Problem 1. 20%

Let G be a simple graph with n vertices and m edges. Show that

(a1) $\sum_{uv \in E(G)} (d(u)+d(v)) = \sum_{v \in V(G)} d^2(v)$

(a2) $\sum_{v \in V(G)} d^2(v) \geq \frac{4|E(G)|^2}{n}$

(b) Let G be a simple graph with n vertices and m edges. Show that G contains a quadrilateral if $m > (n\sqrt{4n-3}+1)/4$.

Problem 2. 20%

(1) Theorem 2.4 can be found from Bondy and Murty's new edition textbook.

Deduce from Theorem 2.4 that every loopless graph G contains a spanning bipartite subgraph F with $e(F) \geq \frac{1}{2}e(G)$.

(2)

Let $G[X,Y]$ be a bipartite graph, each vertex of which is joined to at least one, but not all, vertices in the other part. Suppose that $d(x) \geq d(y)$ for all $xy \notin E$. Show that $|Y| \geq |X|$, with equality if and only if $d(x) = d(y)$ for all $xy \notin E$ with $x \in X$ and $y \in Y$.

Problem 3. 20%

(1) Show that all eigenvalues of a graph G are zero if and only G is a null graph.

(2)

If G is bipartite, and λ is an eigenvalue of adjacency matrix A, then so is $-\lambda$.

Problem 4. 20% Let $(d_1, d_2, \ldots, d_n)$ be a non-increasing sequence of nonnegative integers.

Prove that there exists a loopless graph with degree sequence $(d_1, d_2, \ldots, d_n)$ if and only if

$\sum_{i=1}^{n} d_i$ is even and $d_1 \le \sum_{i=2}^{n} d_i$.

Problem 5. 20%

Let P stand for Petersen graph. Let P* stand for the graph by deleting a vertex from P.

Show that

(a1) The edge chromatic index of P is 4.

(a2) Deduce that P is non-Hamiltonian.

(b) The edge chromatic index of P* is 4.