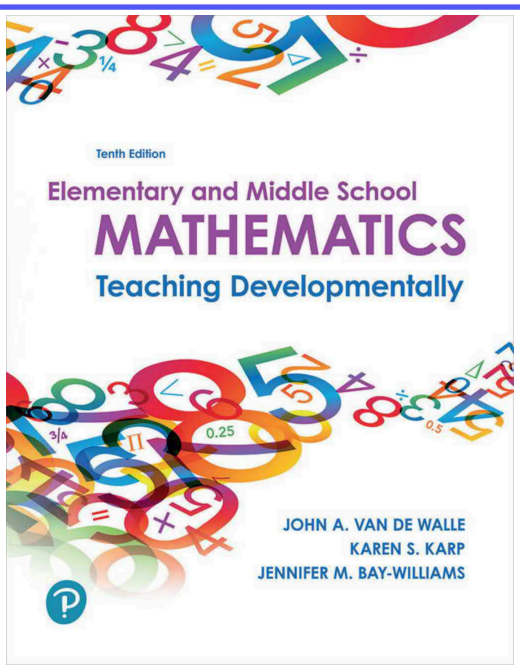
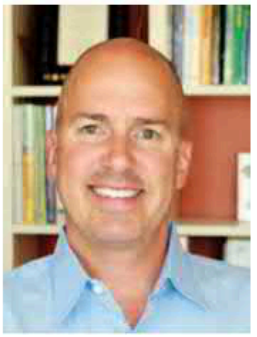


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Brief Contents

PART I Teaching Mathematics: Foundations and Perspectives

CHAPTER 1 Teaching Mathematics in the 21st Century 1

CHAPTER 2 Exploring What It Means to Know and Do Mathematics 13

CHAPTER 3 Teaching through Problem Solving 30

CHAPTER 4 Planning in the Problem-Based Classroom 55

CHAPTER 5 Creating Assessments *for* Learning 83

CHAPTER 6 Teaching Mathematics Equitably to All Students 103

PART II Development of Mathematical Concepts and Procedures

CHAPTER 7 Developing Early Number Concepts and Number Sense 125

CHAPTER 8 Developing Meanings for the Operations 153

CHAPTER 9 Developing Basic Fact Fluency 183

CHAPTER 10 Developing Whole-Number Place-Value Concepts 211

CHAPTER 11 Developing Strategies for Addition and Subtraction Computation 238

CHAPTER 12 Developing Strategies for Multiplication and Division Computation 273

CHAPTER 13 Algebraic Thinking, Equations, and Functions 299

CHAPTER 14 Developing Fraction Concepts 337

CHAPTER 15 Developing Fraction Operations 373

CHAPTER 16 Developing Decimal and Percent Concepts and Decimal Computation 405

CHAPTER 17 Ratios, Proportions, and Proportional Reasoning 435

CHAPTER 18 Developing Measurement Concepts 460

CHAPTER 19 Developing Geometric Thinking and Geometric Concepts 500

CHAPTER 20 Developing Concepts of Data and Statistics 543

CHAPTER 21 Exploring Concepts of Probability 580

CHAPTER 22 Developing Concepts of Exponents, Integers, and Real Numbers 604

APPENDIX A Standards for Mathematical Practice 632

APPENDIX B NCTM Mathematics Teaching Practices from *Principles to Actions* 635

APPENDIX C Guide to Blackline Masters 637

APPENDIX D Activities at a Glance 643

Contents

Preface xiii

PART I Teaching Mathematics: Foundations and Perspectives

The fundamental core of effective teaching of mathematics combines an understanding of how students learn, how to promote that learning by teaching through problem solving, and how to plan for and assess that learning daily. That is the focus of these first six chapters, providing discussion, examples, and activities that develop the core ideas of learning, teaching, planning, and assessment for each and every student.



CHAPTER 1
Teaching Mathematics in the 21st Century 1

Becoming an Effective Teacher of Mathematics 1
A Changing World 2
Factors to Consider 3
The Movement toward Shared Standards 4
Mathematics Content Standards 5
The Process Standards and Standards for Mathematical Practice 6
How to Effectively Teach the Standards 8
An Invitation to Learn and Grow 9
Becoming a Teacher of Mathematics 10
Resources for Chapter 1 12



CHAPTER 2
Exploring What It Means to Know and Do Mathematics 13

What Does It Mean to Do Mathematics? 13
Goals for Students 14
An Invitation to Do Mathematics 14
Where Are the Answers? 19
What Does It Mean to Know Mathematics? 19
Relational Understanding 20
Mathematical Proficiency 22
How Do Students Learn Mathematics? 25
Constructivism 25
Sociocultural Theory 25
Implications for Teaching Mathematics 26
Connecting the Dots 28
Resources for Chapter 2 29



CHAPTER 3
Teaching through Problem Solving 30

Problem Solving 30
Teaching *for* Problem Solving 31
Teaching *about* Problem Solving 31
Teaching *through* Problem Solving 34
Teaching Practices for Teaching through Problem Solving 35
Ensuring Success for Every Student 35
Tasks That Promote Problem Solving 37
High-Level Cognitive Demand 37
Multiple Entry and Exit Points 38
Relevant Contexts 40
Evaluating and Adapting Tasks 42
Developing Procedural Fluency 45
Example Tasks 45
What about Drill and Practice? 47
Orchestrating Classroom Discourse 48
Classroom Discussions 48
Questioning Considerations 51
How Much to Tell and Not to Tell 52
Writing 52
Resources for Chapter 3 54



CHAPTER 4
Planning in the Problem-Based Classroom 55

A Three-Phase Lesson Format 55
The *Before* Lesson Phase 55
The *During* Lesson Phase 58
The *After* Lesson Phase 60

Process for Preparing a Lesson 62
Step 1: Determine the Learning Goals 62
Step 2: Consider Your Students' Needs 63
Step 3: Select, Design, or Adapt a Worthwhile Task 63
Step 4: Design Lesson Assessments 64
Step 5: Plan the *Before* Phase 64
Step 6: Plan the *During* Phase 65
Step 7: Plan the *After* Phase 65
Step 8: Reflect and Refine 66
High-Leverage Routines 66
3-Act Math Tasks 67
Number Talks 67
Worked Examples 67
Warm-ups and Short Tasks 68
Learning Centers 69
Differentiating Instruction 69
Open Questions 70
Tiered Lessons 70
Parallel Tasks 73
Flexible Grouping 73
Planning for Family Engagement 74
Communicating Mathematics Goals 74
Family Math Nights 75
Homework Practices 78
Resources for Families 79
Involving *All* Families 80
Resources for Chapter 4 81



CHAPTER 5
Creating Assessments *for* Learning 83

Integrating Assessment into Instruction 83
What Are the Main Assessment Types? 84
What Should Be Assessed? 85
Assessment Methods 86
Observations 86
Questions 88
Interviews 88
Tasks 91
Rubrics and Their Uses 95
Generic Rubrics 95
Task-Specific Rubrics 96
Student Self-Assessment 98
Tests 99
Expanding the Usefulness of Tests 99
Improving Performance on High-Stakes Tests 100
Communicating Grades and Shaping Instruction 100
Grading 100
Shaping Instruction 101
Resources for Chapter 5 102



CHAPTER 6
Teaching Mathematics Equitably to All Students 103

Mathematics for Each and Every Student 104
Providing for Students Who Struggle and Those with Special Needs 105
Multitiered System of Support: Response to Intervention 105
Implementing Interventions 106
Teaching and Assessing Students with Learning Disabilities 110
Adapting for Students with Moderate/Severe Disabilities 112
Culturally and Linguistically Diverse Students 112
Funds of Knowledge 113
Mathematics as a Language 113
Culturally Responsive Mathematics Instruction 114
Teaching Strategies That Support Culturally and Linguistically Diverse Students 116
Focus on Academic Vocabulary 116
Foster Student Participation during Instruction 118
Implementing Strategies for English Learners 120
Providing for Students Who Are Mathematically Gifted 121
Acceleration and Pacing 122
Depth 122
Complexity 122
Creativity 122
Strategies to Avoid 123
Reducing Resistance and Building Resilience 123
Give Students Choices That Capitalize on Their Unique Strengths 123
Nurture Traits of Resilience 123
Make Mathematics Irresistible 124
Give Students Leadership in Their Own Learning 124
Resources for Chapter 6 124

PART II Teaching Student-Centered Mathematics

Each of these chapters *applies* the core ideas of Part I to the content taught in K–8 mathematics. Clear discussions are pro vided for how to teach the topic, what a learning progression for that topic might be, and what worthwhile tasks look like Hundreds of problem-based, engaging tasks and activities are provided to show how the concepts can be developed witl students. These chapters are designed to help you develop pedagogical strategies now, and serve as a resource and referenc for your teaching now and in the future.



CHAPTER 7
Developing Early Number Concepts and Number Sense 125

Promoting Good Beginnings 126
The Number Core: Quantity, Counting, and Cardinality 127
Quantity and the Ability to Subitize 127
Counting 128
Cardinality 130
Thinking about Zero 131
Numeral Writing and Recognition 131
Counting On and Counting Back 133
The Relations Core: More Than, Less Than, and Equal To 134
Developing Number Sense by Building Number Relationships 136
Relationships between Numbers 1 through 10 136
Relationships for Numbers 10 through 20 and Beyond 145
Number Sense in Their World 147
Calendar Activities 147
Estimation and Measurement 148
Represent and Interpret Data 149
Resources for Chapter 7 151



CHAPTER 8
Developing Meanings for the Operations 153

Developing Addition and Subtraction Operation Sense 154
Addition and Subtraction Problem Structures 155
Teaching Addition and Subtraction 158
Properties of Addition and Subtraction 164
Developing Multiplication and Division Operation Sense 166
Multiplication and Division Problem Structures 166
Teaching Multiplication and Division 169
Properties of Multiplication and Division 174
Strategies for Teaching Operations through Contextual Problems 175
Resources for Chapter 8 182



CHAPTER 9
Developing Basic Fact Fluency 183

Teaching and Assessing the Basic Facts 184
Developmental Phases for Learning Basic Facts 184
Approaches to Teaching Basic Facts 184
Teaching Basic Facts Effectively 186
Assessing Basic Facts Effectively 188
Reasoning Strategies for Addition Facts 189
One More Than and Two More Than (Count On) 190
Adding Zero 191
Doubles 192
Combinations of 10 193
10 + _____ 193
Making 10 193
Use 10 195
Using 5 as an Anchor 195
Near-Doubles 195
Reasoning Strategies for Subtraction Facts 196
Think-Addition 196
Down under 10 198
Take from 10 198
Reasoning Strategies for Multiplication and Division Facts 199
Foundational Facts: 2, 5, 10, 0, and 1 199
Nines 201
Derived Multiplication Fact Strategies 202
Division Facts 203
Reinforcing Basic Fact Mastery 205
Games to Support Basic Fact Fluency 205
About Drill 207
Fact Remediation 208
Resources for Chapter 9 210



CHAPTER 10
Developing Whole-Number Place-Value Concepts 211

Pre-Place-Value Understandings 212
Developing Whole-Number Place-Value Concepts 213
Integrating Base-Ten Groupings with Counting by Ones 213
Integrating Base-Ten Groupings with Words 214
Integrating Base-Ten Groupings with Place-Value Notation 215
Base-Ten Models for Place Value 215
Groupable Models 216
Pregrouped Models 216
Nonproportional Models 217
Activities to Develop Base-Ten Concepts 217
Grouping Activities 218
Grouping Tens to Make 100 220
Equivalent Representations 221
Reading and Writing Numbers 223
Two-Digit Number Names 223
Three-Digit Number Names 225
Written Symbols 225
Place Value Patterns and Relationships—A Foundation for Computation 227
The Hundreds Chart 227
Relative Magnitude Using Benchmark Numbers 230

Approximate Numbers and Rounding 232
Connections to Real-World Ideas 232
Numbers Beyond 1000 232
Extending the Place-Value System 232
Conceptualizing Large Numbers 234
Resources for Chapter 10 237



CHAPTER 11
Developing Strategies for Addition and Subtraction Computation 238

Toward Computational Fluency 239
Connecting Addition and Subtraction to Place Value 240
Three Types of Computational Strategies 246
Direct Modeling 246
Invented Strategies 247
Standard Algorithms 249
Development of Invented Strategies in Addition and Subtraction 251
Creating a Supportive Environment for Invented Strategies 251
Models to Support Invented Strategies 252
Adding and Subtracting Single-Digit Numbers 254
Adding Multidigit Numbers 256
Subtraction as "Think-Addition" 258
Take-Away Subtraction 259
Extensions and Challenges 260
Standard Algorithms for Addition and Subtraction 261
Standard Algorithm for Addition 261
Standard Algorithm for Subtraction 263
Introducing Computational Estimation 265
Understanding Computational Estimation 265
Suggestions for Teaching Computational Estimation 265
Computational Estimation Strategies 267
Front-End Methods 267
Rounding Methods 267
Compatible Numbers 268
Resources for Chapter 11 272



CHAPTER 12
Developing Strategies for Multiplication and Division Computation 273

Invented Strategies for Multiplication 274
Useful Representations 274
Multiplication by a Single-Digit Multiplier 275
Multiplication of Multidigit Numbers 276
Standard Algorithms for Multiplication 279
Begin with Models 279
Develop the Written Record 282
Invented Strategies for Division 283
Standard Algorithm for Division 286
Begin with Models 287
Develop the Written Record 288
Two-Digit Divisors 290
A Low-Stress Approach 291
Computational Estimation 292
Teaching Computational Estimation 292
Computational Estimation Strategies 293
Resources for Chapter 12 298



CHAPTER 13
Algebraic Thinking, Equations, and Functions 299

Strands of Algebraic Thinking 300
Connecting Number and Algebra 300
Number Combinations 300
Place-Value Relationships 301
Algorithms 303
Properties of the Operations 303
Making Sense of Properties 303
Applying the Properties of Addition and Multiplication 306
Study of Patterns and Functions 307
Repeating Patterns 308
Growing Patterns 309
Relationships in Functions 310
Graphs of Functions 312
Linear Functions 314
Meaningful Use of Symbols 317
Equal and Inequality Signs 318
The Meaning of Variables 325
Mathematical Modeling 330
Algebraic Thinking across the Curriculum 332
Geometry, Measurement and Algebra 332
Data and Algebra 333
Algebraic Thinking 334
Resources for Chapter 13 335



CHAPTER 14
Developing Fraction Concepts 337

Meanings of Fractions 338
Fraction Constructs 338
Fraction Language and Notation 339
Fraction Size Is Relative 340
Models for Fractions 341
Area Models 341

Length Models 343
Set Models 344
Fractions as Numbers 346
Partitioning 346
Iterating 354
Magnitude of Fractions 357
Equivalent Fractions 359
Conceptual Focus on Equivalence 359
Equivalent Fraction Models 360
Fractions Greater than 1 363
Developing an Equivalent-Fraction Algorithm 365
Comparing Fractions 367
Comparing Fractions Using Number Sense 367
Using Equivalent Fractions to Compare 370
Teaching Considerations for Fraction Concepts 370
Fraction Challenges and Misconceptions 370
Resources for Chapter 14 372

CHAPTER 15
Developing Fraction Operations 373

Understanding Fraction Operations 374
Effective Teaching Process 374
Addition and Subtraction 376
Contextual Examples 376
Models 377
Estimation 380
Developing the Algorithms 381
Fractions Greater Than One 384
Challenges and Misconceptions 384
Multiplication 386
Contextual Examples and Models 386
Estimation 392
Developing the Algorithms 393
Factors Greater Than One 393
Challenges and Misconceptions 394
Division 394
Contextual Examples and Models 395
Answers That Are Not Whole Numbers 400
Estimation 401
Developing the Algorithms 401
Challenges and Misconceptions 403
Resources for Chapter 15 404

CHAPTER 16
Developing Decimal and Percent Concepts and Decimal Computation 405

Extending the Place-Value System 406
The 10-to-1 Relationship—Now in Two Directions! 406
The Role of the Decimal Point 407
Measurement and Monetary Units 408
Precision and Equivalence 409
Connecting Fractions and Decimals 409
Say Decimal Fractions Correctly 409
Use Visual Models for Decimal Fractions 410
Multiple Names and Formats 412
Developing Decimal Number Sense 414
Familiar Fractions Connected to Decimals 414
Comparing and Ordering Decimal Fractions 418
Computation with Decimals 421
Addition and Subtraction 421
Multiplication 423
Division 426
Introducing Percents 428
Physical Models and Terminology 428
Percent Problems in Context 430
Estimation 432
Resources for Chapter 16 434

CHAPTER 17
Ratios, Proportions, and Proportional Reasoning 435

Ratios 436
Types of Ratios 436
Ratios Compared to Fractions 437
Two Ways to Think about Ratio 437
Proportional Reasoning 438
Types of Comparing Situations 439
Covariation 443
Strategies for Solving Proportional Situations 449
Rates and Scaling Strategies 450
Ratio Tables 453
Tape or Strip Diagram 454
Double Number Line Diagrams 455
Equations (Cross Products) 456
Percent Problems 457
Teaching Proportional Reasoning 458
Resources for Chapter 17 459

CHAPTER 18
Developing Measurement Concepts 460

The Meaning and Process of Measuring 461
Concepts and Skills 462
Introducing Nonstandard Units 463
Introducing Standard Units 464
Developing Unit Familiarity 464

Measurement Systems and Units 466
The Role of Estimation and Approximation 466
Strategies for Estimating Measurements 467
Measurement Estimation Activities 468
Length 469
Comparison Activities 469
Using Physical Models of Length Units 470
Making and Using Rulers 471
Conversion 473
Area 474
Comparison Activities 474
Using Physical Models of Area Units 475
The Relationship between Area and Perimeter 477
Developing Formulas for Perimeter and Area 479
Volume and Capacity 484
Comparison Activities 485
Using Physical Models of Volume and Capacity Units 486
Developing Formulas for Volumes of Common Solid Shapes 487
Weight and Mass 489
Comparison Activities 489
Using Physical Models of Weight or Mass Units 489
Angles 490
Comparison Activities 490
Using Physical Models of Angular Measure Units 490
Using Protractors 491
Time 492
Comparison Activities 492
Reading Clocks 493
Elapsed Time 494
Money 495
Recognizing Coins and Identifying Their Values 495
Resources for Chapter 18 499



CHAPTER 19
Developing Geometric Thinking and Geometric Concepts 500

Geometry Goals for Students 501
Developing Geometric Thinking 501
The van Hiele Levels of Geometric Thought 502
Implications for Instruction 506
Shapes and Properties 507
Sorting and Classifying 508
Composing and Decomposing Shapes 510
Categories of Two- and Three-Dimensional Shapes 512
Construction Activities 515
Applying Definitions and Categories 516
Exploring Properties of Triangles 516
Midsegments of a Triangle 518
Exploring Properties of Quadrilaterals 519
Exploring Polygons 521
Circles 521
Investigations, Conjectures, and the Development of Proof 522
Transformations 523
Symmetries 525
Composition of Transformations 526
Congruence 528
Similarity 529
Dilation 529
Location 530
Coordinate Plane 531
Measuring Distance on the Coordinate Plane 534
Visualization 534
Two-Dimensional Imagery 535
Three-Dimensional Imagery 536
Resources for Chapter 19 542



CHAPTER 20
Developing Concepts of Data and Statistics 543

What Does It Mean to Do Statistics? 544
Is It Statistics or Is It Mathematics? 545
The Shape of Data 545
The Process of Doing Statistics 546
Formulating Questions 549
Classroom Questions 549
Questions beyond Self and Classmates 549
Data Collection 552
Sampling 552
Using Existing Data Sources 553
Data Analysis: Classification 554
Attribute Materials 554
Data Analysis: Graphical Representations 557
Creating Graphs 557
Bar Graphs 558
Pie Charts and Circle Graphs 560
Continuous Data Graphs 561
Bivariate Data 564
Data Analysis: Measures of Center and Variability 568
Measures of Center 568
Understanding the Mean 569
Choosing a Measure of Center 572
Variability 574
Analyzing Data 576
Interpreting Results 577
Resources for Chapter 20 578



CHAPTER 21
Exploring Concepts of Probability 580

ntroducing Probability 581
Likely or Not Likely 581
The Probability Continuum 584
heoretical Probability and Experiments 585
Process for Teaching Probability 586
Theoretical Probability 587
Experiments 589
Why Use Experiments? 592
Use of Technology in Experiments 592
ample Spaces and the Probability of Compound Events 593
Independent Events 593
Area Representation 596
Dependent Events 597
imulations 598
tudent Assumptions Related to Probability 601
esources for Chapter 21 602



CHAPTER 22
Developing Concepts of Exponents, Integers, and Real Numbers 604

Exponents 605
Exponents in Expressions and Equations 605
Order of Operations 606
Exploring Exponents on the Calculator 609
Integer Exponents 610
Scientific Notation 611
Positive and Negative Numbers 614
Contexts for Exploring Positive and Negative Numbers 614
Meaning of Negative Numbers 616
Tools for Illustrating Positive and Negative Numbers 618
Operations with Positive and Negative Numbers 619
Addition and Subtraction 619
Multiplication 623
Division 624
Real Numbers 626
Rational Numbers 626
Irrational Numbers 627
Supporting Student Reasoning about Number 629
Resources for Chapter 22 631

APPENDIX A Standards for Mathematical Practice 632
APPENDIX B NCTM Mathematics Teaching Practices from *Principles to Actions* 635
APPENDIX C Guide to Blackline Masters 637
APPENDIX D Activities at a Glance 643

References 662
ndex 687
Credits 697