.