The Empty Number Line:

A Useful Tool or Just Another Procedure?

A grade 4 mathematics homework question instructed students, "Use the empty number line to solve the following," causing nine-year-old Emily to respond, "No one should tell me what strategy to use. I should be allowed to make up my own mind!" Emily found it easier to solve some two-digit addition and subtraction computations by splitting them into tens and ones rather than using a strategy that could be recorded on a number line.

Recently, a new mathematics syllabus for K–6 students was introduced in New South Wales, Australia (Board of Studies 2002). This document reflected an international trend in its emphasis on the development of mental strategies in the early elementary years (e.g., NCTM 2000); it also formally delayed instruction of traditionally taught algorithms for the four operations and introduced some new instructional "tools." One such tool was the empty, or blank, number line. This article is the result of my daughter Emily's and my decision to explore for ourselves the origins and potential benefits of using an empty number line.

By Janette Bobis

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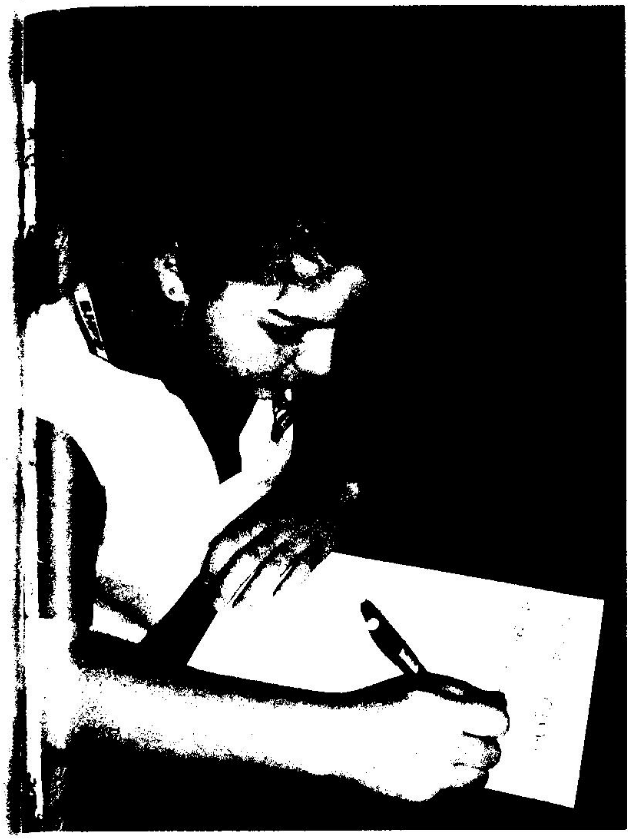
What Is an Empty Number Line?

The empty number line—a number line that is presented with no numbers or markers—is a visual representation for recording and sharing students' thinking strategies during mental computation (NSW Department of Education and Training 2002). Starting with an empty number line, students mark only the numbers they need for their calculation. For example, **figure 1** shows Emily's strategy for solving the problem 53 – 26 as she recorded it on an empty number line.

Why Use an Empty Number Line?

The first recorded use of the empty number line that we could find occurred in the Netherlands in the 1970s. According to Gravemeijer (1994), the empty number line was developed as a "new" tool to help overcome difficulties associated with the common "procedure only" use of base-ten materials regularly used when modeling the standard written algorithms. Early experiments with the empty number line were not as successful as hoped, possibly because it was introduced in a measurement context and its similarity to a standard ruler, with its rigid calibrations, made students feel uncertain when approximating the position of numbers on a line with no given calibrations. However, Treffers (1991) and Beishuizen (2001) found that students could successfully use the empty number line to record and make sense of a variety of solution



strategies for two-digit addition and subtraction. The empty number line is now widely accepted in the Netherlands as an important didactical tool for working with numbers up to 100 and beyond (Van den Heuvel-Panhuizen 2001).

Advantages of using the empty number line, as outlined by Gravemeijer (1994), include—

- the need for a *linear* representation of number;
- the close alignment of young children's intuitive mental strategies with the empty number line; and
- its potential to foster the development of more sophisticated strategies in children.

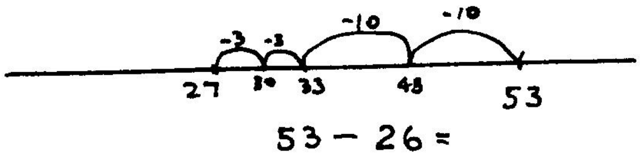
Clearly, base-ten materials, such as Dienes blocks, are best for dealing with quantities and actual problems, but linear representations, such as the empty number line, are best for modeling distance or measurement. The strong links between the empty number line and young children's intuitive mental strategies are evident in the way that children, when attempting to solve number problems up to 100, naturally tend to focus first on counting strategies—counting all, counting on, or counting down. Students who are more proficient mental calculators use a combination of counting strategies (usually in chunks of 10) with partitioning strategies. In partitioning, children "take apart" numbers in flexible ways to make them more convenient to calculate mentally. These strategies typically approximate the jumps on a number line. Note that to solve the problem 53 – 26 (**fig. 1**), Emily used a combination of counting back in chunks of 10 and then partitioning the 6 into 2 lots of 3 to bridge the decade more easily.

The empty number line's potential to support the development of more sophisticated strategies is the result of children's using the line to record their computation strategies, thus revealing to others (and to themselves) their thinking processes. Hence, we see not only the level of thinking but also the errors in thinking that might occur. Instructional decisions can then be made to assist the development of more efficient strategies. Given this last point, it seems logical that such a visual recording of children's thinking strategies can provide a stimulus for classroom discussion and sharing of mental strategies. Students can explain their strategies by "showing" them to others. This fact makes the empty number line a very powerful tool for enhancing communication in the classroom.

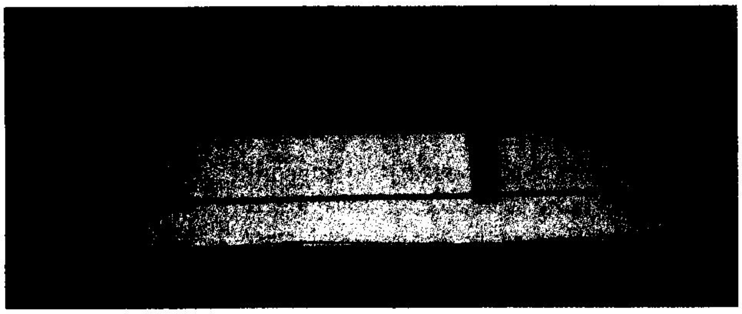
Many of the benefits of using an empty number line are supported by Emily's reflections on its use. Currently in grade 4, she was introduced to the empty number line at the age of 8, when she was in grade 3. She now considers the empty number line "easier to learn and remember than the pencil-and-paper method" because "I get to record the strategy I'm thinking, not what I think the teacher wants." Furthermore, she explains, "If you make a mistake, it's easier to find it." From Emily's perspective, the empty number line is "easier" to use because—

- she understands how it works (it represents *her* thinking);

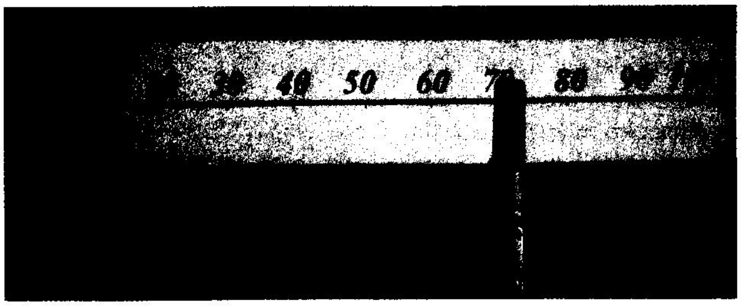
Emily recorded "jumps" on a number line to solve the problem 53 – 26.



A number line indicating only the numbers 0 and 100 (a) and the reverse side of this number line (b), which includes decade numbers added by the teacher

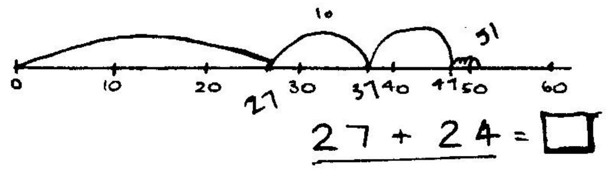


2a.



2b.

A predrawn number line, indicating multiples of 10, is used to solve the problem 27 + 24.



- it keeps a record of each step in her thinking;
- it allows her to track errors; and
- it enables her to think about what to do next when the computation is too demanding to do solely "in her head."

Introducing the Empty Number Line

Before children are introduced to the empty number line, they should already be familiar with a linear representation of number—a number line with numbers. Buys (2001) recommends that the empty number line be introduced through strings of beads that alternate in color every 10 beads. Although Emily's teachers did not use this particular approach, they used another type of number line that supported the linear representation and assisted Emily's and her classmates' understanding of an empty number line. **Figure 2a** shows a number line that displays only the numbers 0 and 100. Emily was asked to move a clip along this number line to a designated number determined by a teacher or another child. After she positioned the clip on the number line at the point she estimated to be the designated number, Emily "flipped" the number line to reveal a more complete number line (see **fig. 2b**). She could then check how close her estimate was to the designated number.

Before children use the empty number line to record more complex mental strategies involving two-digit addition and subtraction, certain counting skills and knowledge are required. Two essential strategies that children must understand and use effectively before they can use the more sophisticated empty number line include (1) counting by tens, both on and off the decade; and (2) jumping across tens (or bridging tens).

Counting by tens, both on and off the decade, allows a student to start with any number and count forward or backward in multiples of 10. When first introduced to this skill, children can use manipulatives, such as strings of 10 beads or bundles of 10 popsicle sticks, to model the counting process and record their counting on either a hundreds chart or as jumps on an empty number line.

Bridging tens requires that children be able to flexibly partition numbers. For example, to solve the problem 8 + 5, the first number remains as a whole and the second number, the 5, is partitioned (for example, into 2 and 3) and added in parts. The process is easier if a part of the 5 (the 2) is added to the 8 to "make 10" before the final part (the 3) is added; hence, 8 + 2 = 10 and then 10 + 3 = 13. This same strategy can then be applied when bridging tens in higher decades (e.g., 38 + 5 = (38 + 2) + 3 = 40 + 3 = 43).

After learning the strategies for counting by tens and bridging tens, students can be introduced to the *jump strategy* (or sequential strategy) for two-digit addition and subtraction. The fundamental characteristic of this strategy is that one number is treated as a whole while the second number is added or subtracted in manageable chunks of tens and ones In calculating a problem such as 66 – 29, it is often more efficient to apply a *compensation strategy*—for example, by subtracting 30 and adding 1 (66 – 29 = (66 – 30) + 1).

For instructional purposes, the children should record only the relevant numbers on the empty number line. As their strategies become more sophisticated, the number of jumps should decrease. Emily recalls some confusing experiences with the empty number line during grade 3, when she was provided with a predrawn number line that started at zero and had all the multiples of 10 marked (see **fig. 3**). Still a relative novice at using the empty number line, Emily thought that she needed to start at the first number marked on the line—zero. The result was the meaningless, "procedure-like" execution of unnecessary jumps. Hence, while a proven powerful tool for developing mental computation in young children, the empty number line and associated jump strategy must be introduced thoughtfully and with well-chosen examples. Inappropriate use could result in the execution of a less sophisticated strategy or in learning a meaningless procedure. Recent work by Van den Heuvel-Panhuizen (forthcoming) refers to similar concerns about the introduction and appropriate application of the empty number line with children in the Netherlands.

Another basic strategy for mental computation of two-digit addition and subtraction is referred to as the *split strategy*. This strategy involves "splitting" the tens and ones and addressing these separately. Emily considered the split strategy a better ("easier") method to use for two homework problems—42 + 26 and 56 + 32—because "the digits [in the ones place] don't add up past ten." Hence, to calculate 42 + 26, Emily performed two addition operations—40 + 20 = 60 and 2 + 6 = 8—and then combined the tens and ones again to get 68. No written recording was necessary because the numbers were simple enough for her to do completely "in her head."

Conclusion

We teachers need to be aware that children will vary their strategy use according to the numbers involved. The rigid application of just one tool or one procedure will severely limit their ability to apply mental strategies in flexible and fluent ways. Hence, children must be given opportunities to develop a variety of strategies and representational tools on their road to fluency. Eventually, children also need to make decisions about what strategy to use and be accountable for these decisions.

The empty number line was introduced into our curriculum to help children move away from meaningless manipulations of algorithms using only pencil and paper, but we must be careful that we do not unintentionally adopt a similar "procedural only" approach toward mental strategies. The development of mental computation strategies could be prone to the same difficulties we face in teaching the standard pencil-and-paper algorithm unless we learn to listen to our greatest critics—our students—and let them make up their own minds about the best strategy to use.

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