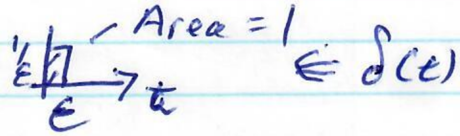
SYSC 4505: Review of Linear Systems 1

Review of Laplace Transforms,

$$\mathcal{L}\{x(t)\} = X(s) = \int_0^\infty x(t)e^{-st}\,dt$$

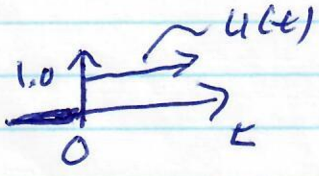
The important L.T. are.

1) L.T. of Unit Impulse



$$\mathcal{L}\{\delta(t)\} = 1$$

2) L.T. of Unit Step.

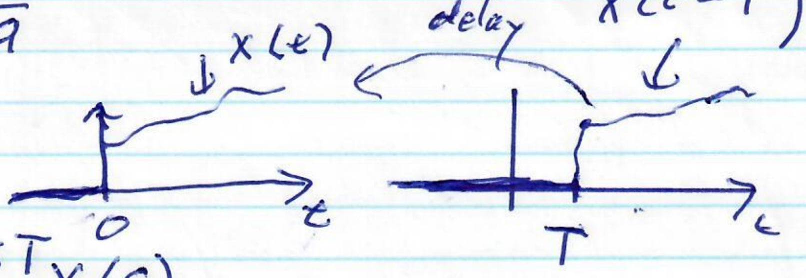


$$\mathcal{L}\{u(t)\} = 1/s$$

3) L.T. of the exponential.

$$\mathcal{L}\{e^{-at}\} = \frac{1}{s+a}$$

4) L.T. of delay



$$\mathcal{L}\{x(t-T)\} = e^{-sT}X(s)$$

$v = m/s$ $\quad l/v$ sec. delay

②

5) L.T. of integral

$$\mathcal{L}\left\{\int x(t)\,dt\right\} = \frac{1}{s} X(s)$$

6) L.T. of the derivative of Signal

$$\mathcal{L}\{\dot{x}(t)\} = s X(s) - X(0)$$

In general, n^{th} derivative

$$\mathcal{L}\{x^{n}(t)\} = s^{n} X(s) - s^{n-1} X(0) - s^{n-2}\dot{x}(0) - \dots - X^{n-1}(0)$$

7) Final Value Theorem

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

8) L.T. of the convolution

(To find the output of a system one convolves the unit impulse response with the input)

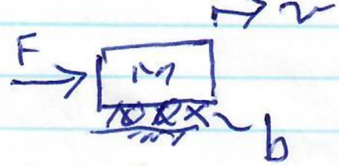
$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(t-\tau)\, x(\tau)\, d\tau$$

$$\mathcal{L}\{y(t)\} = Y(s) = h(s)\, X(s)$$ multiplication in the Laplace Domain.

9) L.T of multiplication with time

$$\mathcal{L}\{t^{n} x(t)\} = (-1)^{n} \frac{d^{n}X(s)}{ds^{n}} \Rightarrow$$ ramp input

$x(t) = t\,u(t)$ $\Rightarrow \frac{1}{s^{2}}$, Using L.T. to Solve D.E.



Viscous friction

$$M\dot{v} = F - bv$$

$$\dot{v} + \frac{b}{M}v = \frac{1}{M}F$$ 1st order dynamic system

In general a First order Dynamic System

$$\dot{x}(t) + a x(t) = b \overbrace{u(t)}^{\text{input}}$$ x(t) is output.

We take L.T. of both sides.

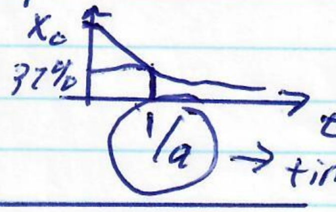
$$s X(s) - X(0) + a X(s) = b u(s)$$

$$(s+a) X(s) = X(0) + b u(s) \quad \mathcal{L}^{-1}$$

$$X(s) = \frac{b}{s+a} u(s) + \boxed{\frac{X(0)}{s+a}}$$ complete Solution

What is I.C. response $u(t) = 0$, $X(0) = X_0$

$$X(t) = e^{-at} X_0$$



→ time constant!

Transfer Function ⇒ L.T. of Unit Impulse Resp.

$$\frac{X(s)}{u(s)} = H(s) = \frac{b}{s+a}$$

Unit step Response. Convolve H(s) with $\mathcal{L}\{u(t)\} = 1/s$

$$X(s) = \boxed{\frac{b}{s+a}} \cdot \boxed{1/s}$$

Unit Step Response

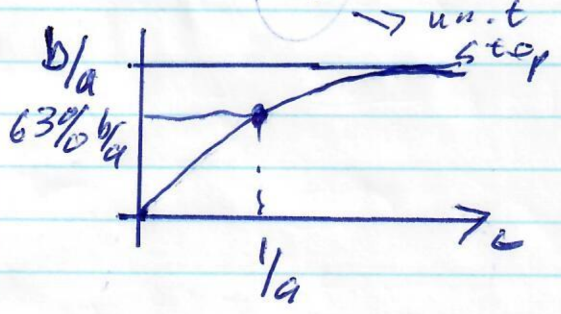
$X(s) = \frac{b}{s(s+a)}$ Take the inverse L.T.

$\mathcal{L}^{-1}\{X(s)\} = \frac{A}{s} + \frac{B}{s+a}$ } partial fraction expansion $\rightarrow \frac{b/a}{s} - \frac{b/a}{s+a}$

$A = sX(s)\Big|_{s=0} = \frac{\cancel{s}b}{\cancel{s}(s+a)}\Big|_{s=0} = b/a$

$B = (s+a)X(s)\Big|_{s=-a} = -b/a$

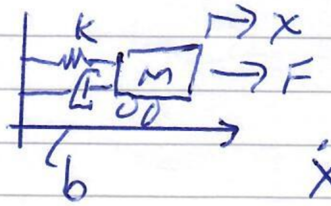
$x(t) = \frac{b}{a} - b/a\, e^{-at} = \frac{b}{a}(1-e^{-at}) \quad t>0$



↙ dynamic of 1st order.

SYSC 4505: Review Lecture 2

Dynamics of Second order (underdamped $0<\zeta<1$)



$$M\ddot{x} = F - kx - b\dot{x}$$

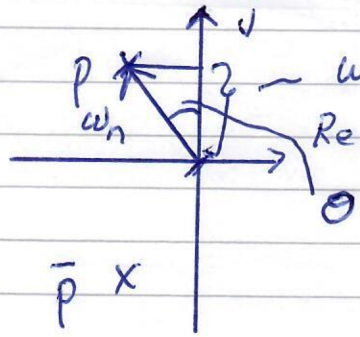
$$\ddot{x} + \frac{b}{M}\dot{x} + \frac{k}{M}x = \frac{1}{M}F$$

Let's write the standard Form of 2nd order dynamic system, in unity gain

$$H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad 0<\zeta<1 \text{ } \} \text{ complex roots.}$$

$$= \frac{\omega_n^2}{(s-p)(s-\bar{p})} \qquad p, \bar{p} = -\zeta\omega_n + \omega_n\sqrt{1-\zeta^2}\,j$$

plot the poles on the s-plane



$\sim \omega_d = \omega_n\sqrt{1-\zeta^2}$

$\sin\theta = \zeta$

$\omega_n \Rightarrow$ undamped Natural freq.

$\omega_d \Rightarrow$ damped Natural freq.

$\zeta \Rightarrow$ Damping ratio.

For unit step input, what is output

$$Y(s) = H(s)\cdot \boxed{X(s)} \qquad H(s) = \frac{Y(s)}{X(s)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

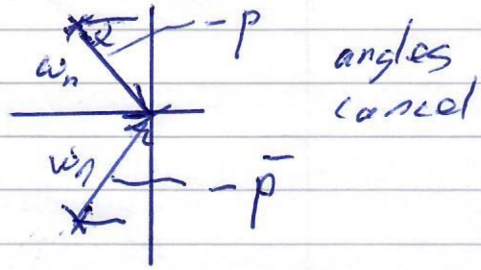
$$= \frac{\boxed{1}\;\omega_n^2}{\boxed{s}\,(s-p)(s-\bar{p})}$$

⑥

Review of 2nd order Systems.

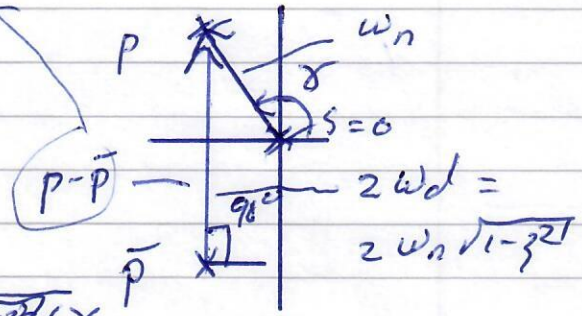
$$y(s) = \frac{\omega_n^2}{s(s-p)(s-\bar{p})} = \frac{A}{s} + \frac{B}{s-p} + \frac{\bar{B}}{s-\bar{p}}$$

$$A = s\,y(s)\Big|_{s=0} = 1$$



$$B = (s-p)\,y(s)\Big|_{s=p} = \frac{\omega_n^2}{p(p-\bar{p})}$$

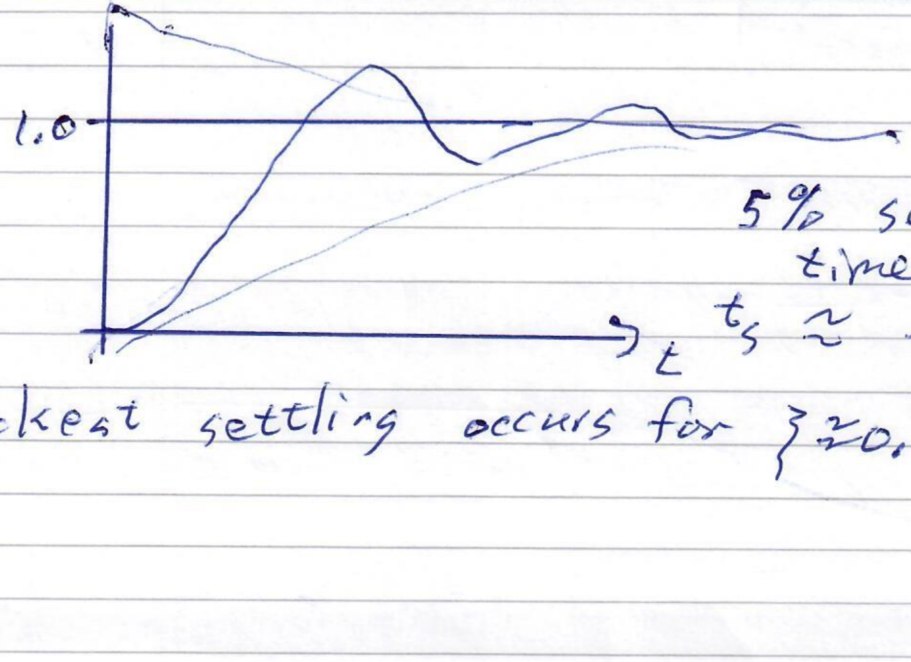
$$p, \bar{p} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\ \checkmark$$



$$B = (s-p)\,y(s)\Big|_{s=p} = \frac{\omega_n^2}{\omega_n \underbrace{2\omega_n\sqrt{1-\zeta^2}}_{\angle 90^\circ}\angle\gamma}$$

$$= |B|\angle B \qquad |B| = \frac{1}{2\sqrt{1-\zeta^2}} \qquad \angle B = -\angle\gamma - 90^\circ$$

$$\underline{y(t) = 1 + 2|B|e^{-\zeta\omega_n t}\cos(\omega_d t + \angle B)}$$



5% settling time

$$t_s \approx \frac{3}{\zeta\omega_n}$$

Quickest settling occurs for $\zeta \approx 0.707$.

Response of higher order Dynamic Systems.

n^{th} order D.E

$$\overset{m^{th}\text{ derivative}}{\swarrow}$$

$$x^n(t) + a_{n-1}x^{n-1}(t) + \dots + a_0 x(t) = b_m u^m(t) + b_{m-1}u^{m-1}(t) + \dots + b_0 u(t)$$

The input/output T.F. is

$$\frac{X(s)}{u(s)} = H(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0} \Big\} \text{ all coefficients are real}$$

Let's say I want the unit impulse response.
Do partial Fraction Expansion.
(roots are real or complex conjugate pairs)

$$H(s) = \frac{b_m(s-z_1)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_i)(s-\bar{p}_i)\cdots(s-p_n)} \quad \xleftarrow{} \text{zero's} \quad m<n$$

$$= \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} + \frac{c_3}{s-p_3} + \frac{\bar{c}_3}{s-\bar{p}_3} + \dots + \frac{c_n}{s-p_n} \quad \Rightarrow \text{poles}$$

Take inverse L.T.

$$h(t) = c_1 e^{p_1 t} + c_2 e^{p_2 t} + 2|c_3| e^{-\zeta_3 \omega_{n_3} t}\cos(\omega_{d_3} t + \angle c_3) + \dots + c_n e^{p_n t}$$

We have a sum of 1^{st} & 2^{nd} order dynamic Systems. If the real parts of the poles are $-ve$, then the system is "Stable"

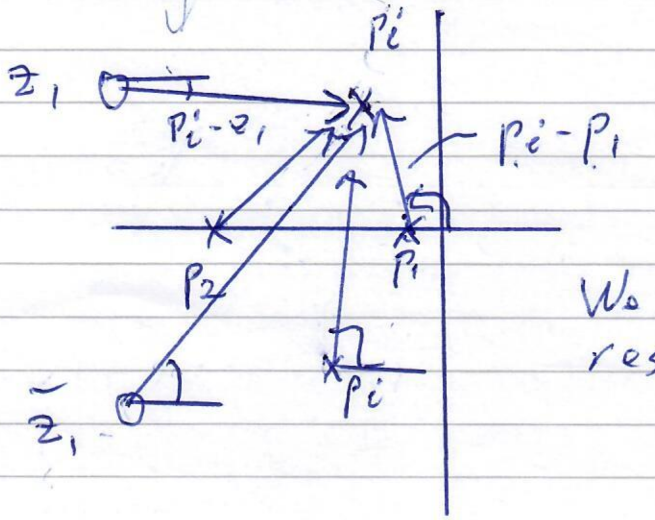
Graphical Computation of Residues

$$H(s) = \frac{c_1}{s-p_1} + \frac{c_2}{s-\overset{*}{p}_2} + \frac{\bar{c}_2}{s-\bar{p}_2} + \frac{c_3}{s-p_3} + \frac{\bar{c}_3}{s-\bar{p}_3} + \cdots$$

Each of the residues is computed as

$$c_i = (s-p_i)H(s)\Big|_{s=p_i}$$

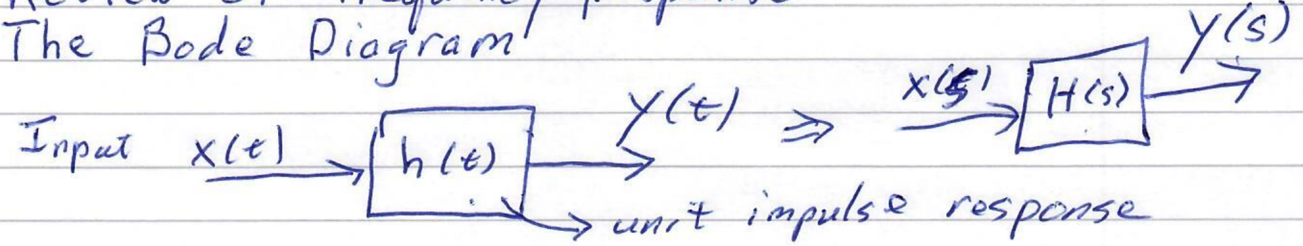
$$= \frac{b_m(p_i-z_1)(p_i-z_2)\cdots(p_i-z_m)}{(p_i-p_1)(p_i-p_2)\cdots(p_i-p_n)}$$



use protractor & ruler

We can see the results.

Review of Frequency Response
The Bode Diagram



My input is $x(t) = A\sin(\omega t)$
The steady state output is

$$y_{ss}(t) = A\underbrace{|H(\omega j)|}\sin(\omega t + \underbrace{\angle H(\omega j)})$$

↳ This is plotted on a logarithmic scale (dB)

gain

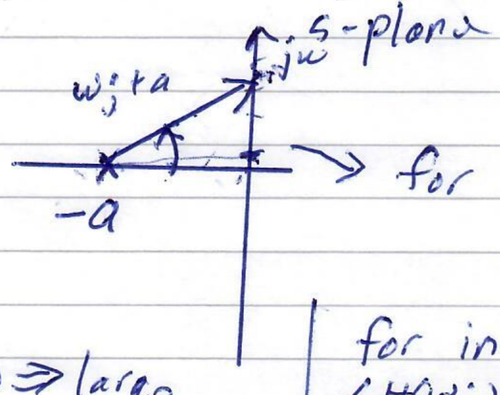
$20\,\text{Log}(k_i) \Rightarrow dB$

Bode Diagram for a first order System

$H(s) = \frac{a}{s+a}$ } unity gain 1^{st} order.

DC Gain $= 1 \Rightarrow 0$ dB, We will graphically

Compute the magnitude & angle of $H(\omega j)$



$\angle H(\omega j)$ $|H(\omega j)|$

→ for small ω, $|H(\omega j)| \approx 1$

$\angle H(\omega j) \approx 0^\circ$

for increasing ω at $\omega = a$ r/s

$\angle H(\omega j) = -45^\circ$ $|H(\omega j)| = -3\,dB$ ←

$20\log\left(\frac{1}{\sqrt{2}}\right)$

as $\omega \Rightarrow$ large

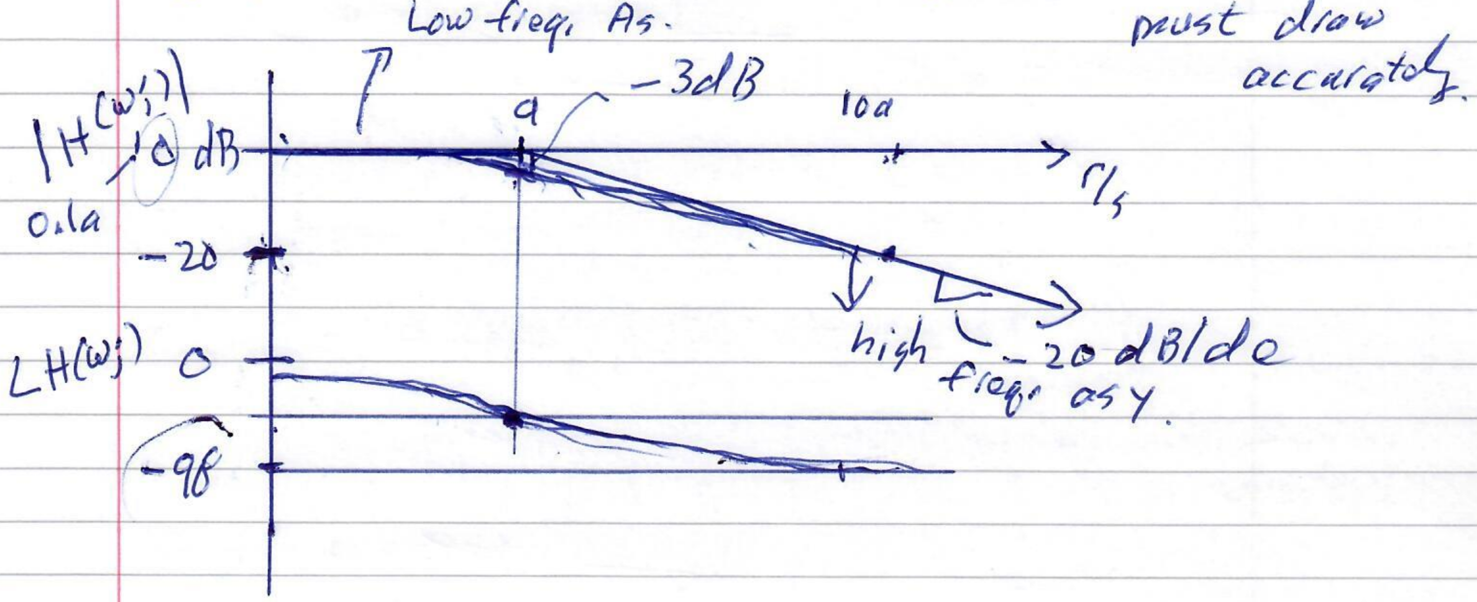
$\angle H(\omega j) \Rightarrow -90^\circ$

drops off at

$-20\,dB/dec$ of increase

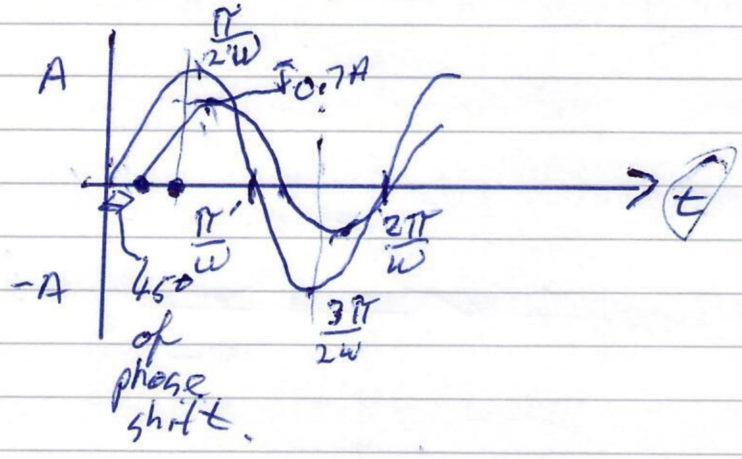
Review of Frequency Response

Rough Sketch. $H(s) = \frac{a}{s+a}$



For higher order dynamic system with real poles & zero we sum the effects of each pole & zero. Zero's are the mirror image of poles.

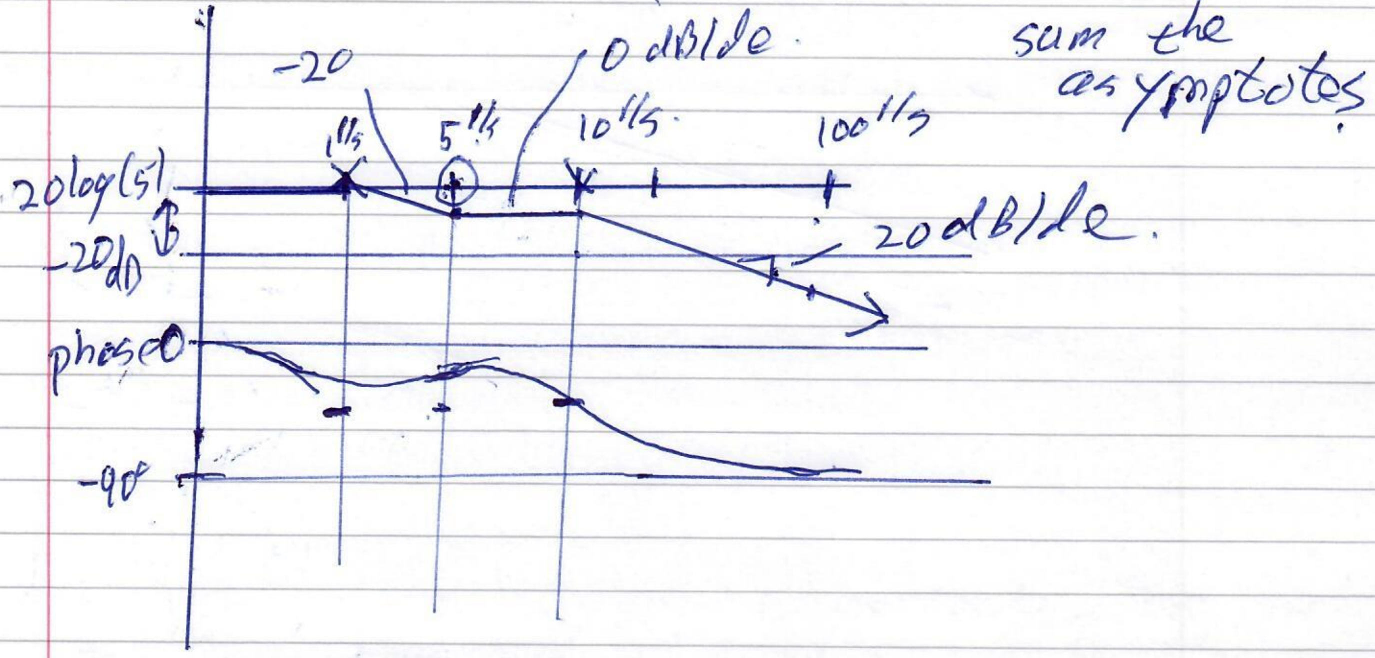
$x(t) = A\sin(\omega t)$



If we have real poles & zero we sum them.

$$H(s) = \frac{\overset{\leq 10}{k}(s+5)}{(s+1)(s+10)}$$

1) DC gain 5
20log(5)



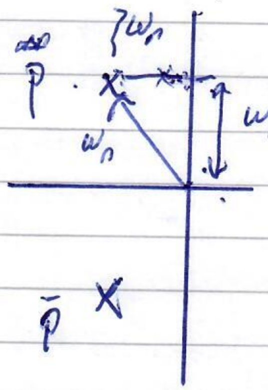
sum the asymptotes.

$$\frac{Y(s)}{X(s)} = H(s) = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad 0<\zeta<1$$

underdampod.

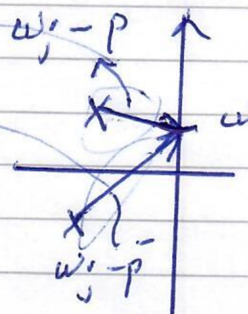
$$p, \bar{p} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\,j$$

$$H(s) = \frac{\omega_n^2}{(s-p)(s-\bar{p})}$$



We get a resonant freq.

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} \Big\} \text{ for small } \zeta \quad \omega_r \approx \omega_n \approx \omega_d$$

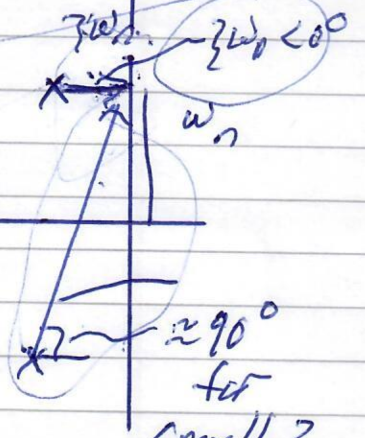


We also know tho height of the peak for small ζ.

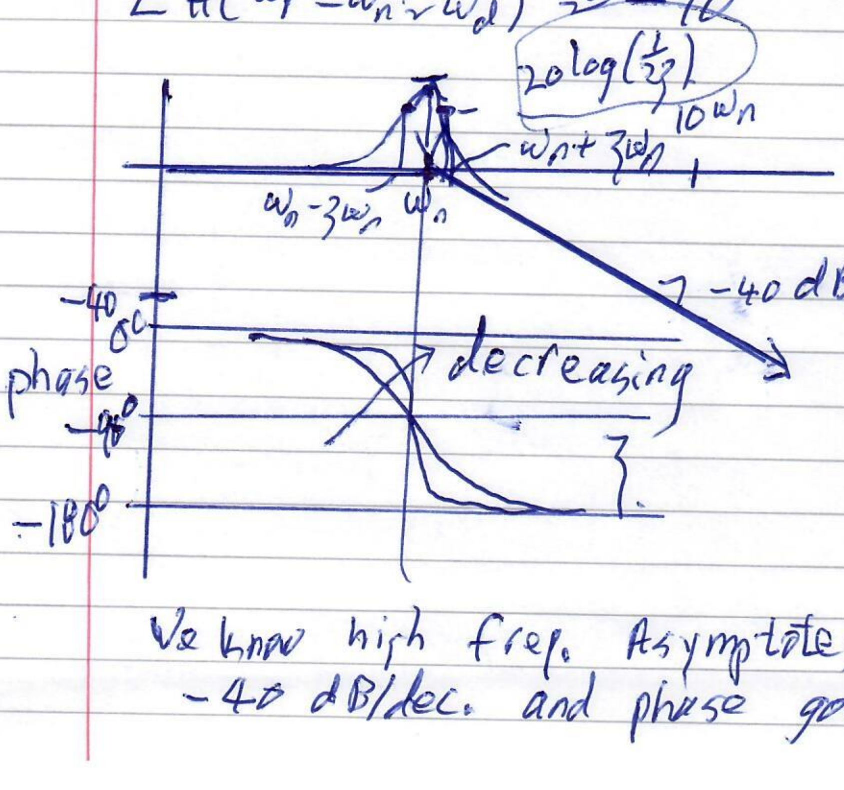
$$|H(\omega_r j)| \approx \frac{1}{2\zeta\sqrt{1-\zeta^2}} \approx \frac{1}{2\zeta}$$

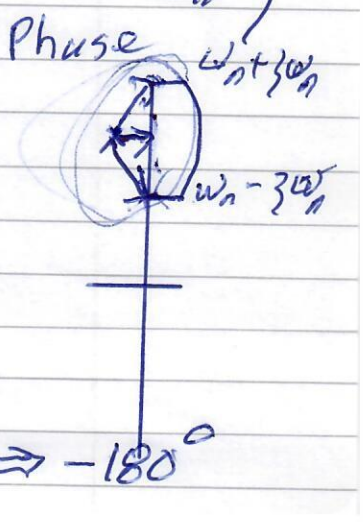
For small ζ

$$|H(\omega_r=\omega_n)| \approx \frac{\omega_n^2}{\zeta\omega_n\angle 0^\circ \; 2\omega_n\angle 90^\circ} = \frac{1}{2\zeta}$$



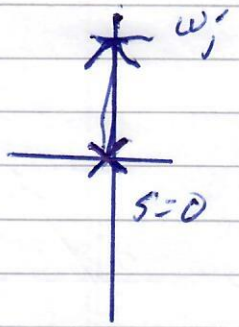
$$\angle H(\omega_r \cong \omega_n \approx \omega_d) \approx -90^\circ$$





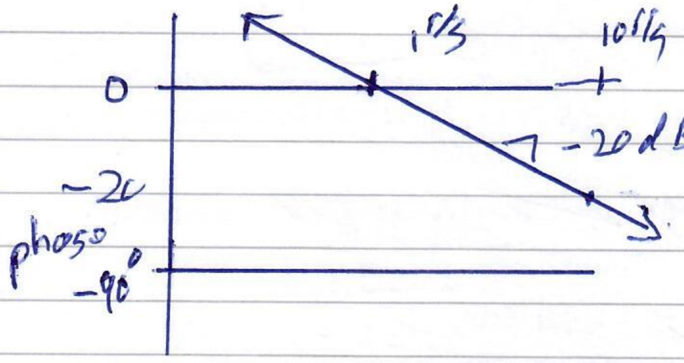
We know high freq. Asymptotes drops at -40 dB/dec. and phase goes from 0 ⇒ -180°

Frequency Response of Integrator.



$H(s) = 1/s \Rightarrow$ it drops @ -20 dB/dec.

$\angle H(\omega j) = -90°$

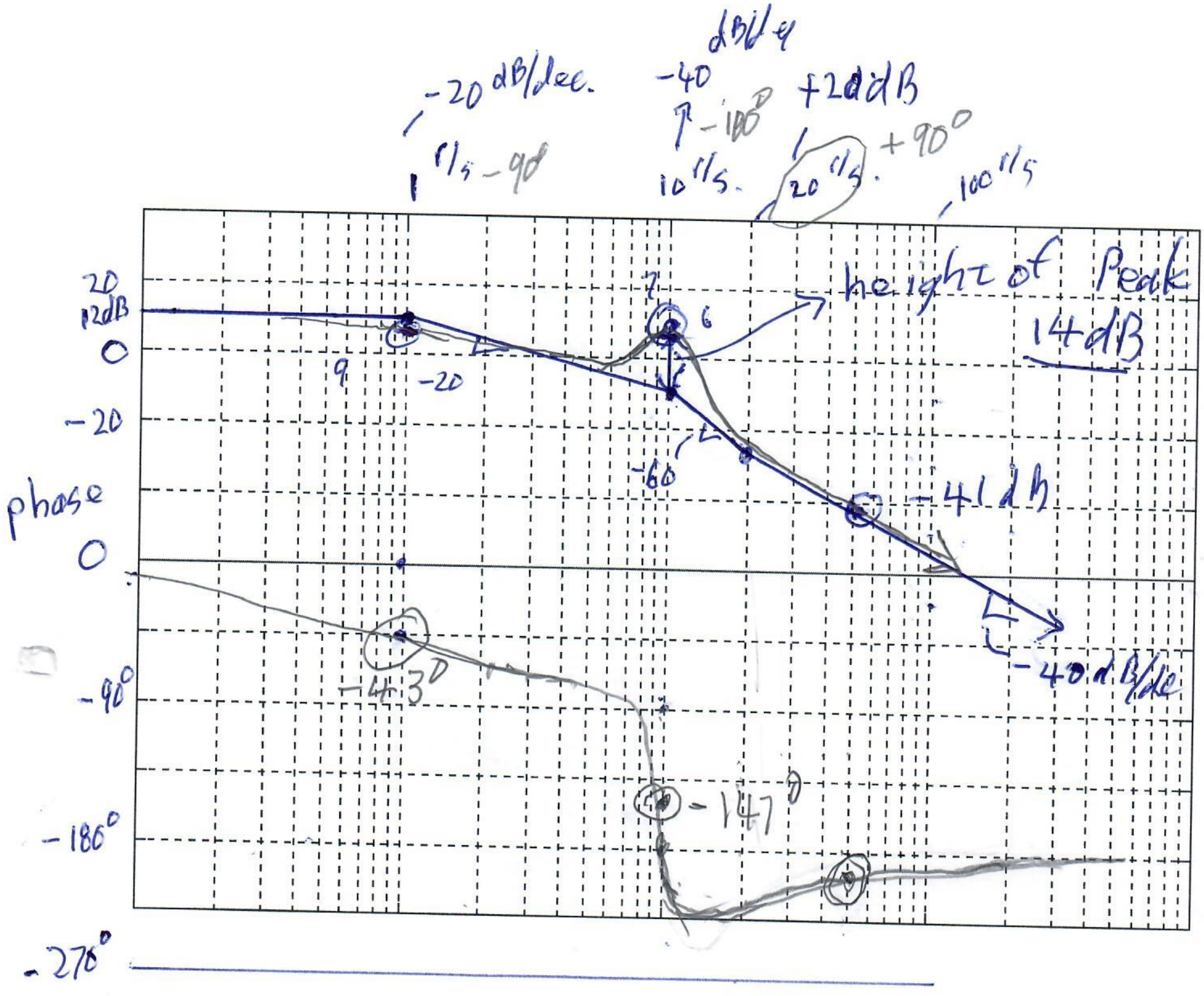


$$H(s) = \frac{20(s+20)}{(s+1)\underbrace{(s-p)(s-\bar{p})}_{\substack{s^2+2s+100 \\ s^2+2\zeta\omega_n s+\omega_n^2}}}$$

DC gain $= 4 = 12$ dB

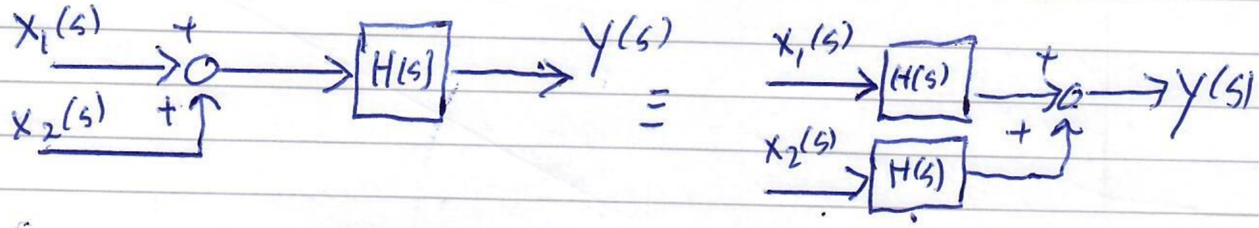
$\omega_n = 10, \ \zeta = 0.1$

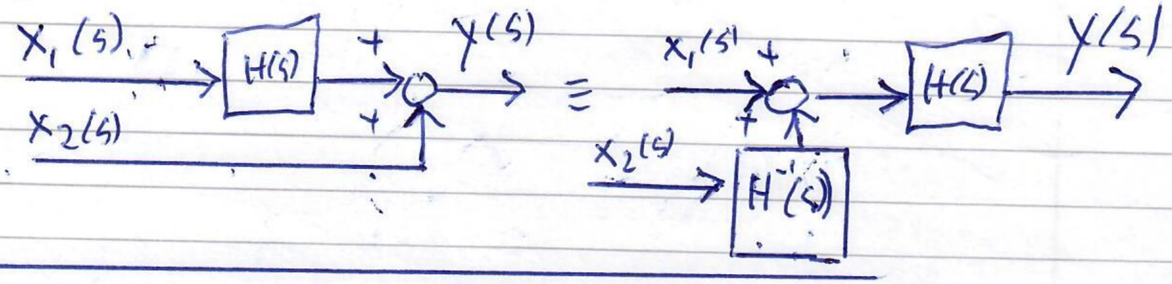
peak $\cong \frac{1}{2\zeta} = 14$ dB


dB/dec
-20 dB/dec.
-40
-180°
+20dB
1 r/s -90°
10 r/s.
20 r/s.
+90°
100 r/s
20
12dB
0
9
-20
7
6
height of Peak
14dB
-20
-60
-41dB
phase
0
-43°
-90°
-147°
-180°
-40dB/dec
-270°

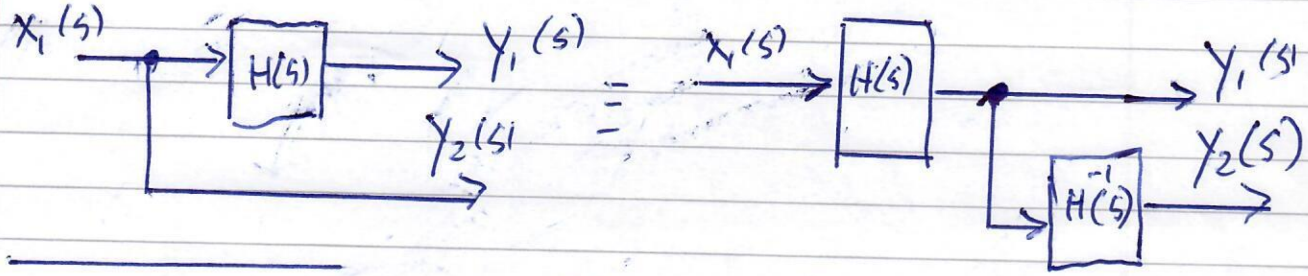
Block Diagram Reduction: Review

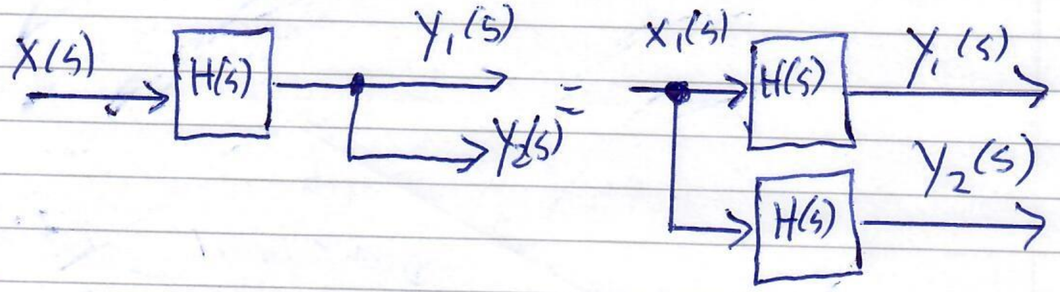
Moving a Summer





Move a Pickoff point

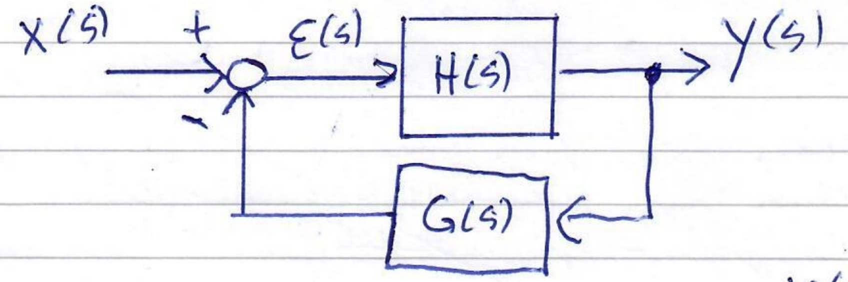




You cannot move a pickoff point across a summer, or vica-versa.

⑯

Reduction of a feedback loop



Closed loop T.F. $H_{CL}(s) = \frac{y(s)}{x(s)}$

$$E(s) = X(s) - G(s)\,y(s) \; ; \quad y(s) = H(s)\,E(s)$$

$$E(s) = X(s) - G(s)\,H(s)\,E(s)$$

$$y(s) = H(s)\big(X(s) - G(s)\,y(s)\big)$$

$$= H(s)\,X(s) - H(s)\,G(s)\,y(s)$$

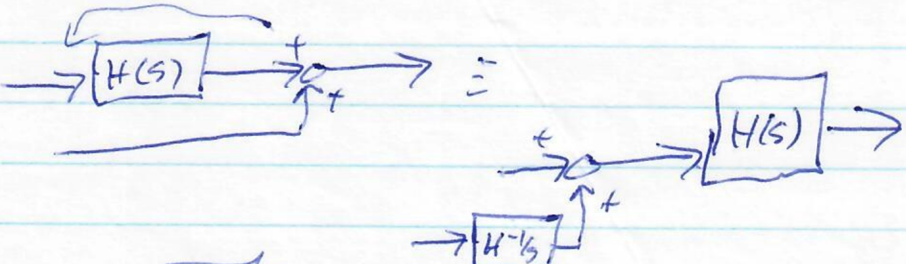
$$\big(1 + H(s)G(s)\big)\,y(s) = H(s)\,X(s)$$

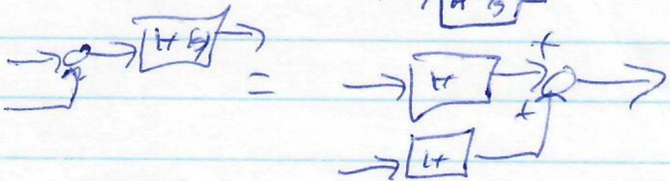
$$\frac{y(s)}{X(s)} = H_{CL}(s) = \frac{H(s)}{1 + H(s)G(s)}$$

Sept. 15, 2011

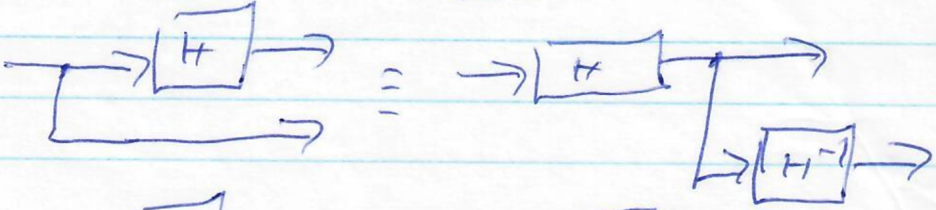
Block Diagram Reduction

Move adders



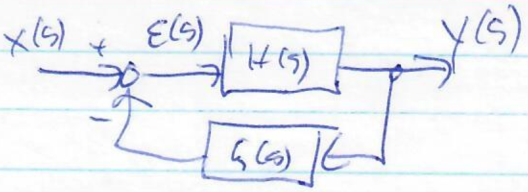


Move pickoff pt.



Cannot move pickoff pt. across summer & Vice Versa.

Feedback loops -



$y(s) = H(s)E(s)$

$E(s) = x(s) - G(s)y(s)$

$y(s) = H(s)x(s) - H(s)G(s)y(s) \qquad (1+H(s)G(s))y(s) = H(s)x(s)$

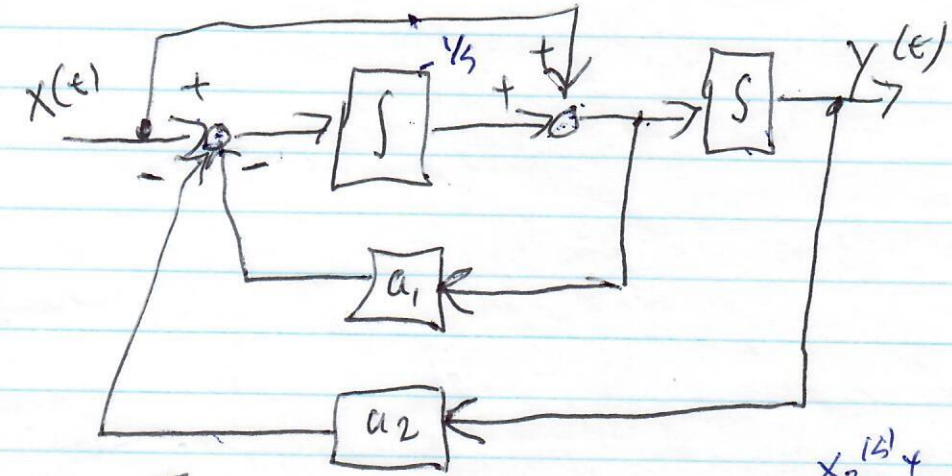
$$\frac{y(s)}{x(s)} = \frac{H(s)}{1+H(s)G(s)}$$

Examples

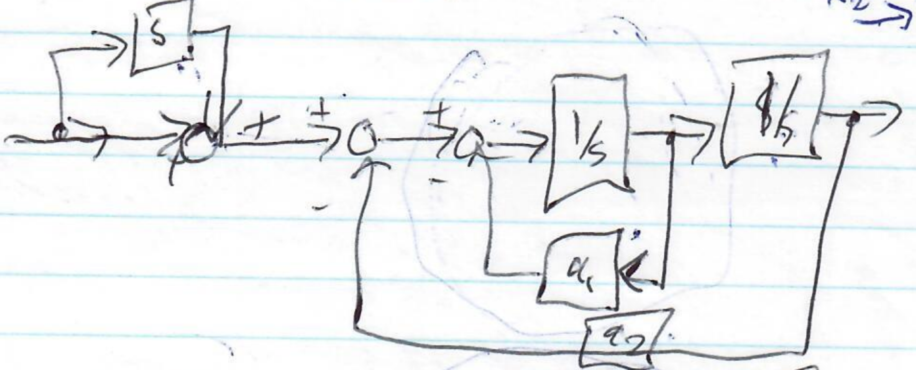
Sept. 19, 2011

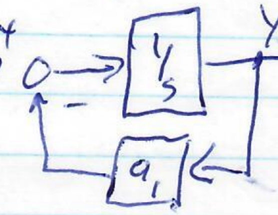
Teaching course review

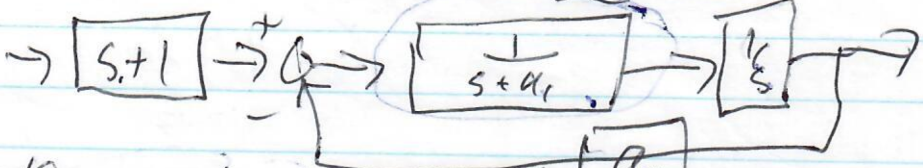
A quick Example



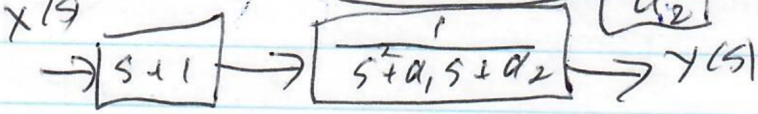
$$\frac{Y_2(s)}{X_2(s)} = \frac{\frac{1}{s}}{1+\frac{a_1}{s}} = \frac{1}{s+a_1}$$







$$\frac{\frac{1}{s(s+a_1)}}{1+\frac{a_2}{s(s+a_1)}} \to \frac{1}{s^2+a_1 s+a_2}$$



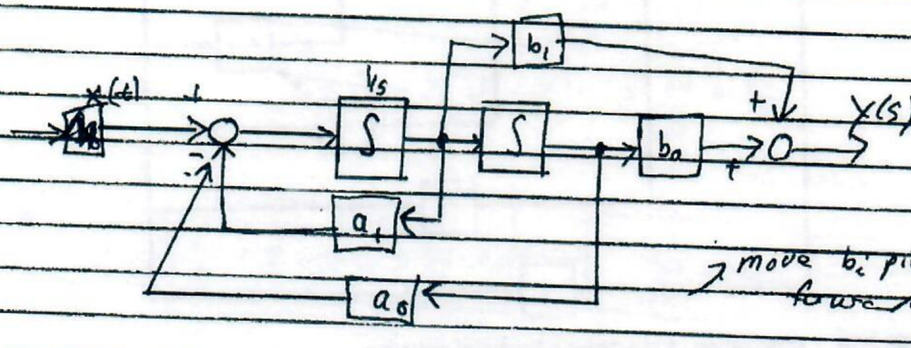
$$\frac{Y(s)}{X(s)} = \frac{s+1}{s^2+a_1 s+a_2} = \frac{s+1}{s^2+a_1 s+a_2}$$

Standard (Canonical) Forms.

Let's say I have an arbitrary D.E. of the form,

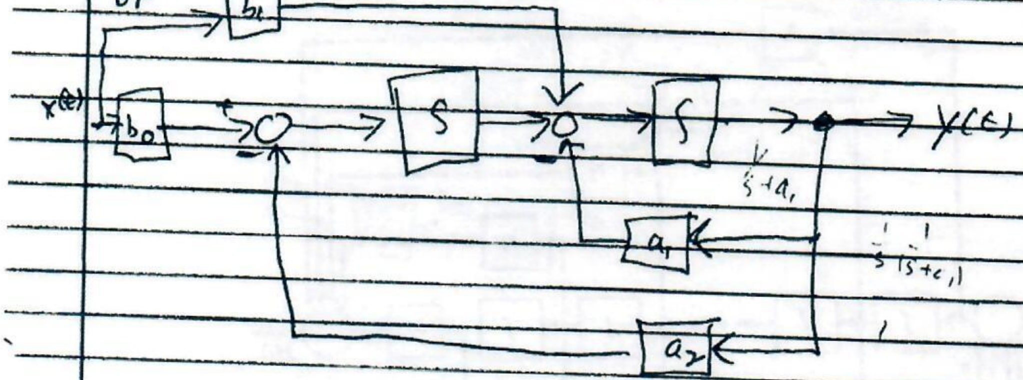
$$\ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = b_1 \dot{x}(t) + b_0 x(t)$$

$$\frac{Y(s)}{X(s)} = \frac{(b_1 s + b_0)}{s^2 + a_1 s + a_0}$$



move b_1 pickoff forward.

or



Modeling of Dynamic Systems.

The order of the systems is equal to the number of independent energy storage devices. We write our equations as a set of 1st order coupled D.E.

m inputs.

n^{th} order.

$$\dot{x}_1(t) = a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2m}u_m$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nm}u_m$$

Outputs

$$y_1 = \overset{c_{11}}{\cancel{a_{11}}}x_1 + c_{12}x_2 + \cdots c_{1n}x_n$$

$$\vdots$$

$$y_k = c_{k1}x_1 + c_{k2}x_2 + \cdots + c_{kn}x_n$$

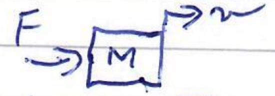
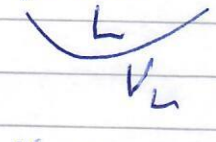
k outputs

We can also write this in matrix form.

$$\dot{\bar{x}} = A\bar{x} + B\bar{u}, \quad A \in R^{n\times n} \quad B \in R^{n\times m}$$

$$\bar{y} = C\bar{x} \quad C \in R^{k\times n}$$

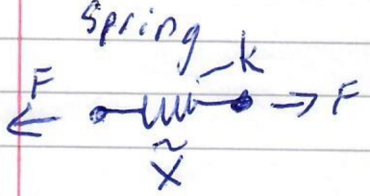
Inertial Elements that store energy.

	State Var.	Elemental Eqn.
	v, m/s	$M\dot{v} = \Sigma F$
	ω 1/s	$\overset{\rightarrow \text{kg}\cdot\text{m}^2}{J}\dot{\omega} = \Sigma \tau$
	i	$L\dot{i} = V_L$

These are the 3 inertial Elements that we will use in this course

Capacitive Elements also Store energy

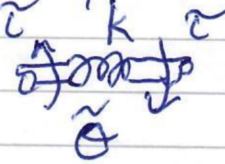
Spring S.V. Elemental Law



$\tilde{x}$ m

$F = k\tilde{x}$

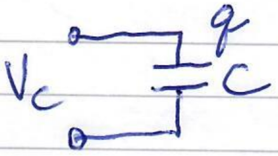
Rotary Spring



$\tilde{\theta}$ rads

$\tilde{\tau} = k\tilde{\theta}$

Electrical Capacitance.

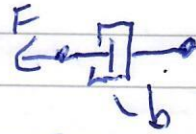


S.V.
C

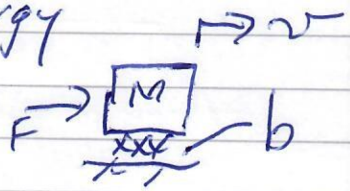
$V_c = \frac{1}{C} q$

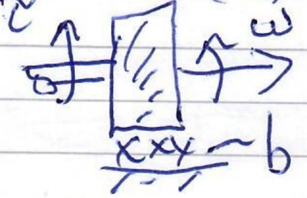
These are the 3 capacitive elements that we will use.

Elements that dissipate Energy

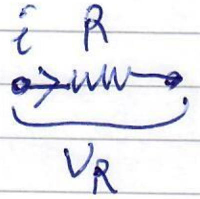


$F = bv$



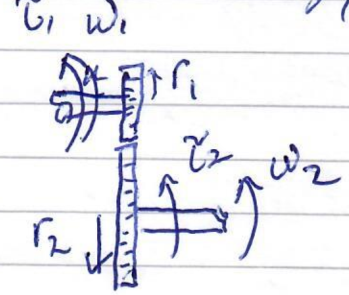
$N/(m/s)$ viscous friction.

$\tau = b\omega$

For electrical Systems.



$V_R = Ri$

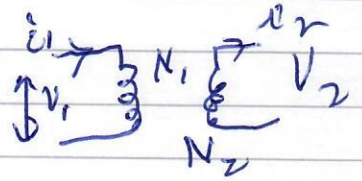
Gears ⇒ Do not store energy, they pass energy right through.



$$\tau_2 = \frac{r_2}{r_1}\tau_1$$

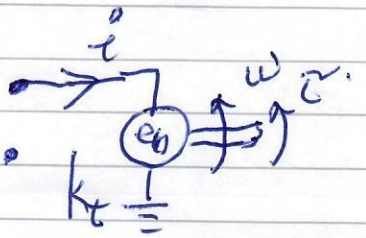
$$\omega_2 = \frac{r_1}{r_2}\omega_1$$

Electrical Transformers.



$$V_2 = \frac{N_2}{N_1} v_1 \qquad i_2 = \frac{N_1}{N_2} i_1$$

Motors:



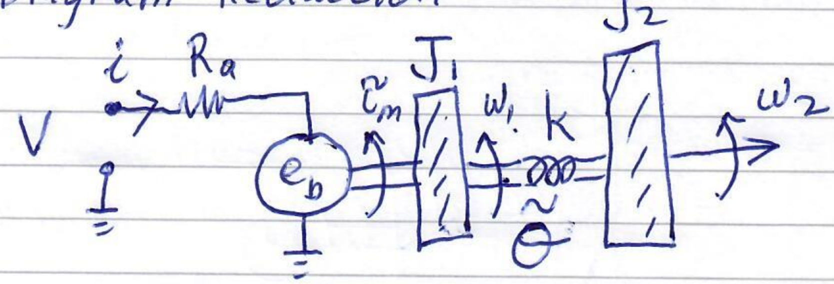
$$e_b i = \tau\omega$$

$$\tau = k_t i$$

→ N·m/Amp

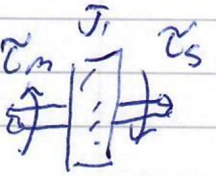
$$e_b = k_t\omega$$

Example of Modeling, Simulation Diagram & Block Diagram Reduction



3rd order system

$$\begin{array}{ccc} J_1 & k & J_2 \\ \omega_1 & \tilde{\theta} & \omega_2 \end{array}$$



$$J_1 \dot{\omega}_1 = \Sigma \tau = \tau_m - \tau_s$$

$$\tau_s = k\tilde{\theta}$$

$$J_1 \dot{\omega}_1 = -\frac{k_t^2}{R_a}\omega_1 - k\tilde{\theta} + \frac{k_t}{R_a}V$$

$$\tau_m = k_t i$$

$$V = R_a i + e_b \Rightarrow e_b = k_t \omega_1$$

$$\dot{\omega}_1 = -\frac{k_t^2}{R_a J_1}\omega_1 - \frac{k}{J_1}\tilde{\theta} + \frac{k_t}{R_a J_1}V$$

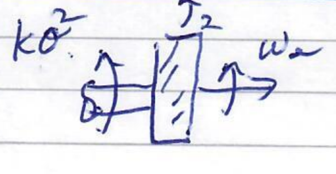
$$i = -\frac{k_t}{R_a}\omega_1 + \frac{1}{R_a}V$$

$$\tau_m = -\frac{k_t^2}{R_a}\omega_1 + \frac{k_t}{R_a}V$$

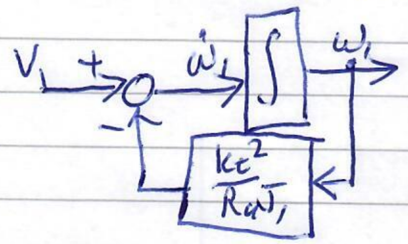
$$\dot{\tilde{\theta}} = \omega_1 - \omega_2$$

$$J_2 \dot{\omega}_2 = k\tilde{\theta}$$

$$\dot{\omega}_2 = \frac{k}{J_2}\tilde{\theta}$$



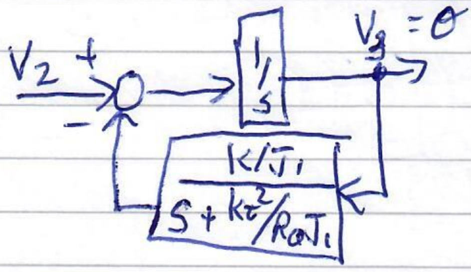
Reduction of Simulation Diagram.



$$= \frac{\omega_1(s)}{V_1(s)} = \frac{1/s}{1 + \frac{1}{s} \cdot k_t^2/R_a J_1}$$

$$= \frac{1}{s + k_t^2/R_a J_1}$$

Reduction of the feedback loop.



$$\equiv \frac{V_3(s)}{V_2(s)} = \frac{1/s}{1+\frac{1}{s}\left(\frac{K/J_1}{s+k_t^2/R_aJ_1}\right)}$$

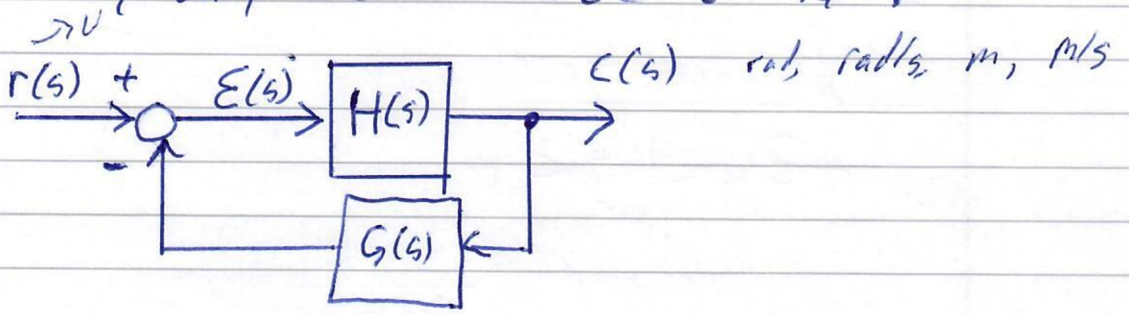
$$= \frac{s+k_t^2/R_aJ_1}{s^2+\frac{k_t^2}{R_aJ_1}s+K/J_1}$$

We get

$$\frac{\omega_2(s)}{V(s)} = \frac{\frac{k k_t}{R_aJ_1J_2}}{s^3+\frac{k_t^2}{R_aJ_1}s^2+\left(\frac{k}{J_1}+\frac{k}{J_2}\right)s+\frac{k k_t^2}{R_aJ_1J_2}}$$

Steady State Errors.

Steady state error is a key specification for control system design. A CNC machine may cut a part to $1/1000$" or a painting robot may only be accurate to $1/4$".



We want to know what $\mathcal{E}(t)$ will be in steady state. We know this from Final Value Theorem.

$$\mathcal{E}_{ss} = \lim_{t \to \infty} \mathcal{E}(s) = \lim_{s \to 0} s\mathcal{E}(s)$$

where $\mathcal{E}(s) = r(s) - G(s)C(s)$, $\quad C(s) = H(s)\mathcal{E}(s)$

$$= r(s) - G(s)H(s)\mathcal{E}(s)$$

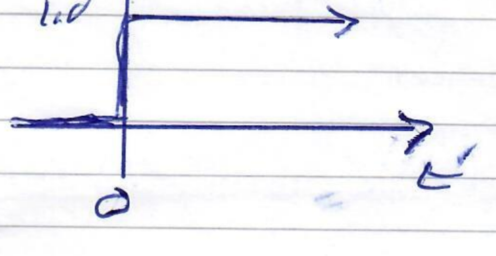
we get

$$\mathcal{E}(s) = \frac{r(s)}{1 + G(s)H(s)} \Rightarrow \frac{\mathcal{E}(s)}{r(s)} = \frac{1}{1 + G(s)H(s)}$$

We want to know the steady state error for different types of input $r(s)$.

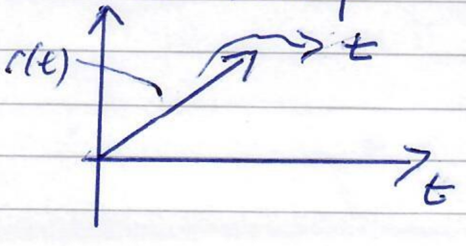
We will use typically the following inputs for $r(s)$.

Unit Step $\swarrow r(s) = 1/s$

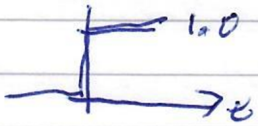


Unit Ramp. $r(s) = \frac{1}{s^2}$

$r(t) = t$
$\forall t > 0$

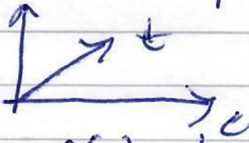


unit step



$r(s) = 1/s$

unit ramp



$r(s) = 1/s^2$

$$r(t) = \begin{cases} t & \forall t > 0 \\ 0 & \text{otherwise} \end{cases}$$

Steady State error is,

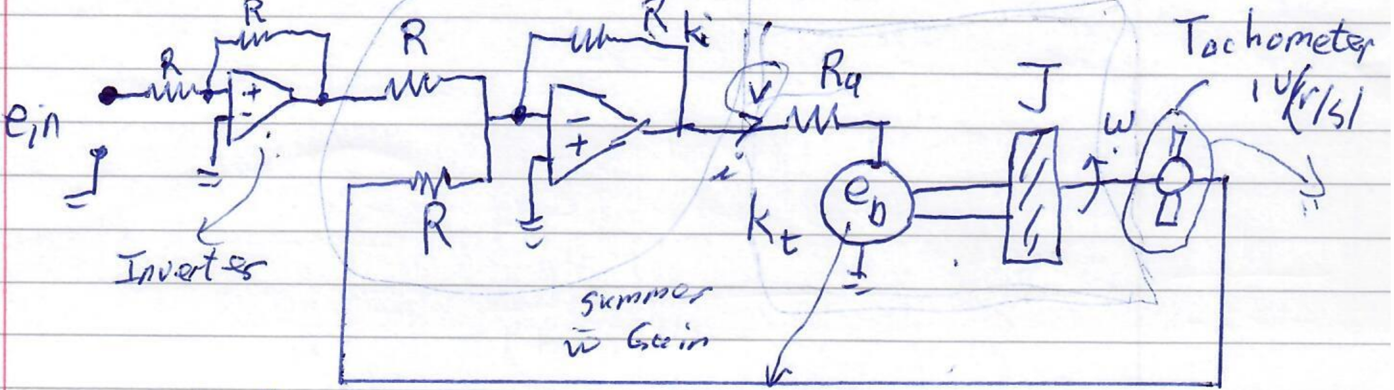
$$E_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s\,r(s)}{1 + G(s)H(s)}$$

if r(s) is unit step. then

$$E_{ss} = \lim_{s \to 0} \frac{\cancel{s}\,{}^1/\cancel{s}}{1 + G(s)H(s)} = \frac{1}{1 + k_p}$$

$$k_p = \lim_{s \to} G(s)H(s)$$

$\hookrightarrow$ coefficient of error due to unit step.

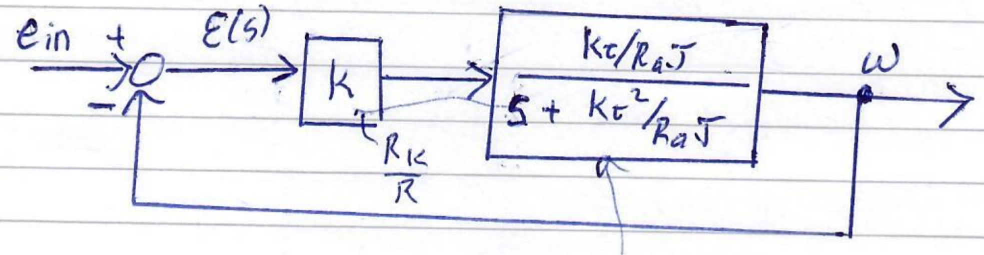


$$J\dot{\omega} = k_t i \qquad V = R_a i + k_t \omega \Rightarrow i = -\frac{k_t \omega}{R_a} + \frac{1}{R_a} V$$

$$J\dot{\omega} = -\frac{k_t^2}{R_a}\omega + \frac{k_t}{R_a} V \Rightarrow \boxed{\dot{\omega} = -\frac{k_t^2}{R_a J}\omega + \frac{k_t}{R_a J} V}$$

$$\frac{\omega(s)}{V(s)} = \frac{\frac{k_t}{R_a J}}{s + \frac{k_t^2}{R_a J}} = H(s)$$

The block Diagram becomes



type "0" system.

Steady State error for a unit step.

$$\mathcal{E}_{ss} = \lim_{s \to 0} \frac{1}{\underset{1}{1} + \underbrace{G(s)H(s)}_{k_p}} = \lim_{s \to 0} \frac{1}{1 + \frac{kk_t/R_aJ}{s + k_t^2/R_aJ}} = \frac{1}{k/k_t}$$

$k_p = k/k_t$ ⇒ For small steady state error kp should be large, which mean the feedback gain "k" should be big!

In general we have following result for steady error.

$$\mathcal{E}_{ss} = \lim_{s \to 0} \frac{s\,r(s)}{1 + G(s)H(s)}$$

The idea of System Type.

Define Open loop Transfer Function

$$S'_{OL}(s) = G(s)H(s)$$

Writing out G(s)H(s) in an arbitrary form.

$$S'_{OL}(s) = \frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{\boxed{s^j}(s-p_1)(s-p_2)\cdots(s-p_n)}$$

→ this is called a type "j" system.

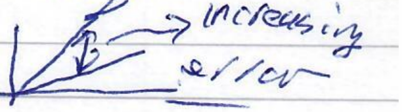
A type "0" system can follow a unit step input with a steady state error of

$$\mathcal{E}_{ss} \approx \frac{1}{1+k_p} \quad \text{where } k_p = \lim_{s\to 0} G(s)H(s)$$

What happen when a type "0" systems tries to follow a ramp.

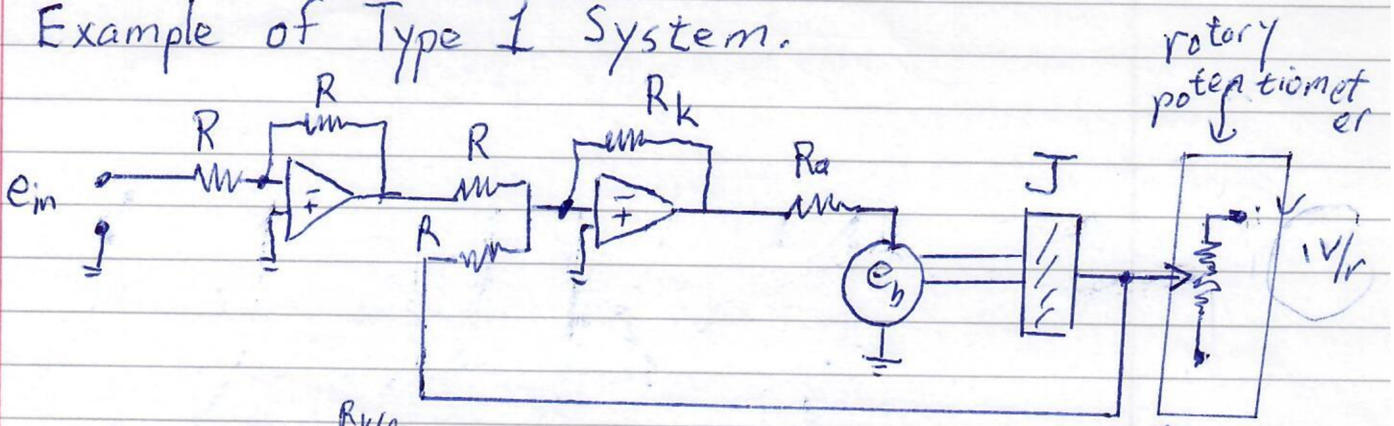
$$\mathcal{E}_{ss} = \lim_{s\to 0} \frac{s\,r(s)}{1+\underbrace{G(s)H(s)}} = \lim_{s\to 0} \frac{s\cdot \boxed{1/s^2}}{1+G(s)H(s)}$$

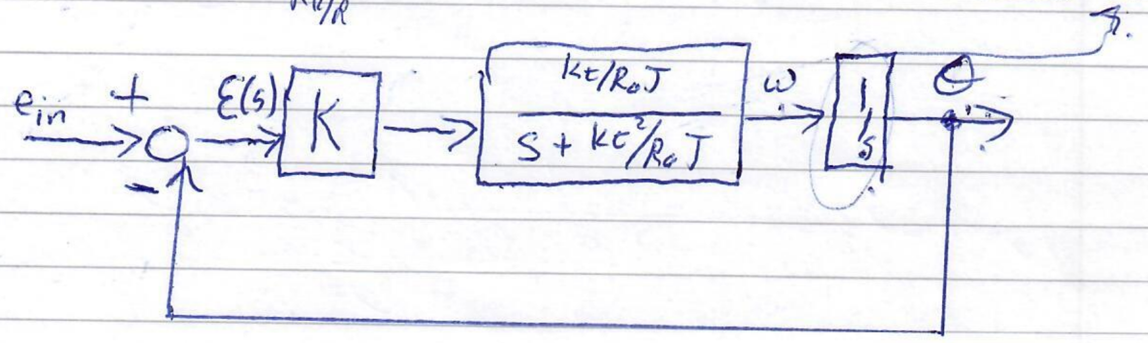
$$= \lim_{s\to 0} \frac{1}{s+\underbrace{sG(s)H(s)}} \Rightarrow \infty$$



A type "0" system cannot follow a ramp input

Example of Type 1 System.





The Steady State Error

$$\mathcal{E}_{ss} = \lim_{s \to 0} \frac{s r(s)}{1 + G(s)H(s)} , \quad G(s)H(s) = \frac{kk_t / R_aJ}{\underbrace{s}_{\text{type "1" system.}}\left(s + k_t^2/R_aJ\right)}$$

If $r(s)$ = unit step = $1/s$.

$$\mathcal{E}_{ss} = \lim_{s \to 0} \frac{1}{1 + \frac{kk_t/R_aJ}{s(s + k_t^2/R_aJ)}} \Rightarrow \infty$$

$$= 0$$

A type 1 system can follow a unit step with "0" steady state error $k_p \Rightarrow \infty$

Ramp Input to Type 1 system.

$$\mathcal{E}_{ss} = \lim_{s \to 0} \frac{s \cdot 1/s^2}{1 + \cancel{s}\frac{kk_t/R_aJ}{s(s + k_t^2/R_aJ)}} = \lim_{s \to 0} \frac{1}{s + \frac{kk_t/R_aJ}{(s + (k_t^2/R_aJ)}}$$

$$= k/k_t$$

We define the coefficient of steady state error due to a ramp input.

$$k_v = \lim_{s \to 0} sG(s)H(s)$$

The steady state error due to a unit ramp input is

$$\mathcal{E}_{ss} = \cancel{\lim_{s}} = \frac{1}{k_v}$$

Let' write a table of System Types and coefficients of steady State errors.

$k_p = \lim_{s \to 0} G(s)H(s)$ error coefficient for a step input

$k_v = \lim_{s \to 0} sG(s)H(s)$ error coefficient for a ramp.

$k_a = \lim_{s \to 0} s^2 G(s)H(s)$ error coefficient for a parabola.

Table

				steady State Error		
System Type	k_p	k_v	k_a	R step	R ramp	R parabola
0	k_p	0	0	$\frac{R}{1+k_p}$	∞	∞
1	∞	$\boxed{k_v}$	0	0	$\frac{R}{k_v}$	∞
2	∞	∞	k_a	0	0	$\frac{R}{k_a}$
3	∞	∞	∞	0	0	0

Stability

For our purposes for a system to be stable the roots of the characteristic eqn. must have -ve real part. The poles must have -ve real parts.

We can use the Routh-Hurwitz method to evaluate whether a polynomial ~~have~~ has
all -ve real parts ~~th~~ for its roots.

For example take the following arbitrary polynomial

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n = 0$$

A necessary condition for stability is that all the a_i are present and +ve. (if all -ve multiply by -1)

We then write out the follow Routh-Hurwitz Table

		a_2	a_4	
s^n	a_0	$a_{[illegible]}$	$a_{[illegible]}$	$\dots$
s^{n-1}	a_1	a_3	a_5	$\dots$
s^{n-2}	b_1	b_2	b_3	
s^{n-3}	c_1	c_2	$\dots$	

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \qquad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1} \quad , \quad "$$

$$a_0 s^n + a_1 s^{n-1} + \dots + a_n = 0$$

Routh - Harwitz

$$\begin{array}{lllll} s^n & a_0 & a_2 & a_4 & \dots \\ s^{n-1} & a_1 & a_3 & a_5 & \dots \\ s^{n-2} & b_1 & b_2 & b_3 & \dots \\ s^{n-3} & c_1 & c_2 & \dots & \end{array}$$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

<u>Example of a</u> 3rd order system.

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

all a_i's are positive (+ve)

Routh-Hurwitz Table

$$\begin{array}{lll} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s & \dfrac{a_1 a_2 - a_0 a_3}{a_1} & 0 \\ s^0 & a_3 & \end{array}$$

For a 3rd order system to be stable then

$$a_1 a_2 > a_0 a_3$$

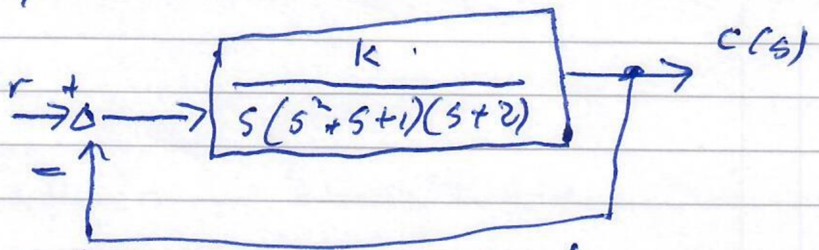
The number of unstable roots is given by the number of sign changes in the 1st column.

Routh-Hurwitz Stability - Continued.

If there is a zero in the 1st column of the table that means you have imaginary roots and if the whole row is zero the you have roots of equal magnitude but opposite sign (Pair of imaginary).

We can use the auxillary eqn. (equation from above row) to find the location of the imaginary roots.

Example:



Find range of k for stability

Doing block diagram reduction

$$\frac{c(s)}{r(s)} = \frac{k}{\textcircled{1}s^4 + 3s^3 + 3s^2 + 2s + k}$$

$$\begin{array}{llll}
s^4 & 1 & 3 & k \\
s^3 & 3 & 2 & 0 \\
s^2 & 7/3 & k & 0 \\
s & \boxed{\frac{14-9k}{7}} & 0 & 0 \\
s^0 & k & &
\end{array}$$

Auxillary equation

$$\frac{7/3 \cdot 2 - 3k}{7/3} = \frac{14-9k}{7}$$

For $14 - 9k > 0$ the system is stable.

Therefore $k < 14/9$

Where are the imaginary roots if $k = 14/9$

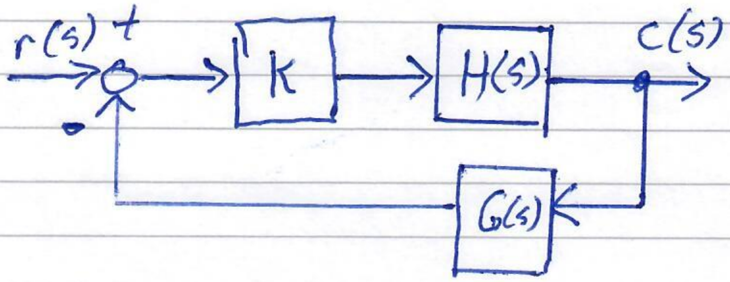
The auxillary equation bocomes,

$$\frac{7}{3}s^2 + \frac{14}{9} = 0 \therefore \Rightarrow s^2 + \frac{2}{3} = 0$$

Imaginary roots at $s = \pm\sqrt{\frac{2}{3}}\, j = \pm 0.82j$

The Root Locus Method

The idea is to determine where the closed loop pole locations will go as a feedback gain is increased. This path is called the root loci, or the root locus.



where do C.L. poles go as K is increased.

Let's only consider K > 0.

The closed loop transfer function.

$$\frac{C(s)}{r(s)} = \frac{KH(s)}{\underbrace{1 + KH(s)G(s)}} \quad \text{when } 1 + KH(s)G(s) = 0$$

that is when we get a closed loop pole location. Recall $KH(s)G(s)$ is a complex number, it has a magnitude & phase. We want to plot all the values for "s" at which.

$$KH(s)G(s) = -1 \Rightarrow |KH(s)G(s)| = 1 \ \& \ \angle H(s)G(s) = \pm 180^\circ.$$

There is the angle criteria.

$$\angle G(s)H(s) = \pm 180^\circ(2i+1) \quad \text{for } i = \overset{0,}{\cancel{1}}, \overset{1,}{\cancel{2}}, 2, \ldots$$
$$= \pm 180^\circ(2i+1) \quad \text{for } i = 0, 1, 2. \ . \ .$$

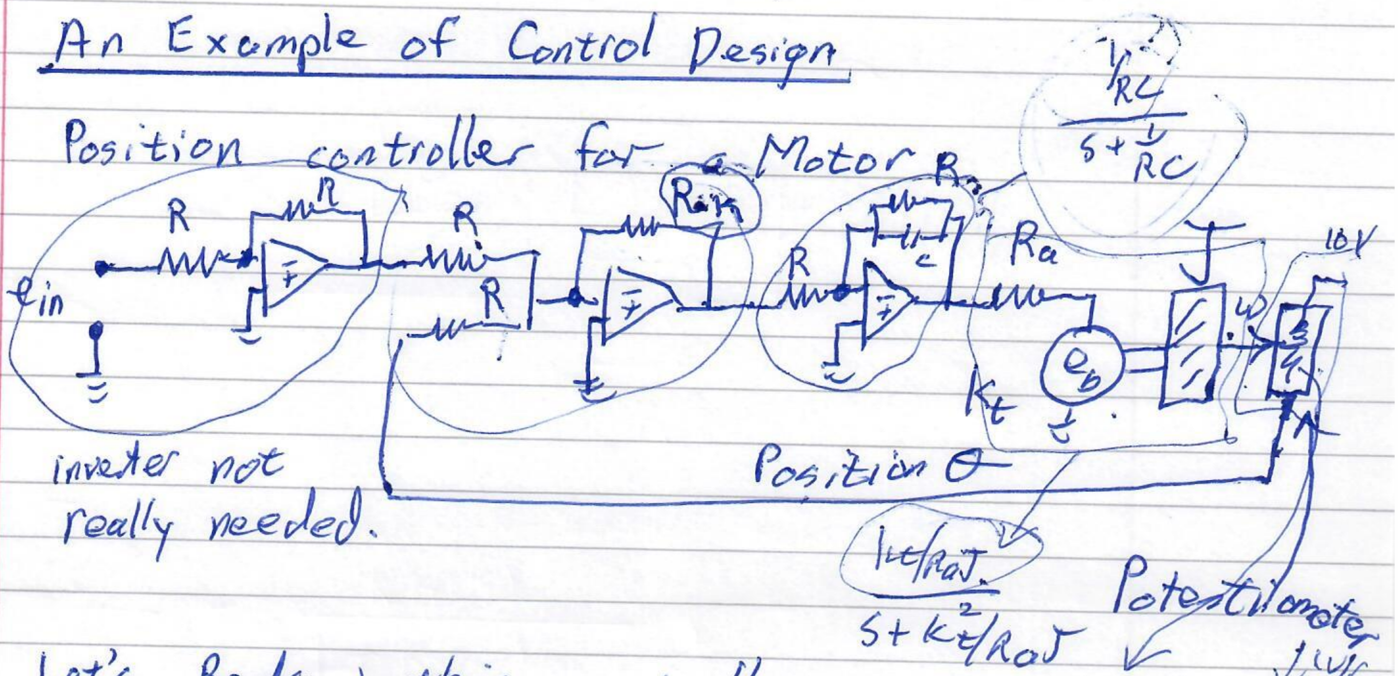
Every point in the s-plane that meets the angle criteria is a possible C.L. pole location.

Magnitude Criteria, (this is used for design to pick "K")

Pick "K" such that $|KH(s)G(s)| = 1$

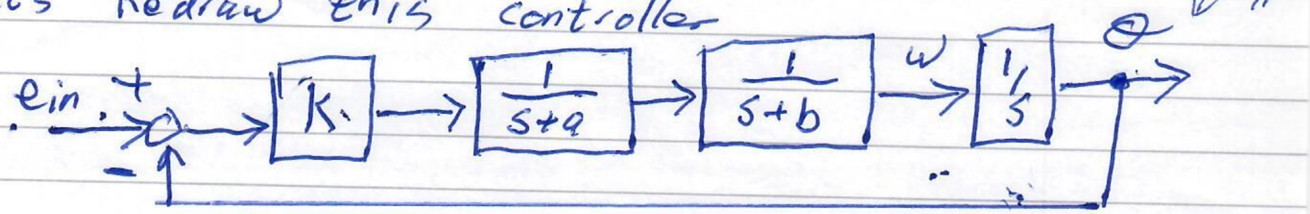
An Example of Control Design

Position controller for a Motor



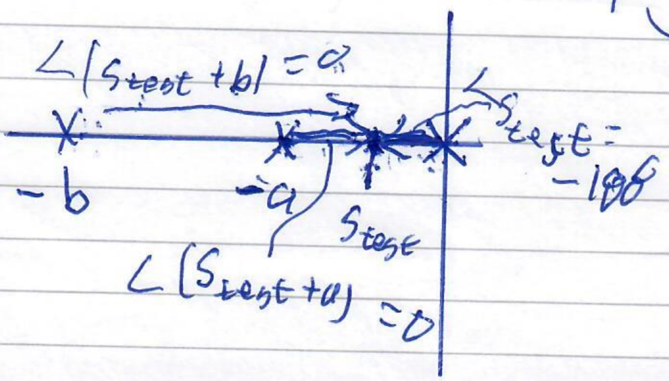
inverter not really needed.

Let's Redraw this controller

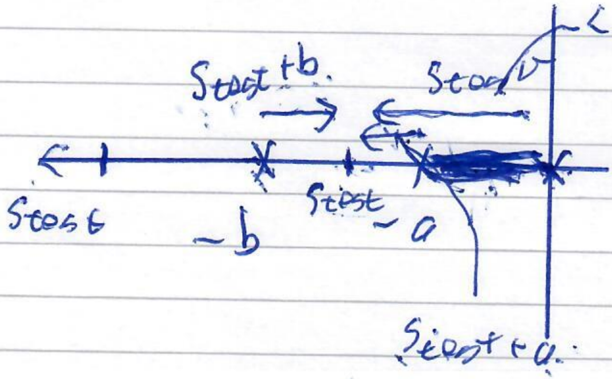


$$\frac{\theta(s)}{e_{in}(s)} = \frac{\frac{K}{s(s+a)(s+b)}}{1+\boxed{\frac{K}{s(s+a)(s+b)}}} \leftarrow \angle = \pm 180^{\circ}$$

$|\bullet| = 1 \rightarrow$ How I choose K



1) Draw poles & zero's on s-plane



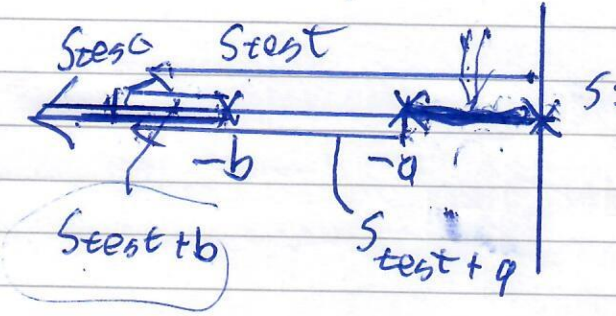
$\angle S_{test} = -180°$

$S_{test}+a = -180°$

$S_{test}+b = 0°$

$\} = -360°$

Does not satisfy angle criteria.

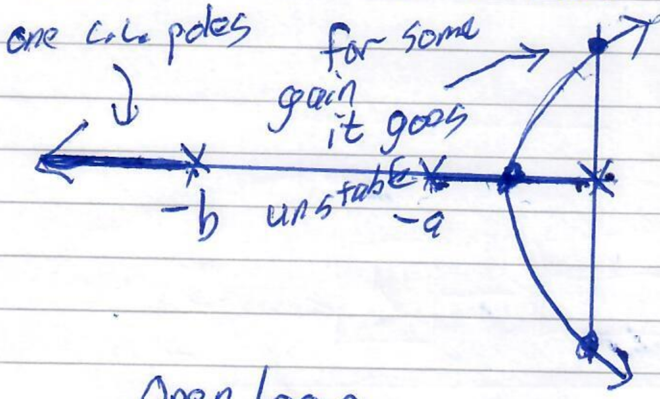


$$\angle \left.\frac{k}{s(s+a)(s+b)}\right|_{s_{test}}$$

$$= -180° -180° -180°$$

$$= \pm 180°(2i+1) \quad i=0,1,2$$

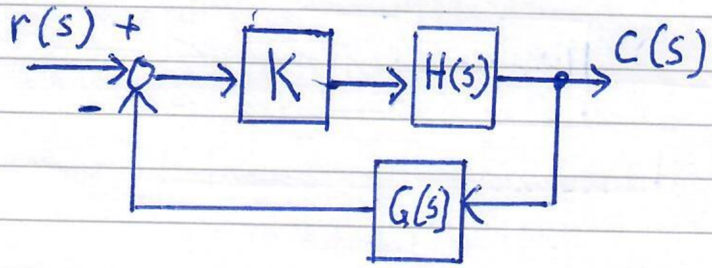
I know what this R.L. will look like



Root locus starts from ~~zer~~ poles and as k increases the R.L. Goes to either ∞ or to Open loop zero locations.

The Open loop poles act as sources & the zero's act as sinks.

Root Locus Plots



Closed Loop Transfer Function

$$H_{CL}(s) = \frac{C(s)}{r(s)} = \frac{KH(s)}{\underbrace{1+\underbrace{KH(s)G(s)}_{S_{OL}}}_{=0}}$$

Where $S_{OL}(s) = KH(s)G(s)$ is the open loop T. F.

Recall 2 criteria 1) Angle Criteria $\angle H(s)G(s) = \pm 180^\circ (2\ell+1)$, $\ell = 0, 1, \ldots$

We then choose "K" to achieve closed loop (specifications) pole location from Magnitude Criteria $|KH(s)G(s)| = 1$

Steps To Root Locus Plot.

1) Plot the poles and zeros on the s-plane
   - Recognize that the root locus will start at the open loop poles and the root locus will go either to infinity or to the open loop zeros.

2) Plot the root locus on the real axis.

3) Compute the angle and intercept of the asymptotes.

3) Compute angle & intercept of asymptotes.

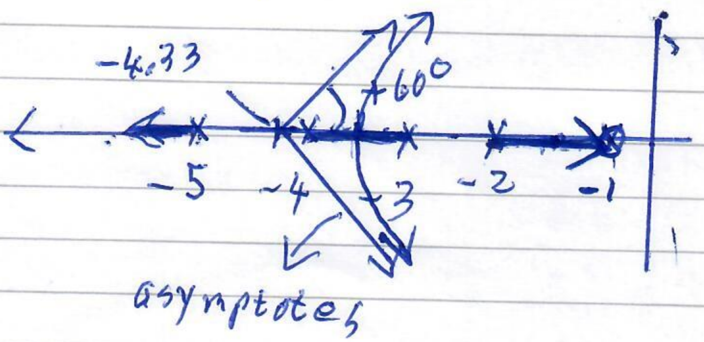
3.1) Angle of asymptotes $= \dfrac{\pm 180^\circ (2i+1)}{n-m}$, $i = 0, 1 \ldots$

n = # poles, m = # of zeros

3.2) Intercept of asymptotes

$$\frac{(p_1 + p_2 + \cdot\cdot\, p_n) - (z_1 + z_2 \cdot\cdot + z_m)}{n-m}$$

For Example: $KG(s)H(s) = \dfrac{K(s+1)}{(s+2)(s+3)(s+4)(s+5)}$



∠ of asymptotes

$$= \frac{\pm 180^\circ}{4-1} = \pm 60^\circ$$

at $i = 1$, $\pm 180^\circ$

Intercept of asymptotes.

$$\sigma_a = \frac{(-2-3-4-5)-(-1)}{\underset{n\ -m}{4-1}} = \frac{-13}{4} = -4.33$$

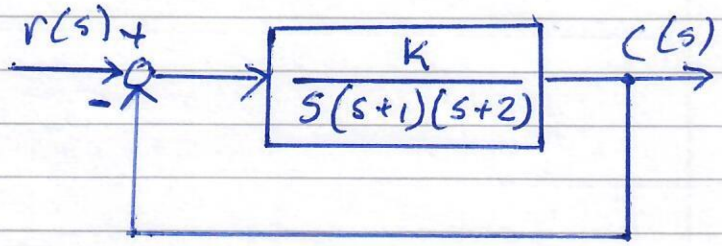
Next Steps.

4) Find Crossing of Imaginary Axis using Routh-Hurwitz Method & Auxillary Equation to find K & crossing point.

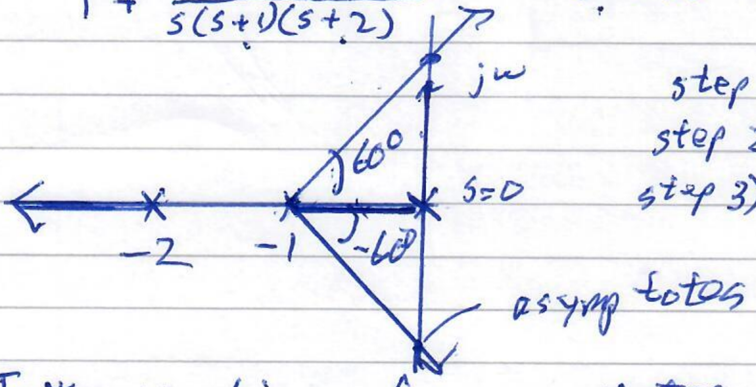
5) Find the breakaway points using
$$\frac{dk}{ds} = 0$$

6) Find the angle of departure and arrival from complex poles & zeros, respectively

Example: 1



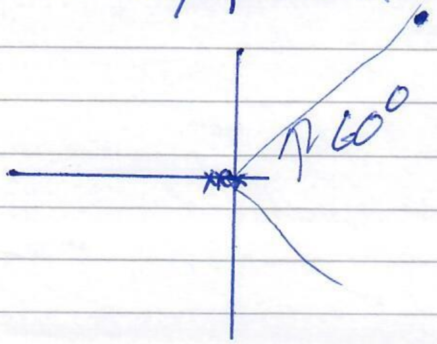
$$\frac{C(s)}{r(s)} = \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}} = \frac{K}{s^3+3s^2+2s+K}$$



step 1) Plot poles/zeros
step 2) R.L. on real Axis
step 3) Asymptotes.

3.1) Intersection of asymptotes.
$$\sigma_a = \frac{(0-1-2)}{\underset{n=3}{3} - m \leftarrow 0} = -1$$

3:1) Angle of asymptotes. $\angle asy = \dfrac{\pm 180^\circ(2i+1)}{3} = \begin{matrix}\pm 60^\circ \\ \pm 180^\circ\end{matrix}$



Example 1 continued

Step 4) Crossing Imaginary Axis use Routh-Harwitz method. C.L. T.F. is $\frac{K}{\textcircled{1}s^3+3s^2+2s+K}$

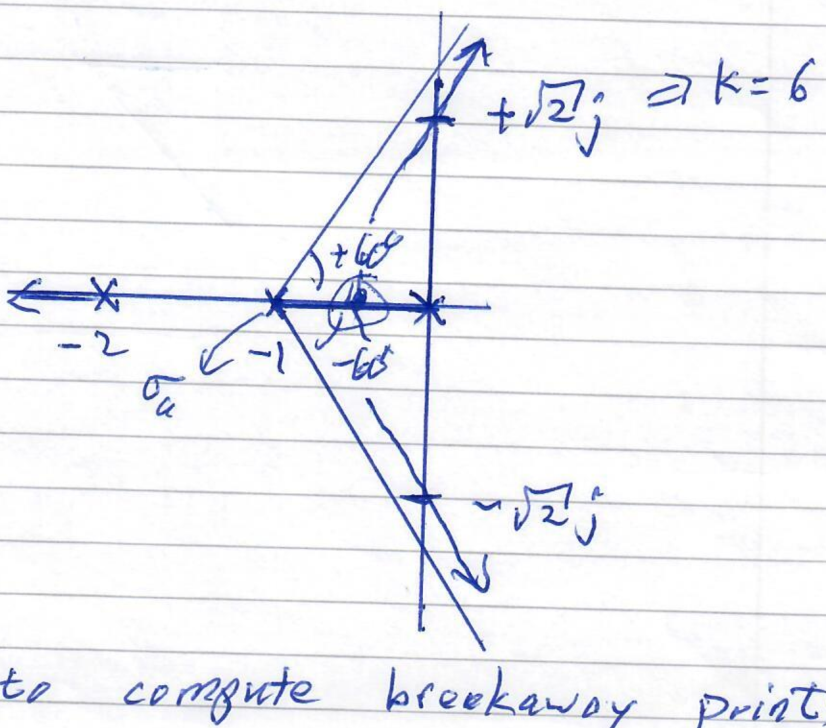
$$\left.\begin{array}{llll} s^3 & 1 & 2 & 0 \\ s^2 & 3 & K & 0 \\ s & \frac{6-K}{3} & 0 & \\ s^0 & K & & \end{array}\right\} \text{For stability } K<6$$

Now let's look at auxillory eqn. for K = 6

$3s^2+6=0 \Rightarrow s^2+2=0 = \boxed{s=\pm\sqrt{2}\,j}$

We get a crossing of the imaginary axis at K = 6, & the root locus crosses the imaginary axis at $s=\pm\sqrt{2}\,j$

So far.



Step 5) Is to compute breakaway points

Breakaway pt.

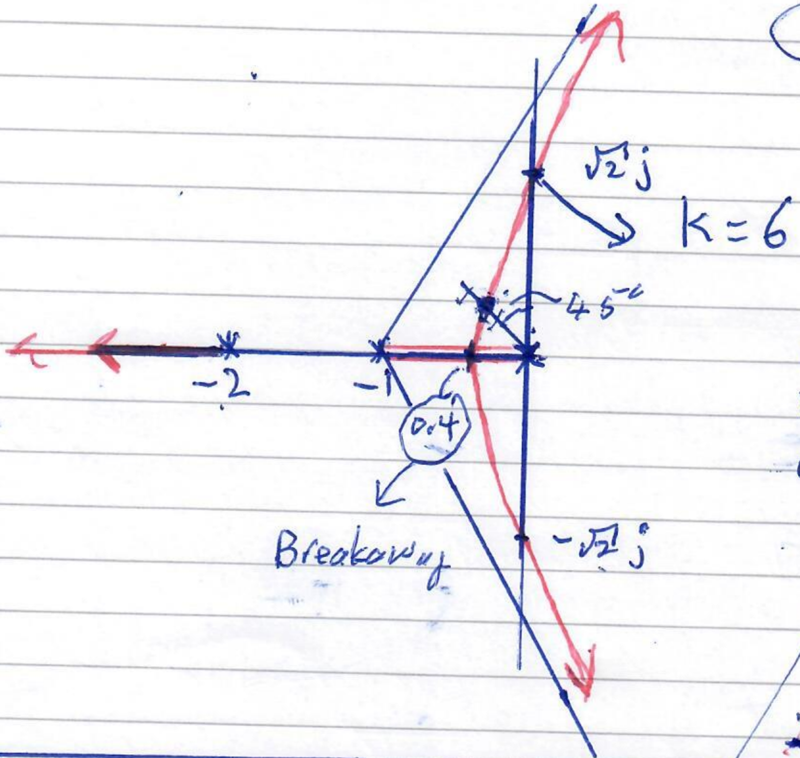
Step 5) $K = -(s^3 + 3s^2 + 2s)$

$$\frac{dK}{ds} = 0 = -(3s^2 + 6s + 2)$$

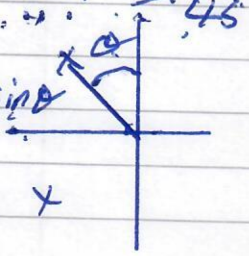
roots of $3s^2 + 6s + 2 \Rightarrow$ $\boxed{s = -0.423}$

$\boxed{s = -1.577}$ ✗

Not on root locus.

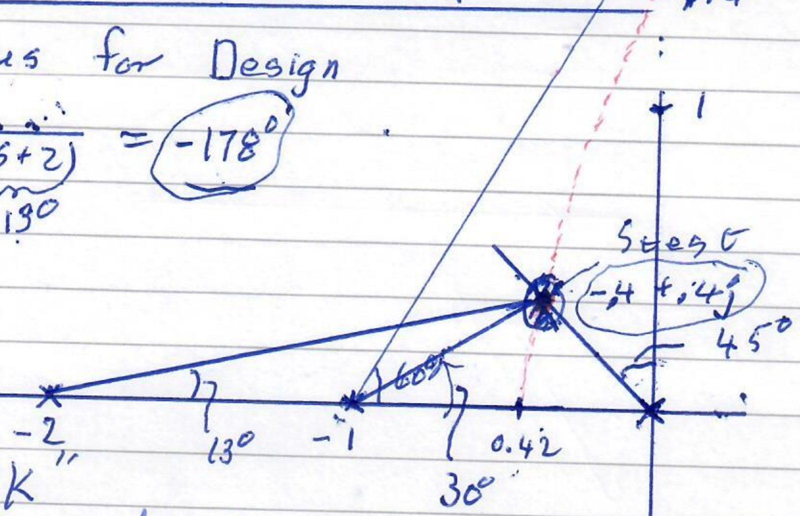


Let's say I wanted a C.L. damping ratio of $\zeta = 0.707$



Root Locus for Design

$$\angle \frac{K}{\underbrace{s}_{135°}\underbrace{(s+1)}_{30°}\underbrace{(s+2)}_{13°}} = \boxed{-178°}$$



What is "K" from Magnitude Criteria

$K = |s||s+1||s+2|$
$= 0.5(0.75)(1.7) = 0.64$

Root Locus Example 2

Autopilot Control for longitudinal mode of an airplane. The 'open loop' transfer function is.

$$s^2+2\zeta\omega_n s+\omega_n^2$$

$$KG(s)H(s) = \frac{K(s+1)}{\boxed{s}(s-1)(s^2+4s+16)} \simeq (s-p_2)(s-\bar{p}_2)$$

step 1: Plot poles & zeros on the s-plane.

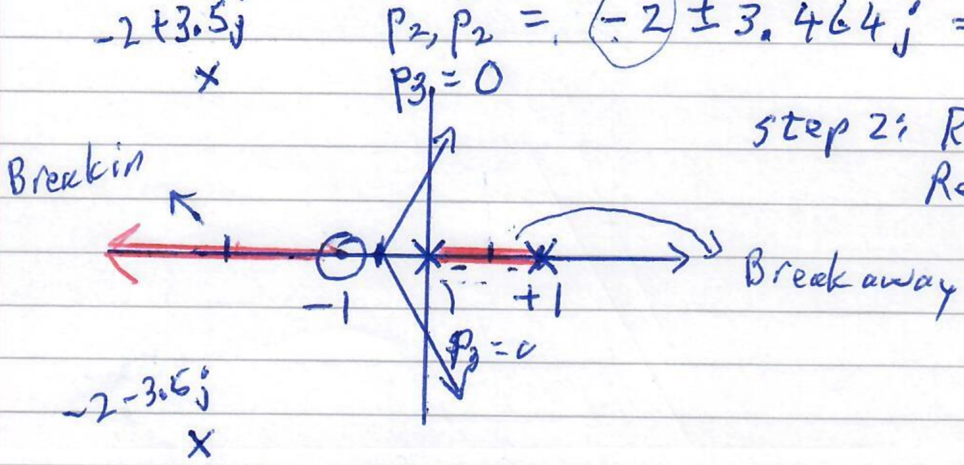
zeros
$z = -1;$

poles
$p_1 = +1$ } unstable

$\zeta = 0.5 \quad \omega_n = 4$

$p_2, \bar{p}_2 = \boxed{-2} \pm 3.464j = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\,j$

$p_3 = 0$



step 2: Root Locus On Real axis

Step 3: Angle & intercept of asymptotes

$$\text{Angle} = \frac{\pm 180(2i+1)}{n-m} = \frac{\pm 180(2i+1)}{4-1} = \pm 60°, \pm 180°$$

The intercept occurs at

$$\sigma_a = \frac{(0+1\boxed{-2}\boxed{-2})-(-1)}{4-1} = \frac{-2}{3}$$

step 4: Crossing of imaginory axis. (Routh-Hurwitz)

$$KG(s)H(s) = \frac{k(s+1)}{s(s-1)(s^2+4s+16)}$$

The characteristic polynomial for C.L. System

$s^4 + 3s^3 + 12s^2 + (K-16)s + k = 0$

s^4	1	12	k	
s^3	3	$k-16$	0	
s^2	$\frac{52-k}{3}$	K		Auxillory Eqn.
s^1	$\frac{-K^2+59k-832}{3(52-k)}$	0		
s^0	K	0		

The roots of the s^1 term $= -K^2 + 59k - 832 = 0$ is $K^* = 23.3$ & $K^{**} = 35.7$

Subtituting into the auxillary eqn. for $K^* = 23.3$

$\frac{52-K}{3} s^2 + K = 0$ $K^* = 23.3$ $s^* = \pm 1.56j$

$K^{**} = 35.7$ $s^{**} = \pm 2.56j$

For $23.3 < K < 35.7$, the system will have stable roots.

Step 5: Find the Breakin - Break away points.

$\frac{dK}{ds} = 0 \qquad 1 + \frac{K(s+1)}{s(s-1)(s^2+4s+16)} = 0$

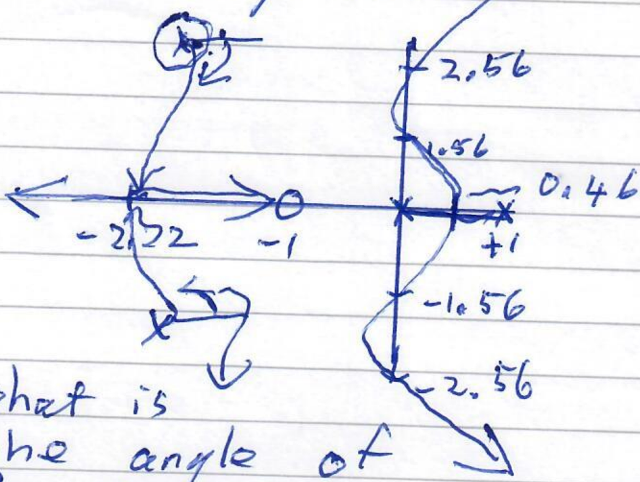
$$K = \frac{-s(s-1)(s^2+4s+16)}{s+1}$$

$$\frac{dK}{ds} = \frac{-(3s^4 + 10s^3 + 21s^2 + 24s - 16)}{(s+1)^2} = 0$$

We get 4 roots, but only 2 will meet the angle criteria.

$\underbrace{s_1 = -2.22}_{\text{break in}}, \quad \underbrace{s_2 = 0.46}_{\text{breakaway.}}$

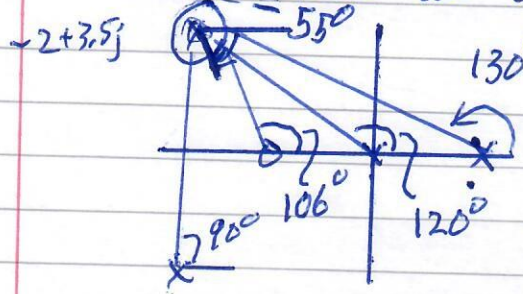
Our current rough sketch.



what is the angle of departure.

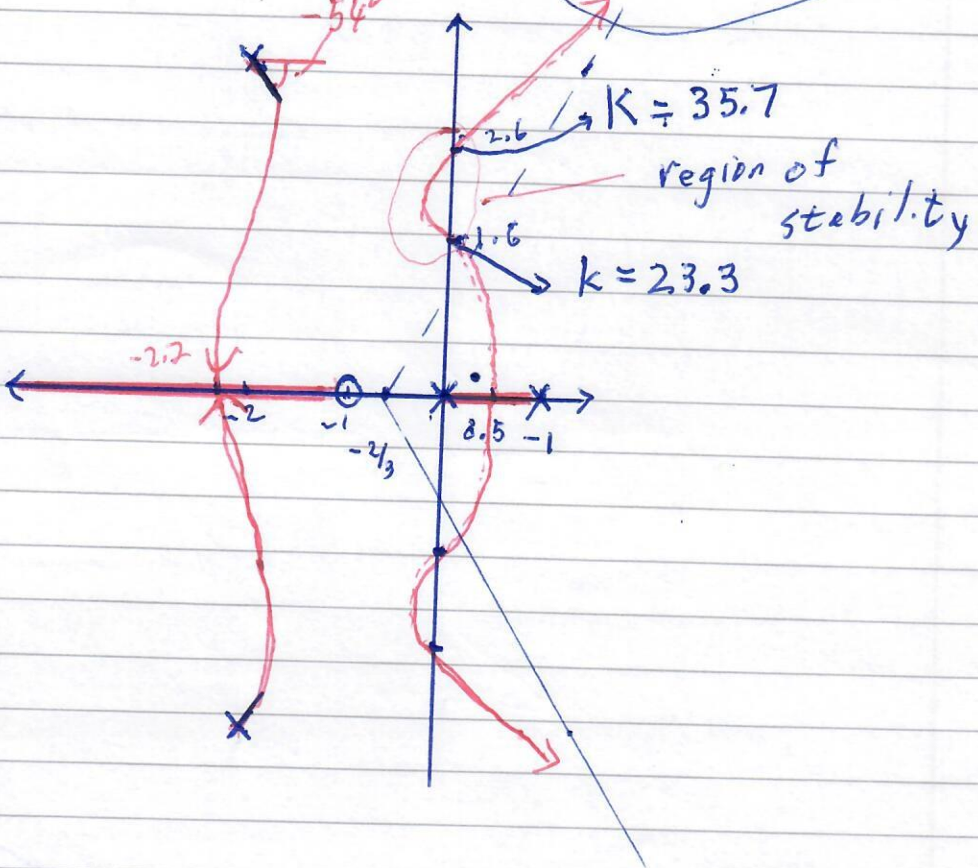
Step 6: Angle of departure & angle of arrival.

departure due to complex poles & angle of arrival to complex zero.
In this example we have complex poles so we have an angle of departure.



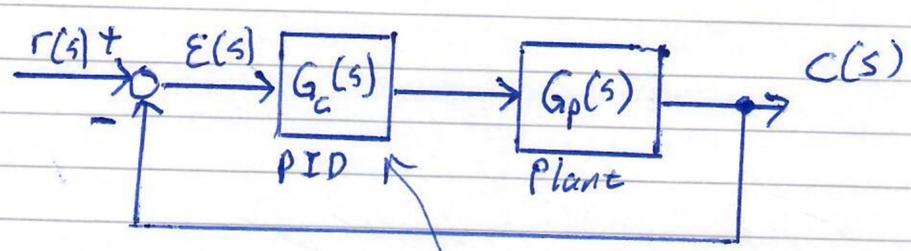
$\angle departure = 180° - 90° - 120° - 130° - 106°$

$= -55°$

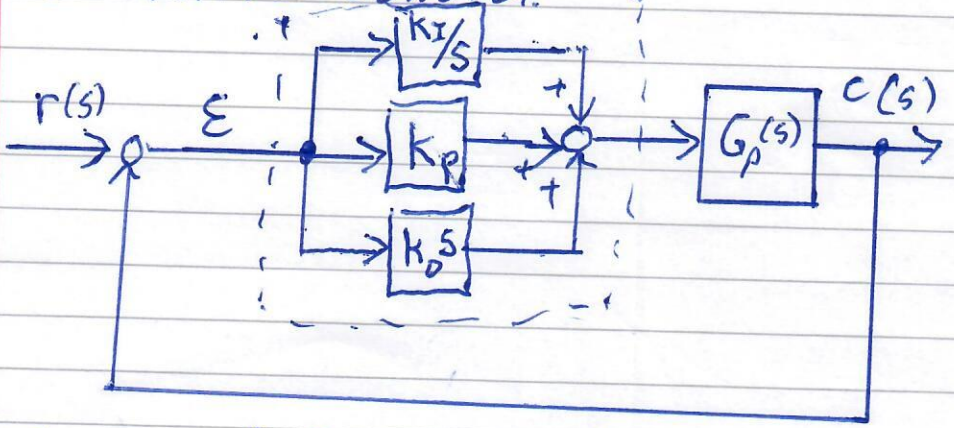


PID controllers.

Used a lot in the process control industry such as oil refineries, pulp & Paper mills, etc.



PID stands for Proportional, Integral and Derivative Control.

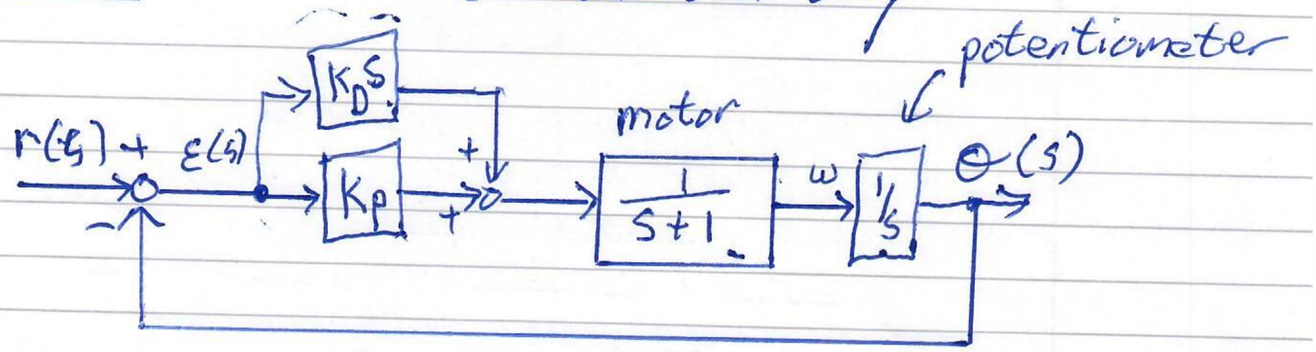


The control Gains are K_I ⇒ Integral, K_p ⇒ Proportional
K_D ⇒ Derivative

Then $G_c(s) = K_p + K_D S + \frac{K_I}{s} \Rightarrow \frac{K_D s^2 + K_p s + K_I}{s}$

Case I: Case of PD controller only

Case I: Case of PD Controller only



We are given the specification that the steady state error due to a unit ramp is $\boxed{\varepsilon_{ss} \le 0.01}$

& a damping ratio of $\zeta = 0.7$.

The steady state error due to a ramp is, $\varepsilon_{ss} = \frac{1}{K_v}$ where k_v is the coefficient of an error due to a ramp. $k_v = \lim_{s \to 0} s\,G(s)H(s)$.

The closed loop transfer function is,

$$\frac{\theta(s)}{r(s)} = \frac{\dfrac{K_D s + k_p}{s(s+1)}}{1 + \dfrac{k_D s + k_p}{s(s+1)}} = \frac{K_D s + k_p}{s^2 + (1+k_D)s + k_p}$$

$$K_D s + k_p = k_D(s + k_p/k_D)$$

Solving for the error specification on k_v

$$k_v = \lim_{s \to 0} s\,G(s)H(s) = \lim_{s \to 0} \frac{\cancel{s}\cdot(k_D s + k_p)}{\cancel{s}(s+1)} = k_p$$

$$\frac{1}{k_v} \le 0.01 \Rightarrow \boxed{k_p = 100} \Rightarrow \text{Specification } \zeta = 0.707 = \frac{1}{\sqrt{2}}$$

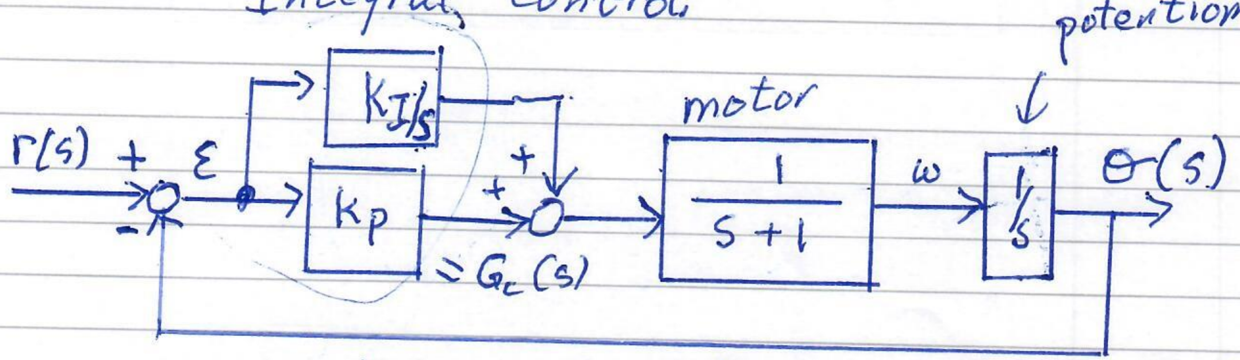
$$s^2 + 2\zeta\omega_n s + \omega_n^2 \Rightarrow s^2 + (1+k_D)s + 100$$

$\omega_n = 10$,

$$1 + k_D = 2\left(\frac{1}{\sqrt{2}}\right)10 \Rightarrow K_D = 10\sqrt{2} - 1$$
$$= 13.14$$

For this PD controller to meet specifications we have to set $K_p = 100$ $K_D = 13.14$

Case 2: In this case only - Proportional + Integral Control



$$G_c(s) = K_p + \frac{K_I}{s} = \frac{K_p s + K_I}{s} = \frac{K_p(s + K_I/K_p)}{s}$$

The system type has increased by 1.

The closed loop transfer function becomes

$$\frac{\theta(s)}{r(s)} = \frac{\dfrac{K_p s + K_I}{s^2(s+1)}}{1 + \dfrac{K_p s + K_I}{s^2(s+1)}} = \frac{K_p s + K_I}{\underset{a_0}{s^3} + \underset{a_1}{s^2} + \underset{a_2}{K_p s} + \underset{a_3}{K_I}}$$

We know from Routh-Hurwitz that a 3rd order system is stable if

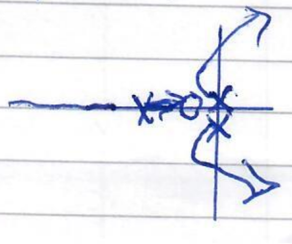
$$a_1 a_2 > a_0 a_3$$
$$K_p > K_I$$

Let's set $K_p = 100$ (Similar to PD case 1) & $K_I = 10$

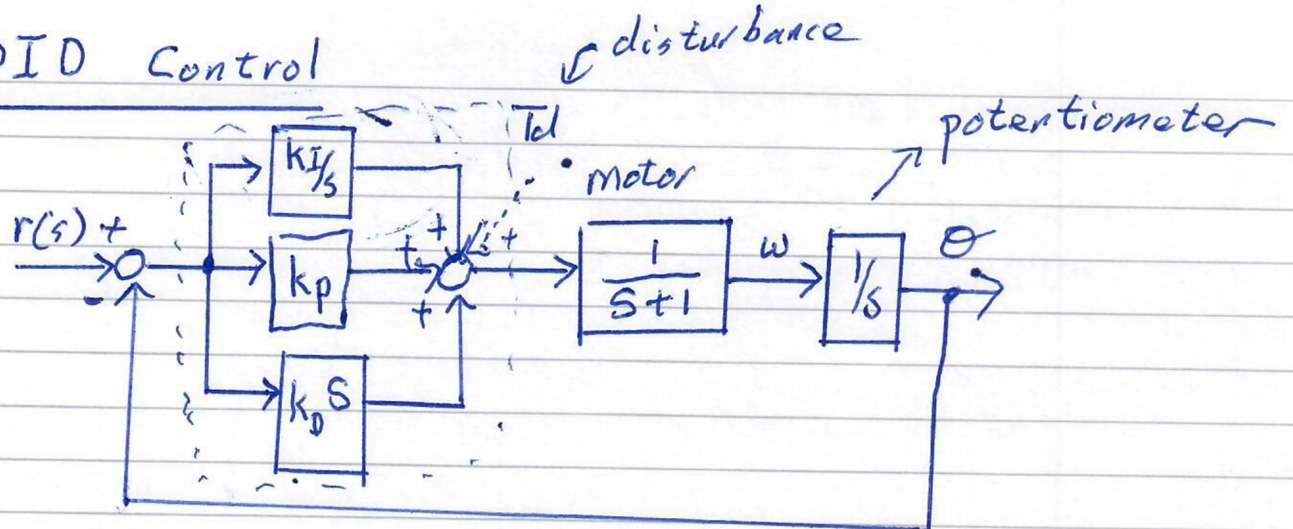
Then the characteristic Eqn. $s^3 + s^2 + 100s + 10 = 0$

The roots are, $p_1, \bar{p}_1 = -0.45 \pm 10j$ → Lightly damped.

$p_3 = -0.1$



PID Control



$$G_c(s) = k_I/s + k_p + k_D s = \frac{k_D s^2 + k_p s + k_I}{s}$$

Let's specify transient dynamics such that all the closed loop poles are at $s_a = -5.0$

The closed loop transfer function for the system

$$\frac{c(s)}{r(s)} = \frac{\frac{k_D s^2 + k_p s + k_I}{s} \cdot \frac{1}{s(s+1)}}{1 + \frac{k_D s^2 + k_p s + k_I}{s^2(s+1)}}$$

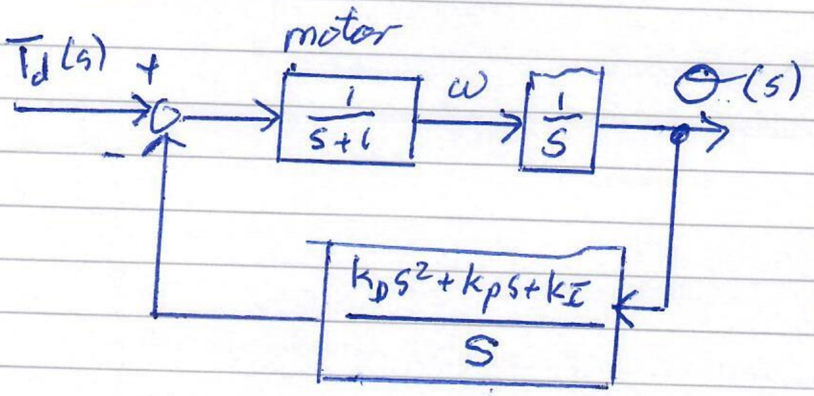
$$= \frac{k_D s^2 + k_p s + k_I}{s^3 + (1+k_D)s^2 + k_p s + k_I}$$

I can independently select k_D, k_p & k_I to achievo any ~~spefi~~ specified closed loop pole location.

$$D(s) = (s+5)^3 = s^3 + 15s^2 + 75s + 125$$

Then set $k_D = 14$, $k_p = 75$, $k_I = 125$.

The Effect of a Disturbance



What is C.L. T.F. from the disturbance to the motor angle.

$$\frac{\Theta(s)}{T_d(s)} = \frac{\frac{1}{s(s+1)}}{1+\frac{k_D s^2+k_P s+k_I}{s^2(s+1)}}$$

$$= \frac{s}{s^3+(1+k_D)s^2+k_P s+k_I \triangleq D(s)}$$

Let's say that the disturbance is a unit step. $T_d(s) = 1/s$. From the Final Value Theorem we can compute the effect in steady state of the disturbance.

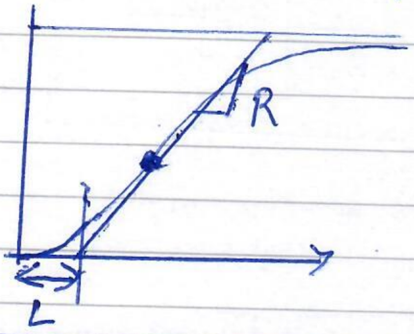
$$\Theta_{ss} = \lim_{s\to 0} s\,\Theta(s) = \lim_{s\to 0} \frac{\boxed{s}\cdot\boxed{s}\cdot\overset{T_d(s)}{\boxed{1/s}}}{D(s)} = \underline{\underline{0}}$$

The integral component will drive the effect of a disturbance to 0.

Ziegler-Nichols Rules

1) Process Reaction Method

Do a test and get results



Proportional $k_p = \frac{1}{RL}$

PI: $k_p = \frac{0.9}{RL}$, $k_I = \frac{1}{3.3L}$

PID: $k_p = \frac{1.2}{RL}$, $k_D = 0.5L$, $k_I = \frac{1}{2L}$

---

2) Ultimate Cycle Method

Set $k_D = k_I = 0$, Increase k_p until sustained oscillation (purely Imaginary roots)
At sustained oscillation set $k_p = k_{pu}$ and measure the period of oscillation P_u

P: $k_p = 0.5 K_{pu}$

PI: $k_p = 0.45 k_{pu}$, $k_I = \frac{1}{0.83 P_u}$

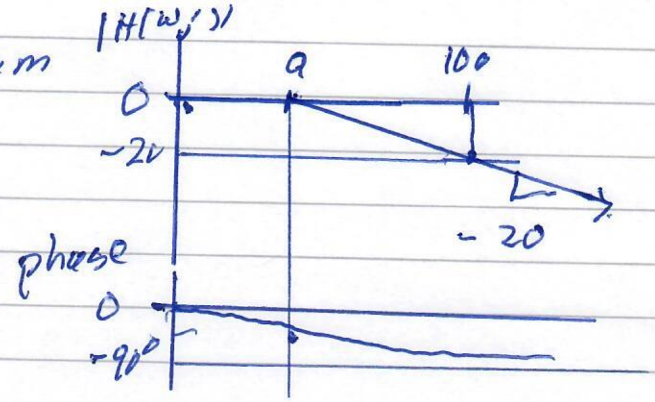
PID: $k_p = 0.6 k_{pu}$, $k_I = \frac{2}{P_u}$, $k_D = 0.125 P_u$

Nyquist / Polar Plots

Are Similar to Bode Diagram another way to illustrate & Graph the frequency Response.

Take transfer Function $H(s) = \frac{a}{s+a}$

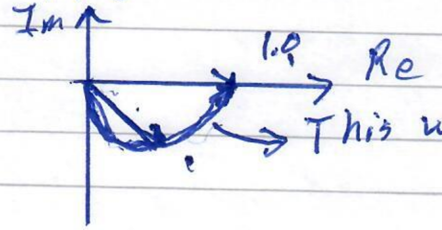
Bode Diagram



The Nyquist/Polar Plot is the Phasor of $H(\omega j)$

It starts at $|H(\omega j)| \to 1$ for small ω and goes to $|H(\omega j)| \to 0$ as $\omega \to \infty$, Phase starts at zero and goes to $-90°$

The Nyquist Plot (1st order unity gain system)



This will map a semicircle.

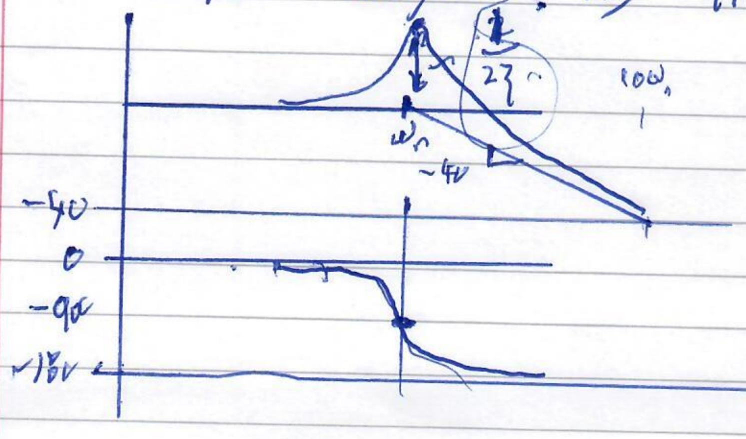
For Second order System we get

The Bode Diagram is,

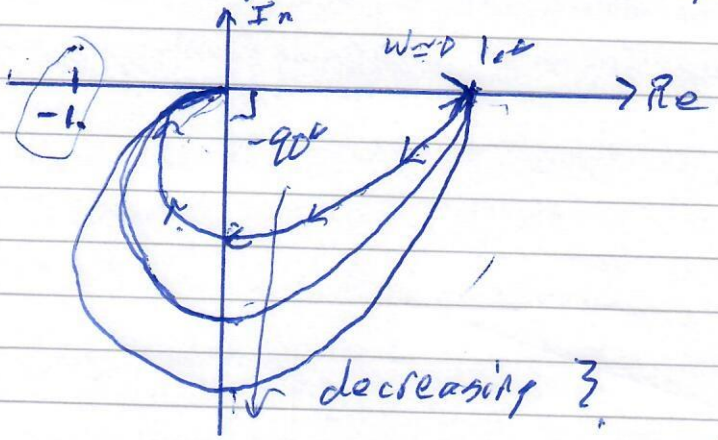
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$= \frac{\omega_n^2}{(s-p)(s-\bar{p})}$$

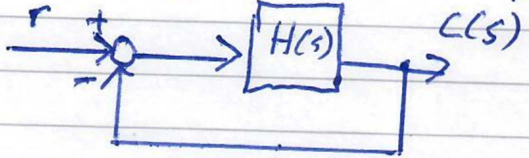
$$p, \bar{p} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\, j$$



Draw the Nyquist/Polar Plot for a 2nd order underdamped system ($\zeta<1$). Unity Gain.



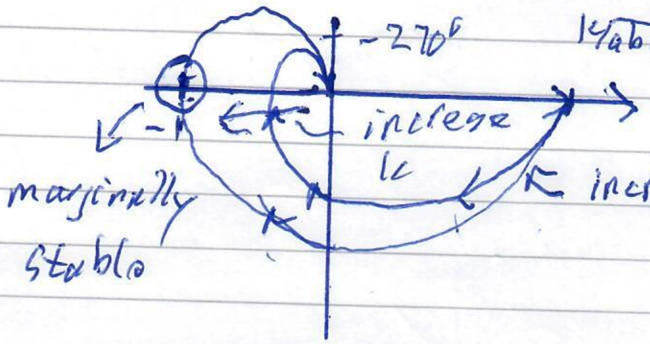
Nyquist & Stability $\quad H(s) = \frac{K}{(s+a)(s+b)(s+c)}$ feedback gain



$\frac{C(s)}{R(s)} = H_{CL}(s) = \frac{H(s)}{1+\boxed{H(s)}}$

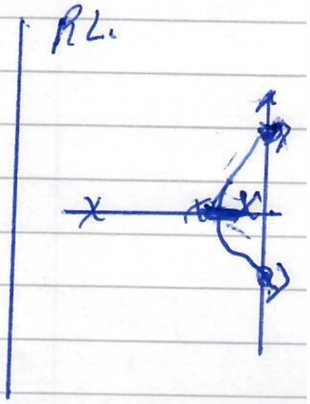
H(s) = -1 we have a closed loop pole

If open loop T.F. Nyquist plot goes through (or around) "-1" then unstable. (-1 purely imaginary roots)



Phase Margin & Gain Margin

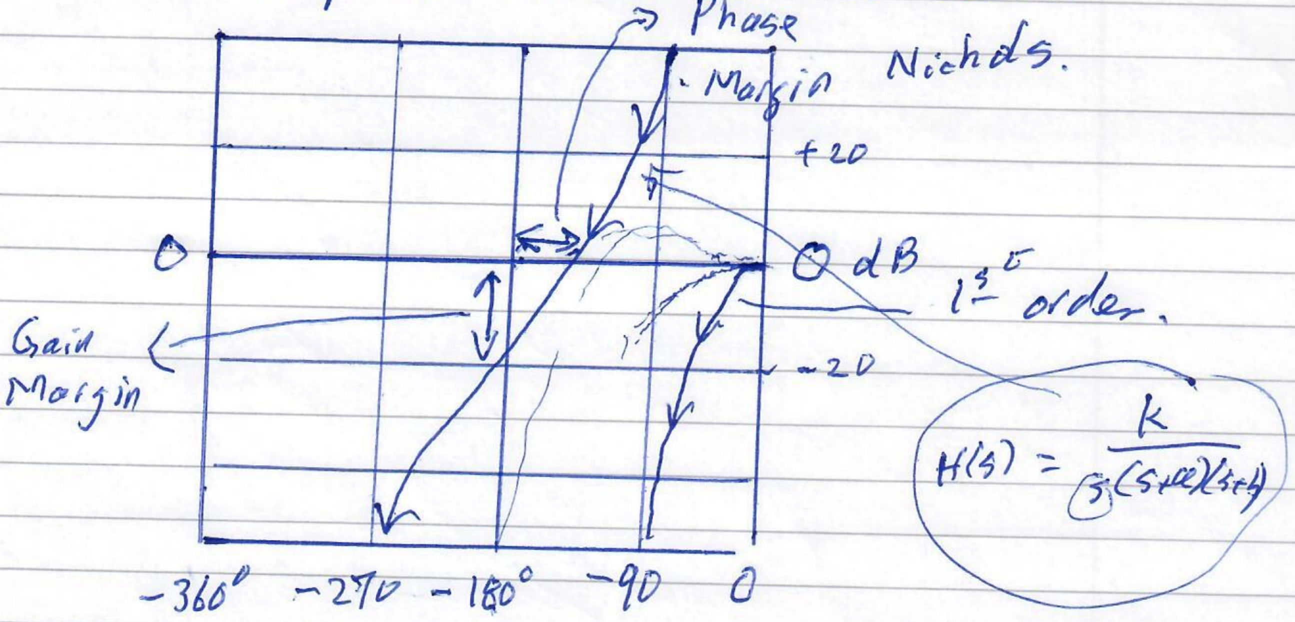
We do not want the Open Loop TF $G(s)H(s) = -1$. The PM & GM are specification on the closed ~~or~~ loop control system.



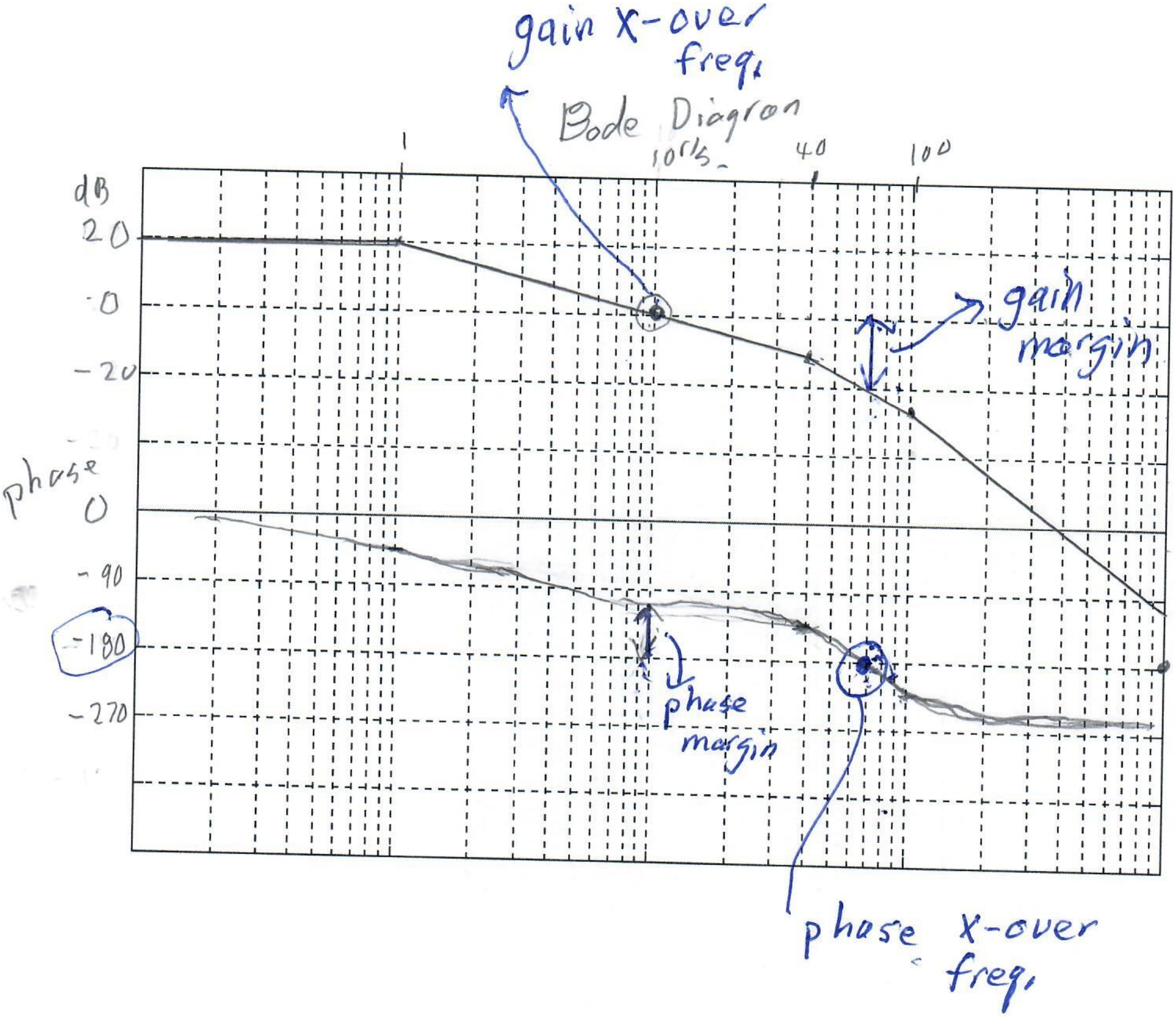
For phase Margin Find gain x-over freq. (where we cross 0 dB) and the phase Margin is addition phase lag to hit -180°. Similarly the Gain Margin (GM) is where phase crosses -180° (Phase x-over freq.) and measure addition gain to reach 0 dB point.

Nichols Chart

- The Nichols is a frequency Diagram for the Open loop T.F. & similar to Bode.

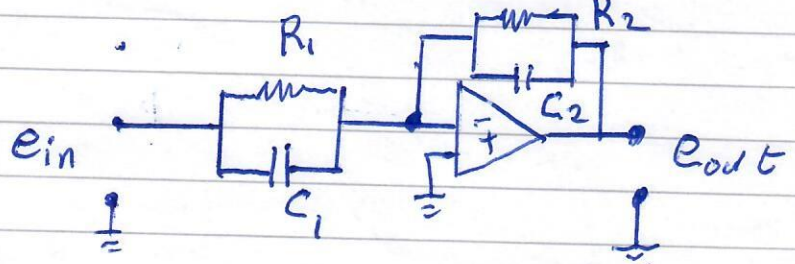


$$H(s) = \frac{K}{s(s+2)(s+4)}$$


gain X-over freq.
Bode Diagram
1
10r/s
40
100
dB
20
0
-20
phase
0
-90
-180
-270
gain margin
phase margin
phase X-over freq.

Phase Lead Compensator Design

Phase lead and lag compensator are built as,



Neglecting Inversion

The T. F $\frac{e_{out}(s)}{e_{in}(s)} = \frac{\left(s + \frac{1}{R_1 C_1}\right)}{\left(s + \frac{1}{R_2 C_2}\right)}$

one can multiply by gain to get compensator

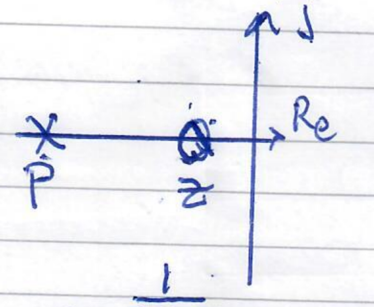
The Lead Compensator.

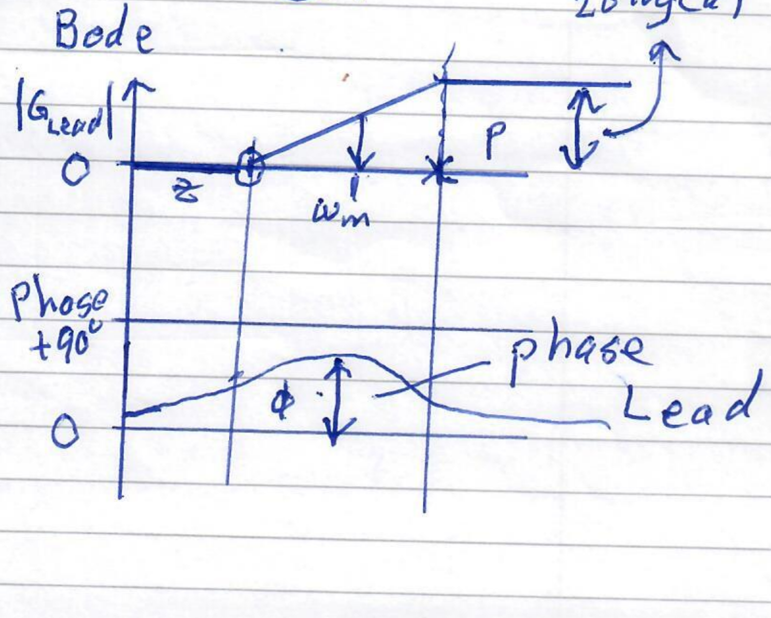
The compensator has T. F. of the following form.

$$G_{Lead}(s) = \frac{aTs+1}{Ts+1} \quad \text{we have } a>1$$

$$= \frac{a\underbrace{\left(s + \frac{1}{aT}\right)}_{z}}{\underbrace{\left(s + \frac{1}{T}\right)}_{P}} \qquad |z|<|p|$$

Plot on s-plane





$$\omega_m = \frac{1}{\sqrt{a}\,T}$$

$$\sin\phi = \frac{a-1}{a+1}$$

We will also write the Lead Compensator

$$G_{Lead}(s) = \frac{Ts+1}{\gamma Ts+1} \qquad \gamma < 1$$

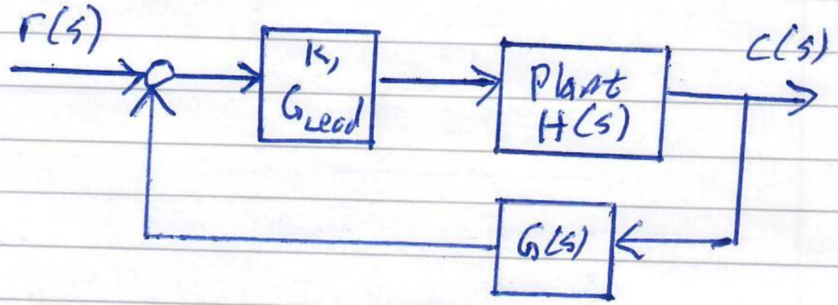
$$= \frac{(s+\frac{1}{T})}{\gamma(s+\frac{1}{\gamma T})} \qquad \sin\phi_m = \frac{1-\gamma}{1+\gamma}$$

$$\omega_m = \frac{1}{\sqrt{\gamma}\,T}$$

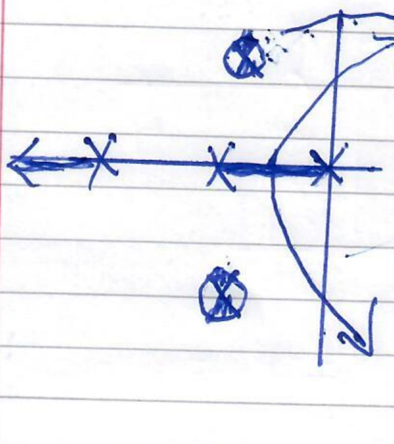
Lead Compensator Design Using root Locus Techiques.

One uses this method when specifications are in terms of rise time, undamped natural freq. & damping ratio, time constants or closed loop pole locations.

For example



Let's draw root locus for $KG(s)H(s)$



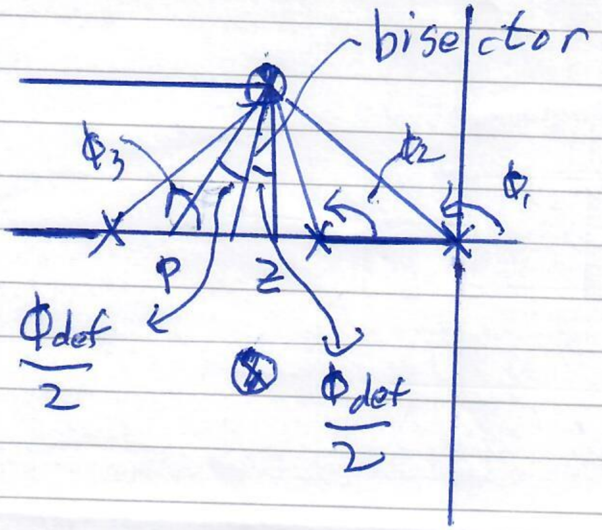
These are my specified C.L. pole location, These C.L. pole location cannot be achieved by a feedback alone we need a Lead Compensator.

Design Procedure for Lead Compensator using root locus Technique

Step 1) Determine desired C.L. pole location from specifications.

Step 2) Plot poles/zeros on s-plane and determine and deficiency at desired C.L. pole locations.

Step 3) Design lead compensator to ~~achive~~ ~~achieve~~ achieve desired C.L. pole location with minimum additional. It is designed as follows,



angle deficiency is ϕ_{def}

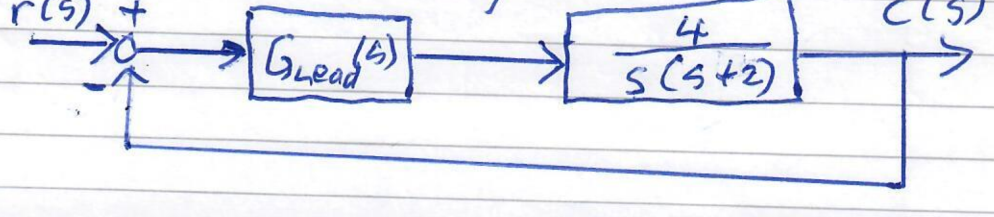
$\phi_{def} = +ve \searrow \ominus$

$$180^\circ = \phi_1 + \phi_2 + \phi_3 \overset{\ominus}{+} \phi_{def}$$

$$\therefore \phi_{def} = 180^\circ - \phi_1 - \phi_2 - \phi_3$$

Example of Lead Compensator using root locus techniques.

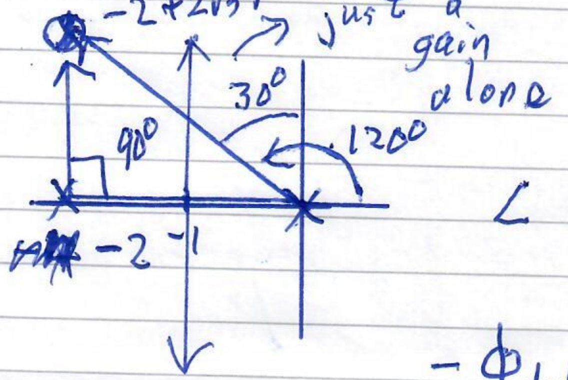
We are give the following



The specification is for the C.L. system to have undamped natural freq. of 4 r/s and a damping ratio of $\zeta = 0.5$.

1) The closed loop pole location must be at $P_{CL}, \bar{P}_{CL} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\,j = -2 \pm 2\sqrt{3}\,j$

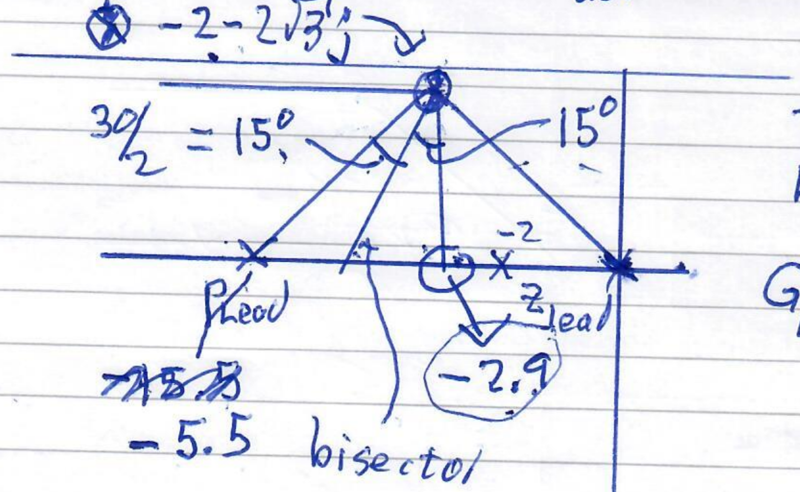
2)



$\zeta = \sin 30^\circ = 0.5$

$$\angle \frac{4}{s(s+2)}\Bigg|_{s=p_{CL}} = -120^\circ - 90^\circ = -210^\circ$$
$$= -210^\circ$$

$$-\phi_{def} = 180^\circ - 210^\circ = \boxed{-30^\circ}$$



$2\sqrt{3} = 3.5$

The Lead Compensator has the form

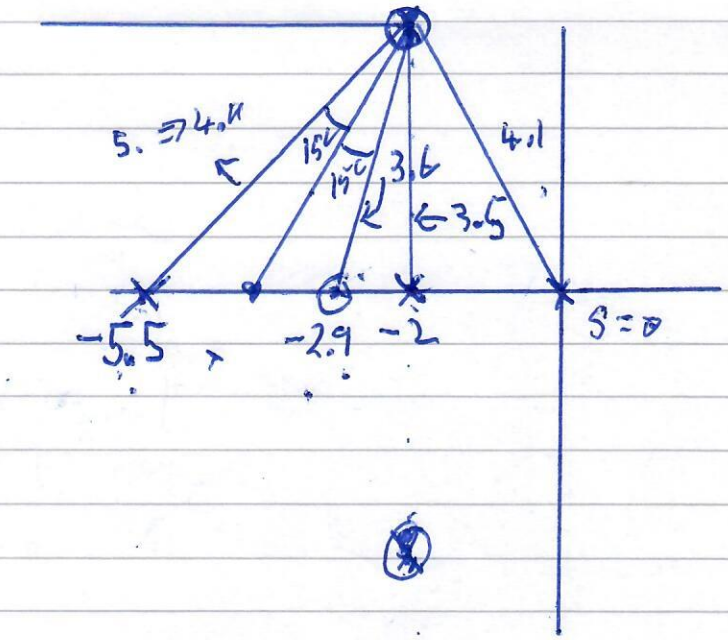
$$G_{Lead}(s) = \frac{K(s+2.9)}{(s+\cancel{15.3}\ 5.5)}$$

Then use magnitude criteria to find the gain "K"

$$\left|\frac{K(s+2.9)\,4}{(s+5.5)\,s(s+2)}\right|_{s=-2+2\sqrt{3}\,j} = 1 \Rightarrow \frac{K(3.6)4}{(4.9)(4.1)(3.5)}$$

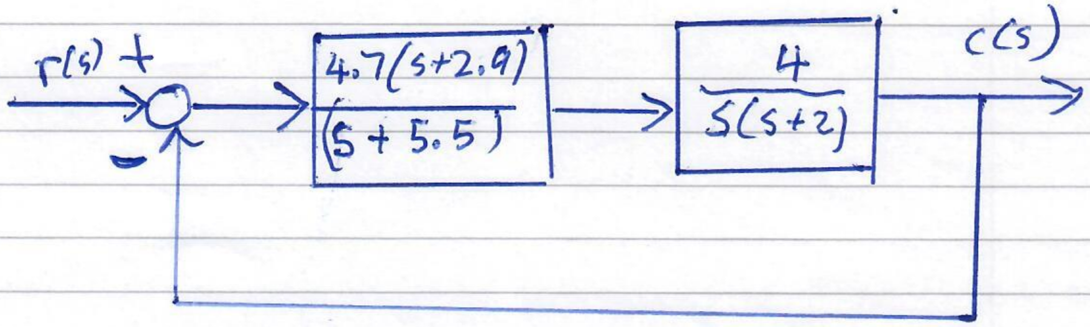
$$\Rightarrow \underline{K = 4.7}$$

Graphical Design
Lead Compensator



The Lead Compensator is

$$G_{Lead} = \frac{4.7(s+2.9)}{(s+5.5)}$$



Lead Compensation Using Frequency Domain.

One uses frequency domain techiques when specification are given in terms of Steady State errors & Phase margin.

We can write the equation for a Lead Compensator.

$$G_{Lead}(s) = K\frac{Ts+1}{\alpha Ts+1} = K_c \frac{\overset{\rightarrow \text{zero}}{(s+\frac{1}{T})}}{\underset{\rightarrow \text{pole}}{(s+\frac{1}{\alpha T})}} \qquad \boxed{\alpha < 1}$$

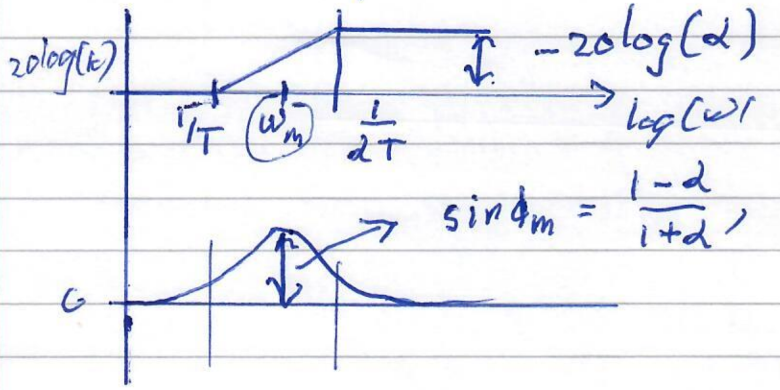
where $K_c = \frac{K}{\alpha}$

Similarly

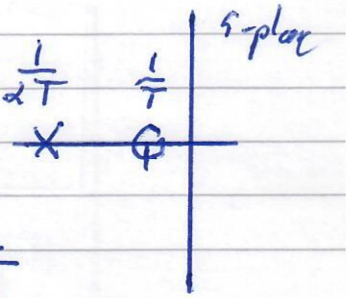
$$G_{Lead}(s) = K\frac{(aTs+1)}{(Ts+1)} = K_c\frac{(s+\frac{1}{aT})}{(s+\frac{1}{T})} \qquad \boxed{a > 1}$$

$K_c = Ka$

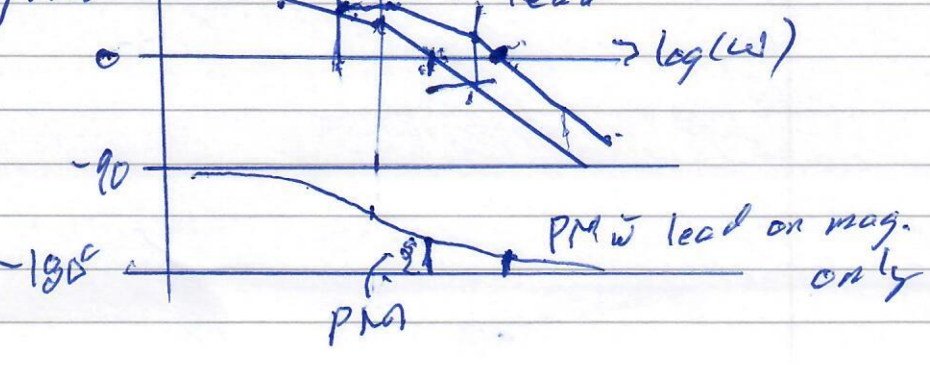
A Lead Compensator



$$\sin\phi_m = \frac{1-\alpha}{1+\alpha}, \quad \omega_m = \frac{1}{\sqrt{\alpha}\,T}$$



The lead Compensator will move the Gain (Mag. of Bode crosses 0 dB) to a higher frequency and further decrease uncompensted phase margin.



Design Steps in the frequency Domain

1) Find O.L. gain to meet steady state error specifications

2) Using Gain from step 1 Plot the Bode Diagram

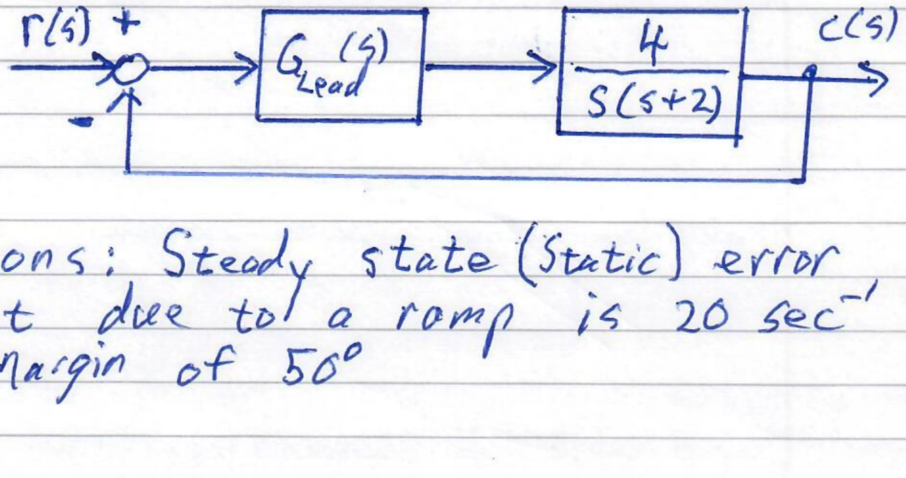
3) Determine necessary additional phase angle (ϕ_m) to meet specification and add $5^\circ \to 10^\circ$ extra to account for Gain X-over shift to higher frequency.

4) Use $\sin\phi_m = \frac{1-\alpha}{1+\alpha}$ to get α. | $\sin\phi_m = \frac{a-1}{a+1}$

* We know ϕ_m from step 3

5) Place the lead compensator (using T) Such that ω_m occurs at the new gain X-over frequency.

Example



Specifications: Steady state (Static) error coefficient due to a ramp is 20 sec^{-1} & Phase Margin of 50°

$\varepsilon_{ss} = \boxed{\frac{R}{K_v}}$

Step 1) Determine Gain k,

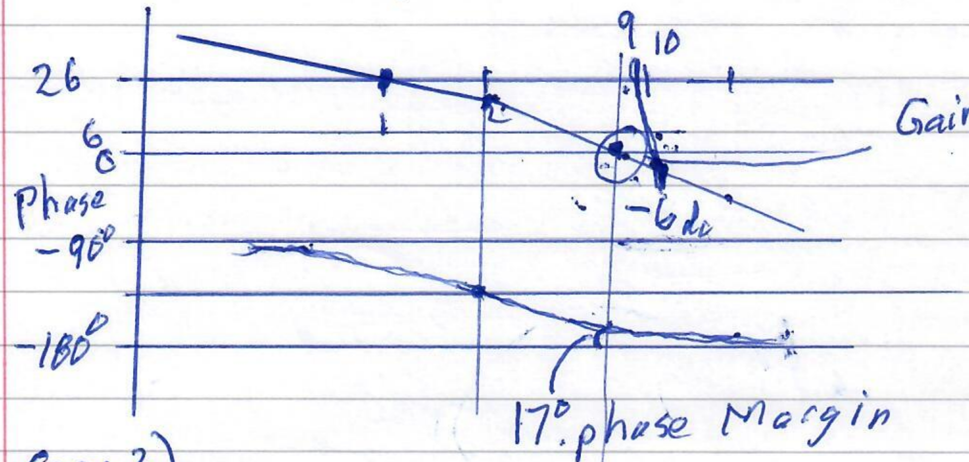
We know that the error coefficient due a ramp is $K_v = \lim_{s \to 0} sG(s)H(s)$

$$= \lim_{s \to 0} \frac{\cancel{s} \cdot K 4}{\cancel{s}(s+2)} = 2k = 20$$

$\therefore K = 10$

Step 2) Plot Bode Diagram. $KG(s)H(s) = \frac{40}{s(s+2)}$

DC gain = 20log(20) = 26 dB



Step 3)

I have a phase margin of 17° Therefore we need additional $50-17+5° = 38°$

↳ added for phase shift to higher freq.

Step 4) $\widehat{\sin\phi_m} = \frac{1-\alpha}{1+\alpha}$ $\phi_m = 38°$

$$0.616 + 0.616\alpha = 1-\alpha \Rightarrow \alpha = 0.238$$

Step 5) Find where to place the compensator
Find "T"

We know the additional Gain at high frequency due to the lead Compensator is

$$|G_{Lead}(w\text{-}hig)| \Rightarrow -20\log(\alpha) = -20\log(0.238)$$

$$= 12.5\ dB$$

The new gain X-over freq. will be at the old (uncompensated Bode with K) at

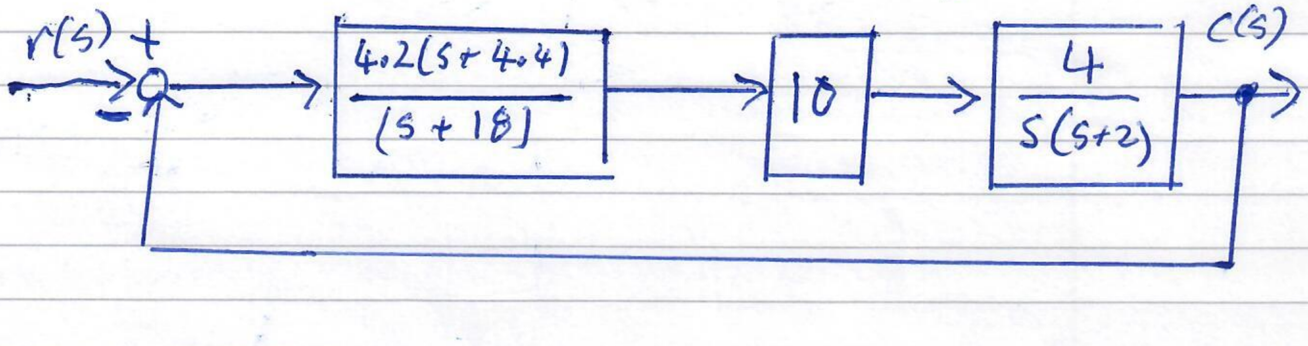
$-\frac{12.5}{2} = -6.25$ dB. From the Bode

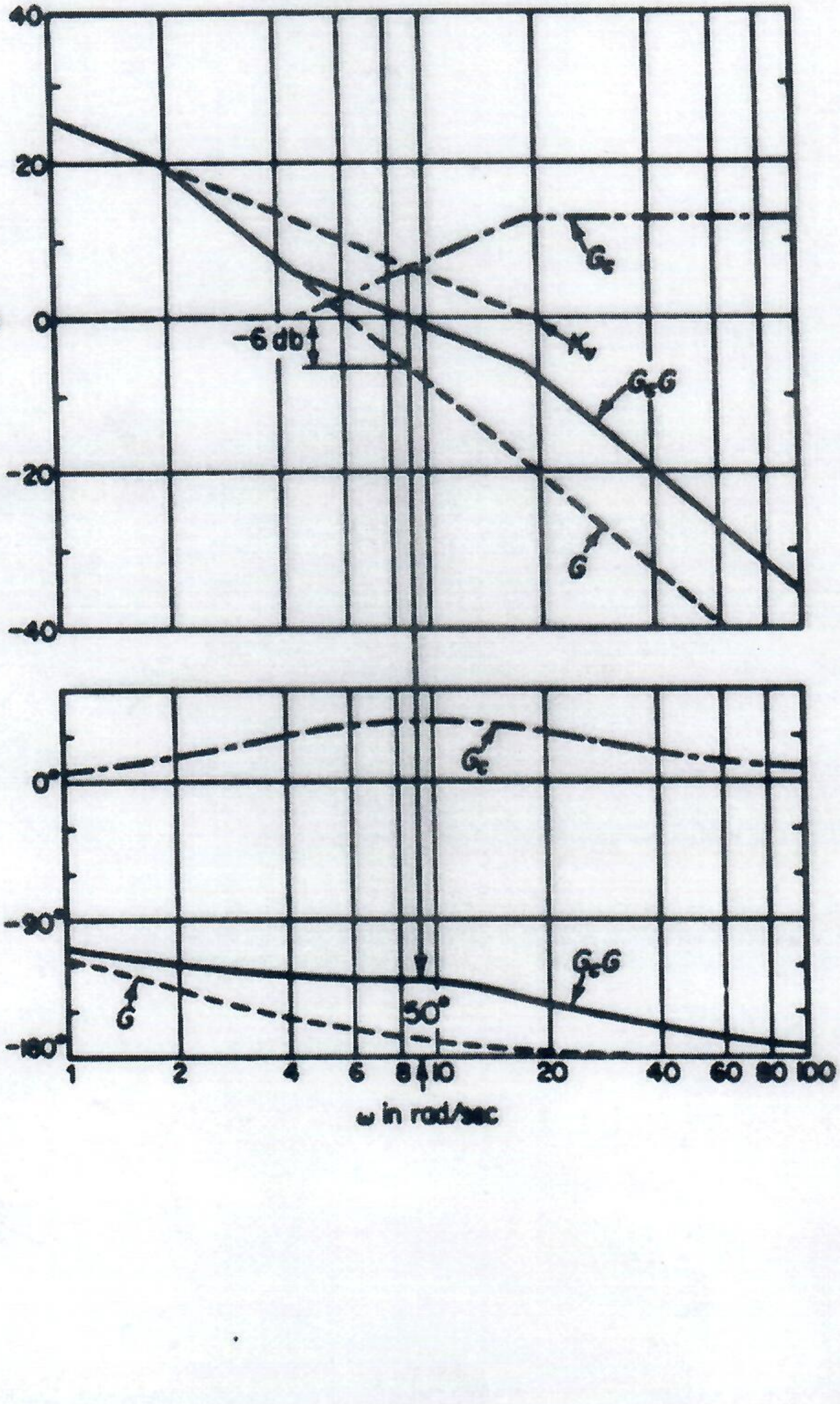
Diagram I know that the new gain x-over freq. is at $\omega_m = 9$ r/s. We then calculate

T from $\omega_m = \frac{1}{\sqrt{\alpha}\,T} = 9$ r/s $\Rightarrow T = 0.23$

The lead Compensator becomes

$$G_{Lead}(s) = \frac{0.23s+1}{0.055s+1} = \frac{4.2(s+4.35)}{(s+18.2)}$$




40
20
db
0
-20
-40
G_c
K_v
G_cG
G
-6 db
0°
-90°
-180°
G_c
G_cG
G
50°
1
2
4
6
8
10
20
40
60
80
100
ω in rad/sec

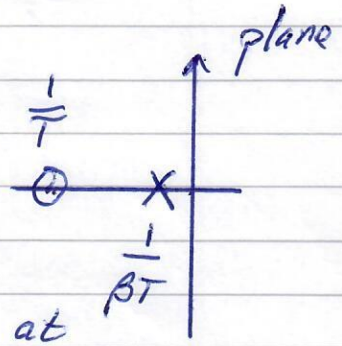
The Phase Lag Compensator

Phase Lag Compensator using root-Locus techinques. One uses a phase Lag compensator when the system has good transient performance (good closed loop pole locations) but does not meet steady state error requirements.

The form of a lag compensator is

$$G_{Lag}(s) = \frac{Ts+1}{\beta Ts+1} \quad \text{where } \beta > 1$$

$$= \frac{1}{\beta}\frac{\overset{\text{zero}}{(s+1/T)}}{\underset{\text{pole}}{(s+\frac{1}{\beta T})}}$$



We will place the pole & zero at low freq. (The pole & zero are small in comparison to transient or dominant C.L. pole location

---

Design Steps.

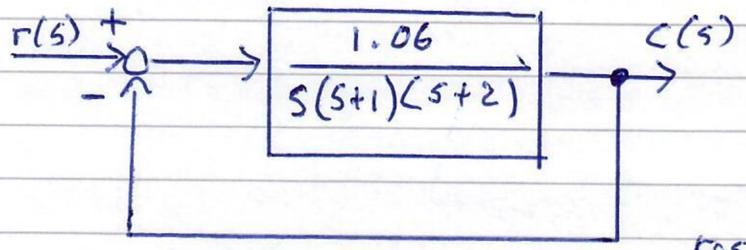
Step 1): (Root locus Design) - Draw root locus for uncompensated System, design feedback to achieve transient response.

Step 2) Calculate additional gain needed to achieve S.S. error performance

Step 3) Determine/Design the pole/zero location for the Lag compensator, such that the C.L. poles are not significantly changed.

Step 4) Redraw root locus make sure all specification are met.

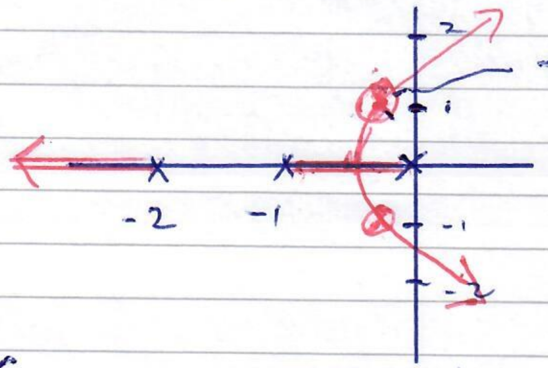
Example: We are given the following system.



We are pleased with the transient response, but the specification for the Steady State ~~error~~ error coefficient due to a ramp is, $k_v = 5 \sec^{-1}$

Recall $k_v = \lim_{s \to 0} sG(s)H(s) = 0.53$

Sketch Root locus for this system



Given our current the closed Loop T.F. is

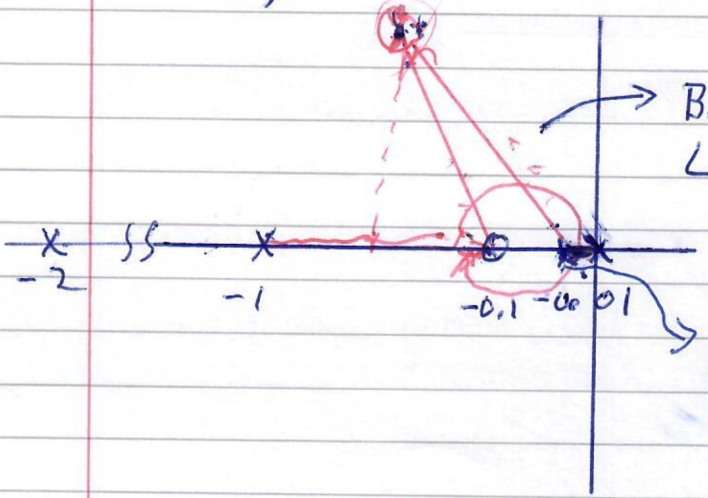
$$\frac{C(s)}{r(s)} = \frac{1.06}{s^3+3s^2+2s+1.06} = \frac{\frac{1.06}{s(s+1)(s+2)}}{1+\frac{1.06}{s(s+1)(s+2)}}$$

The dominant C.L. pole locations are $p, \bar{p} = -0.33 \pm 0.58j$, $\omega_n = 0.7\ r/s.$ $\zeta = 0.5$

Lag Compensator Design Continued.

The current s.s. error coefficient due to a ramp is $k_v = 0.53$ and it is specified to be $k_v = 5.0$. Therefore, set the lag compensator to be. (set $\beta = 10$)

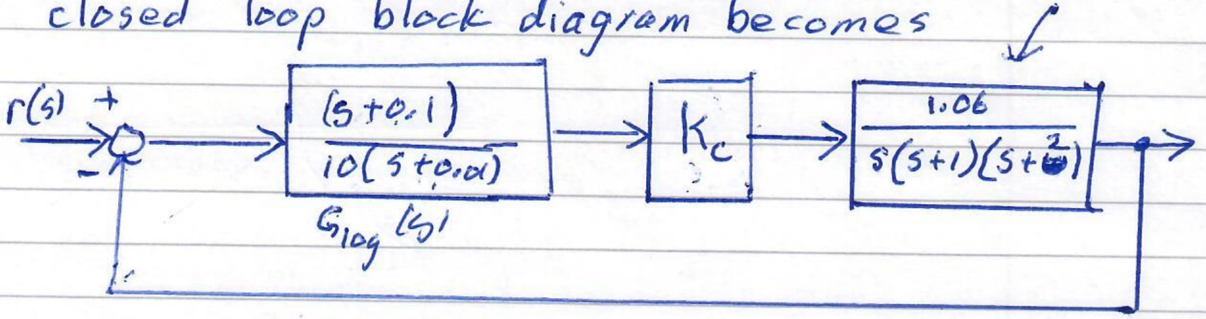
$$G_{Lag} = \frac{Ts+1}{\beta Ts+1} = \frac{10s+1}{100s+1} = \left.\frac{1/10\,(s+0.1)}{(s+0.01)}\right\} \text{Low freq.}$$



Because compensator at Low freq. then contribution to mag. criteria almost cancel from Lag pole/zero

phase Contribution about $\underline{\underline{5^\circ}}$

The closed loop block diagram becomes

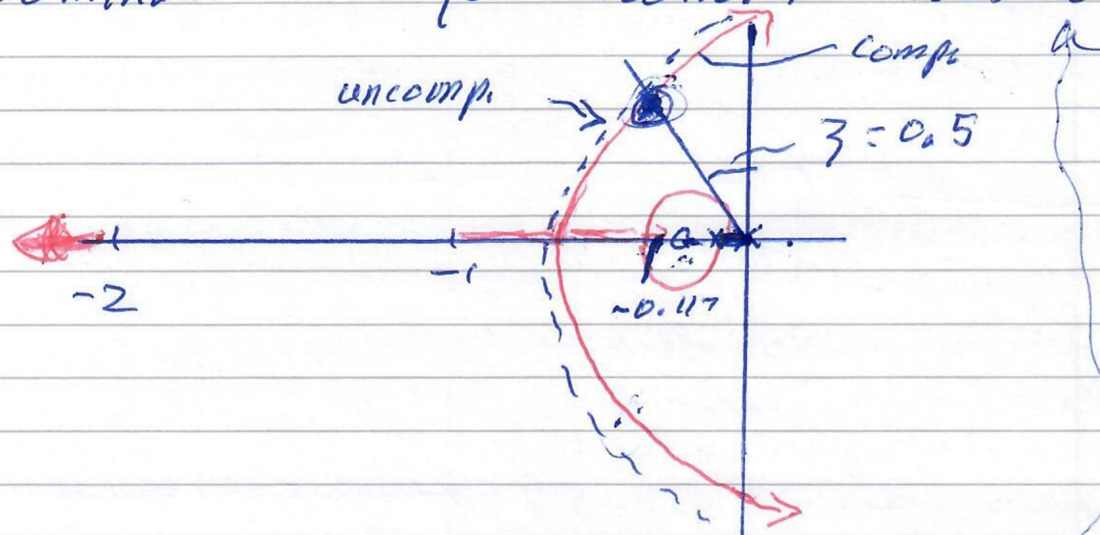


$$G_{openloop}(s) = \frac{K(s+0.1)}{s(s+0.01)(s+1)(s+2)} \qquad K = \frac{K_c(1.06)}{10}$$

We expect $K_c \approx 10$

K should be close to 1

Let's Redraw the root locus for the compensated System, keeping the original damping ratio at $\zeta = 0.5$ (recall the previous dominant C.L. pole locations $-0.33 \pm 0.58j$.)



My new compensated C.L. dominant pole location is ~~at~~ $p_c, \bar{p}_c = -0.28 \pm 0.51j$

To achieve the new C.L. pole location use the Magnitude Criteria.

$$K = \left| \frac{s(s+0.01)(s+1)(s+2)}{(s+0.1)} \right|_{s=-0.28\pm0.51j} = \underline{0.98}$$

We get $K \cong 1$, as expected.

What is the new k_v

$$k_v = \lim_{s \to 0} sG(s)H(s) = \frac{0.98(0.1)}{(0.01)(1)(2)} = \underline{\underline{4.9}}$$

Close enough!

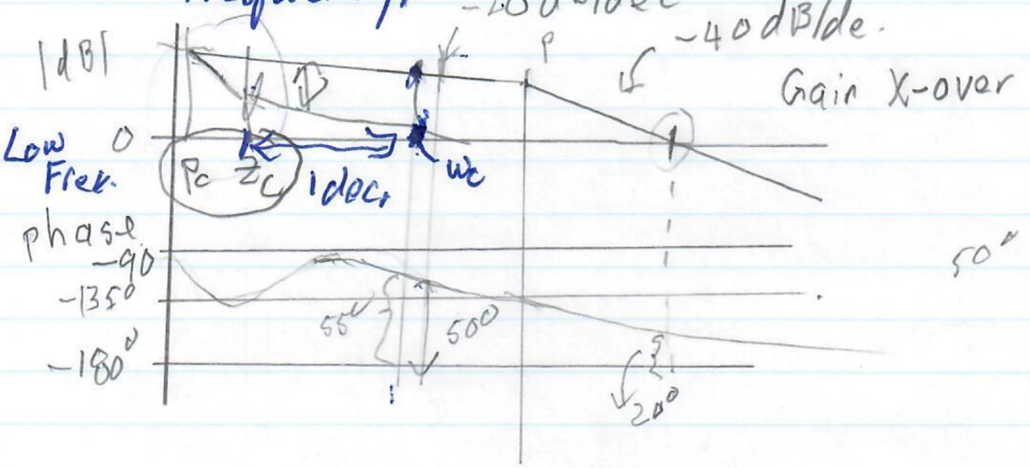
Lag Comp. R.L. continued.

If we wanted K_v exactly = 5, then K_c = 9.44 that would ~~degrees~~ decrease damping ratio a little bit & it would be OK.

We also got a C.L. pole location at $p_{CL_2} = -0.117$, but this close loop pole will be essentially cancelled by the zero at $z = -0.1$

Lag Compensation in the Frequency Domain.

One designs a Lag Compensator using the frequency domain (Bode) diagram when one has a specification on Steady State error (i.e. kv) and a specification on phase margin. The Lag compensator acheives the Phase margin by attenuation of the higher freq. magnitude of the Bode diagram. The phase diagram (Bode Plot) is only affected at low frequency.



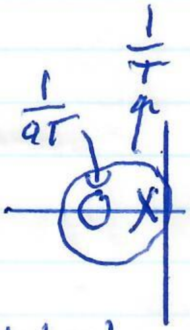
Steps to Design Lag Compensator

Step 1) Compute Gain required to meet S.S. error requirements. Then plot Open Loop T.F. with the necessary.

Step 2) From the Bode Diagram determine the required Gain X-over freq. such that the Phase Margin will be achieved and take into consideration the effect of the Lag Compensator.

Step 3) Determine amount of attenuation needed to bring magnitude plot down to 0 dB at desired x-over frequency

Recall Lag Compensator has the form.



$$G_{Lag}(s) = \frac{aTs+1}{Ts+1} \qquad \boxed{a<1}$$

The magnitude of the Open loop T.F (without Lag) is

$$\left|G_{OL}(\omega_c j)\right| = -20\log(a) \quad dB$$

Then compute $a = 10^{-\boxed{|G_{OL}(\omega_c j)|}/20}$

Then choose "T" such that $\frac{1}{aT}$ is a decade below ω_c. $\frac{1}{aT} = \frac{\omega_c}{10}$.

Step 4) Draw Compensated Bode Diagram and make sure specifications are met.

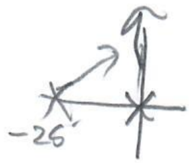
Example: Given

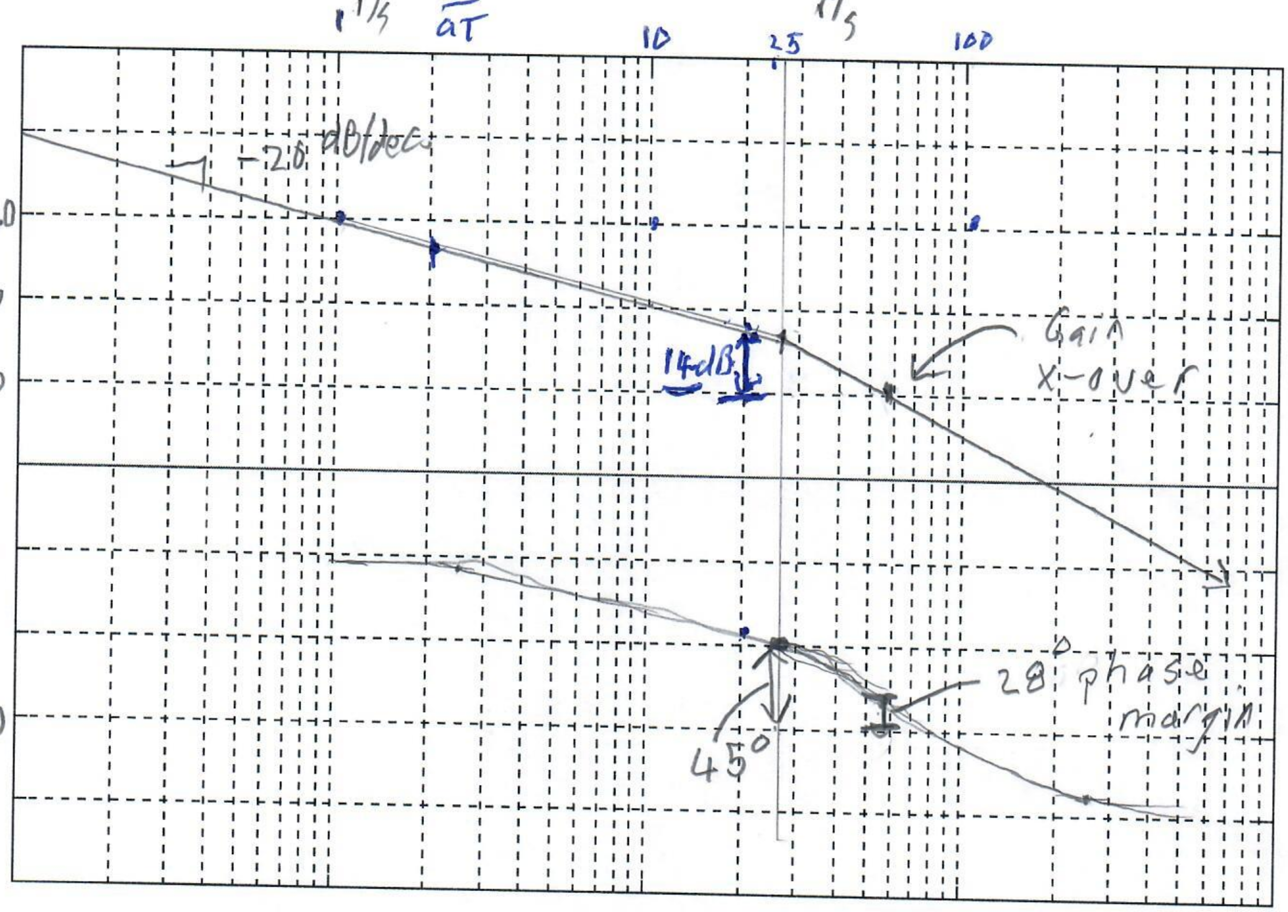
$$\frac{\theta(s)}{\alpha(s)} = \frac{2500k}{s(s+25)}$$

Specifications: ① Steady State Error due to a ramp is, $\varepsilon_{ss} = 0.01$ r/s

② Phase Margin greater than 45°

$$KG(s)H(s) = \frac{2500}{s(s+25)}$$





Stead State error due to aramp

Step 1)

$$E_{ss} = \frac{1}{k_v} \leq 0.01 \Rightarrow K_v = 100$$

$$k_v = \lim_{s \to 0} s\,G(s)H(s) = \lim_{s \to 0} \frac{\cancel{s}\,2500k}{\cancel{s}(s+25)} \Rightarrow k = 1$$

Given this gain sketch the Bode Diagram.

Step 3) We only have 28° of phase margin, but we need at Least 45°. We know at $w = 25\ r/s$ we get 45° phase margin but Lag compensator will add phase ~~d~~ delay. As such place New Gain X-over frequency at $W_c = 20\ r/s$ to adjust for Lag compensator.

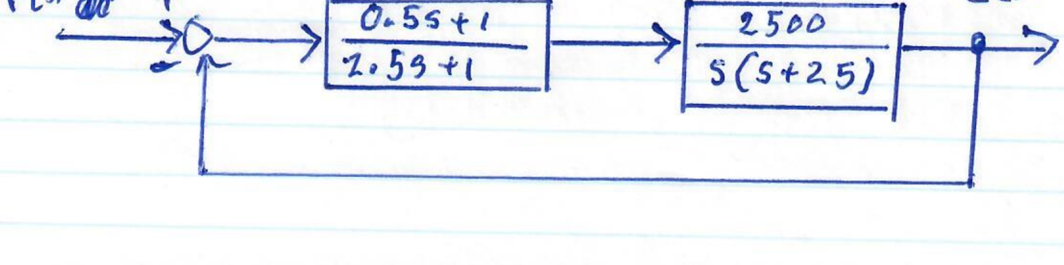
Therefore $a = 10^{-|G_{OL}(20j)|/20}$

where $G_{OL}(20j) = 14\ dB \Rightarrow a = 10^{-0.7} = 0.2$

We want $\frac{1}{aT} = \frac{W_c}{10} = 2 \Rightarrow T = 2.5$

The Lag Compensator is

$$G_{Lag}(s) = \frac{0.5s+1}{2.5s+1}$$



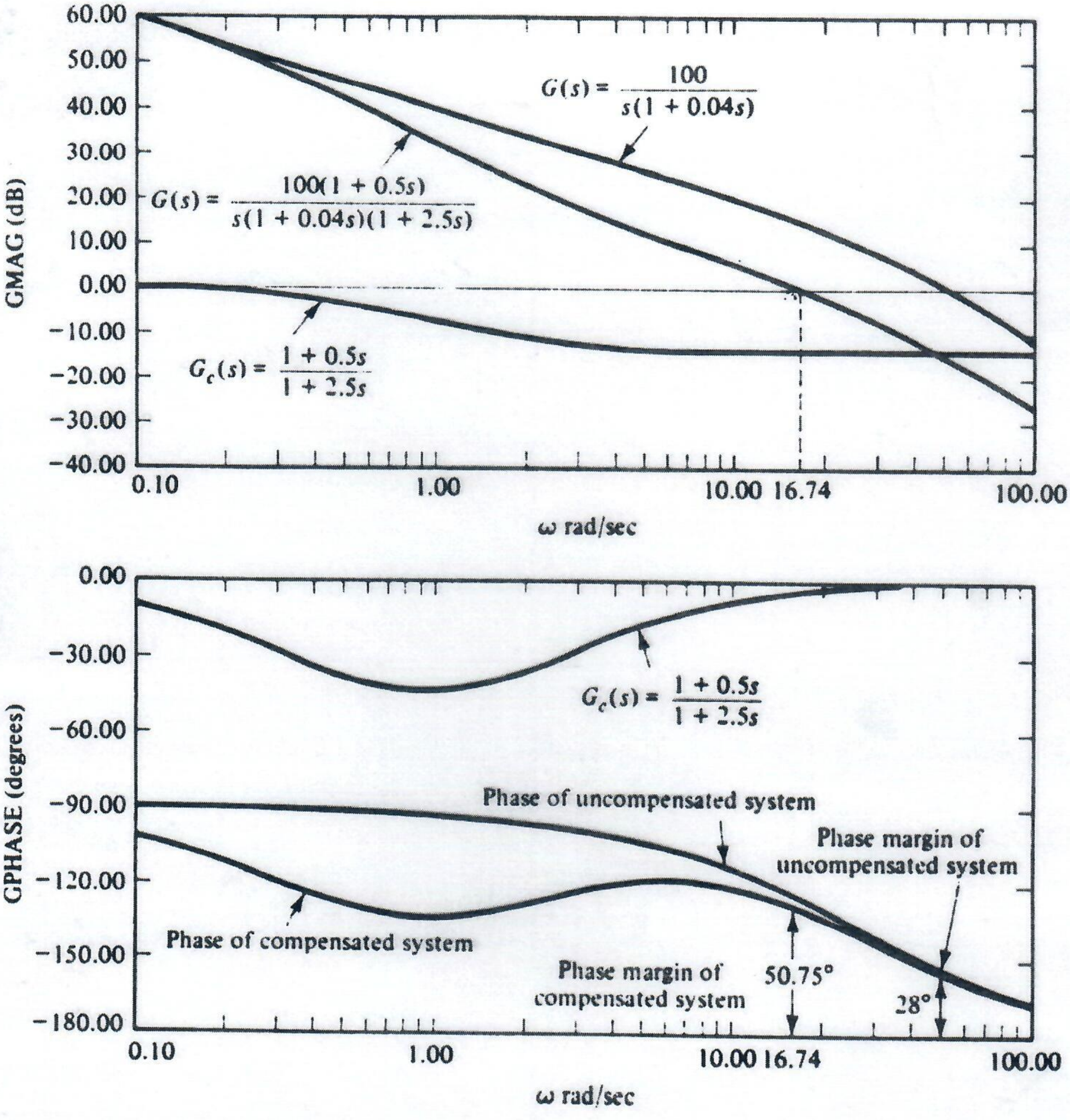


Figure 10-13 Bode plots of compensated and uncompensated systems in Example 10-3.

Lead-Lag Compensation - Root Locus Method

Use this compensator when both transient response & steady state error need to be improved.

A Lead-Lag compensator has the form.

$$G_c(s) = K_c \underbrace{\left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}}\right)}_{\text{Lead}} \underbrace{\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right)}_{\text{Lag}} \qquad \beta > 1, \ \gamma > 1$$

Step 1) Determine desired C.L. pole location

Step 2) Determin angle deficiency & phase Lead compensator.

Step 3) Compute the gain K_c to meet magnitude criteria

$$\left| K_c \left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}}\right) G_p(s) \right| = 1.0$$

Step 4) From Specs on Steady State error, choose β

$$K_v = \lim_{s \to 0} s G_c(s) G_p(s) = s K_c \frac{\beta}{\gamma} G_p(s)$$

Choose T_2 large s.t. phase change at desired C.L. poles is less than $5^\circ \rightarrow 10^\circ$?

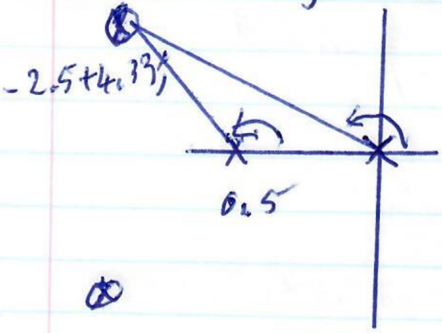
Example ; $G(s) = \frac{4}{s(s+0.5)}$

The specs: $\zeta = 0.5$, $\omega_n = 5$ r/s $k_v = 80$ sec^{-1}

Desired C.L. pole locations, $p, \bar{p} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\,j$

$= -2.5 \pm 4.33j$

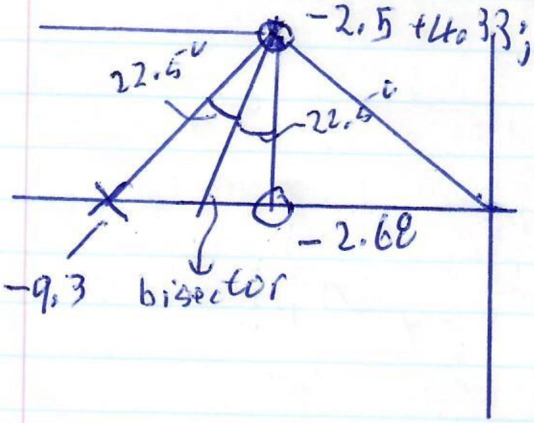
Check angle Deficiency



$$\angle G(s)\Big|_{-2.5+4.33j} = -235^{\circ}$$

Therefore we need $\underline{55^{\circ}}$ of Lead.

Therefore the Lead part of the compensator is,



$$G_{Lead}(s) = K_c\left(\frac{s+2.6\overset{7}{8}}{s+9.3}\right)$$

$$= \frac{K_c}{\cancel{4}}\left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}}\right)$$

$T_1 = \frac{1}{2.7} = 0.37$, $\frac{\gamma}{T_1} = 9.3$

$\gamma = 3.5$

Use Magnitude Criteria to Find K_c

$$4K_c\left|\frac{s+2.7}{(s+9.3)s(s+0.5)}\right|_{s=-2.5+4.33j} = 1$$

$K_c = 11.$

Step 4) Find the Lag Compensator such that $k_v = 80 \text{ sec}^{-1}$

Recall the $k_v = \lim_{s \to 0} sG(s)H(s)$

$$= \lim_{s \to 0} sG_c(s)G_p(s)$$

$$= \lim_{s \to 0} k_c \frac{\beta}{\gamma} G(s) \qquad G(s) = \frac{\boxed{4}}{s(s+\boxed{0.5})}$$

$$80 = \frac{11 \beta 4}{3.5(0.5)} \Rightarrow \beta = 3$$

Choose T_2 largo Such that $\angle G_{lag}(s)\Big| \le \overset{5^{0}}{\cancel{0.5}}$

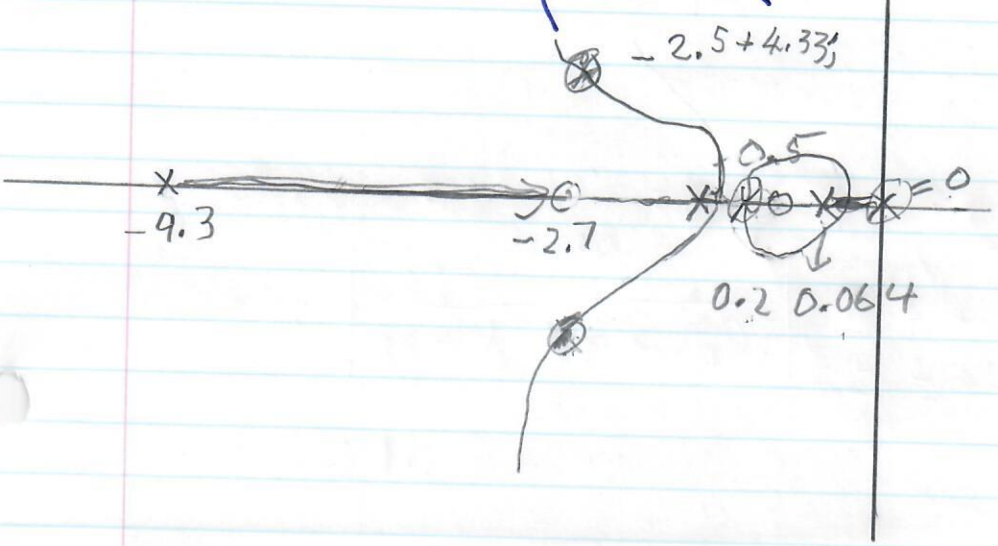
We will choose $T_2 = 5$

s = dominant C.L. pole location

Then $G_{Lag}(s) = \dfrac{s+0.2}{s+0.064} = \dfrac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}$

The Complete Compensator is.

$$G_c(s) = 11 \left(\frac{s+2.7}{s+9.3}\right)\left(\frac{s+0.2}{s+0.064}\right)$$



80

Lead Lag Compensation the frequency Domain.

Recall the general form of a Lead/Lag Compensator is,

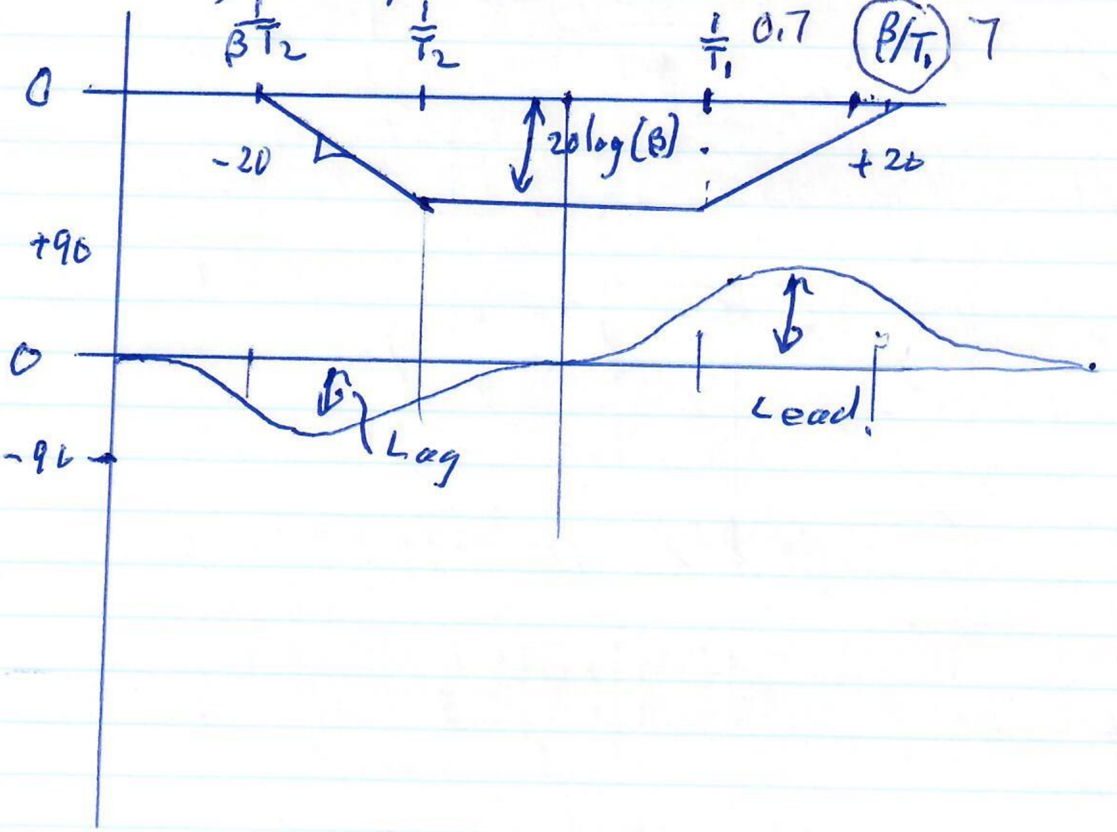
$$G_c(s) = K_c \underbrace{\left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}}\right)}_{\text{lead}} \underbrace{\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\beta T_2}}\right)}_{\text{Lag.}} \qquad \gamma, \beta > 1$$

$$= K\left(\frac{T_1 s+1}{\frac{T_1}{\gamma}s+1}\right)\left(\frac{T_2 s+1}{\beta T_2 s+1}\right)$$

$\frac{1}{T_2}$ is set a lower freq.

When designing in the frequency Domain we set $\gamma = \beta$.

The Bode diagram of the T.F. of the Lead/Lag Compensator has the form



Lead/Lag Compensator in freq. Domain,

Example: Given open loop T.F.

$$G(s) = \frac{K}{(s)(s+1)(s+2)}$$

Specification is $k_v = 10\ s^{-1}$, Phase margin 50°. To meet specifications on k_v,

$$k_v = \lim_{s \to 0} sG(s) = \frac{K}{(1)(2)} = 10 \Rightarrow K = 20$$

Draw Bode Diagram. DC Gain = 10, ≐ 20dB

We would need 50° + 32° of lead but we cannot built a ~~lead~~ Compensator w 82° of Lead (This would require too much high freq. gain).

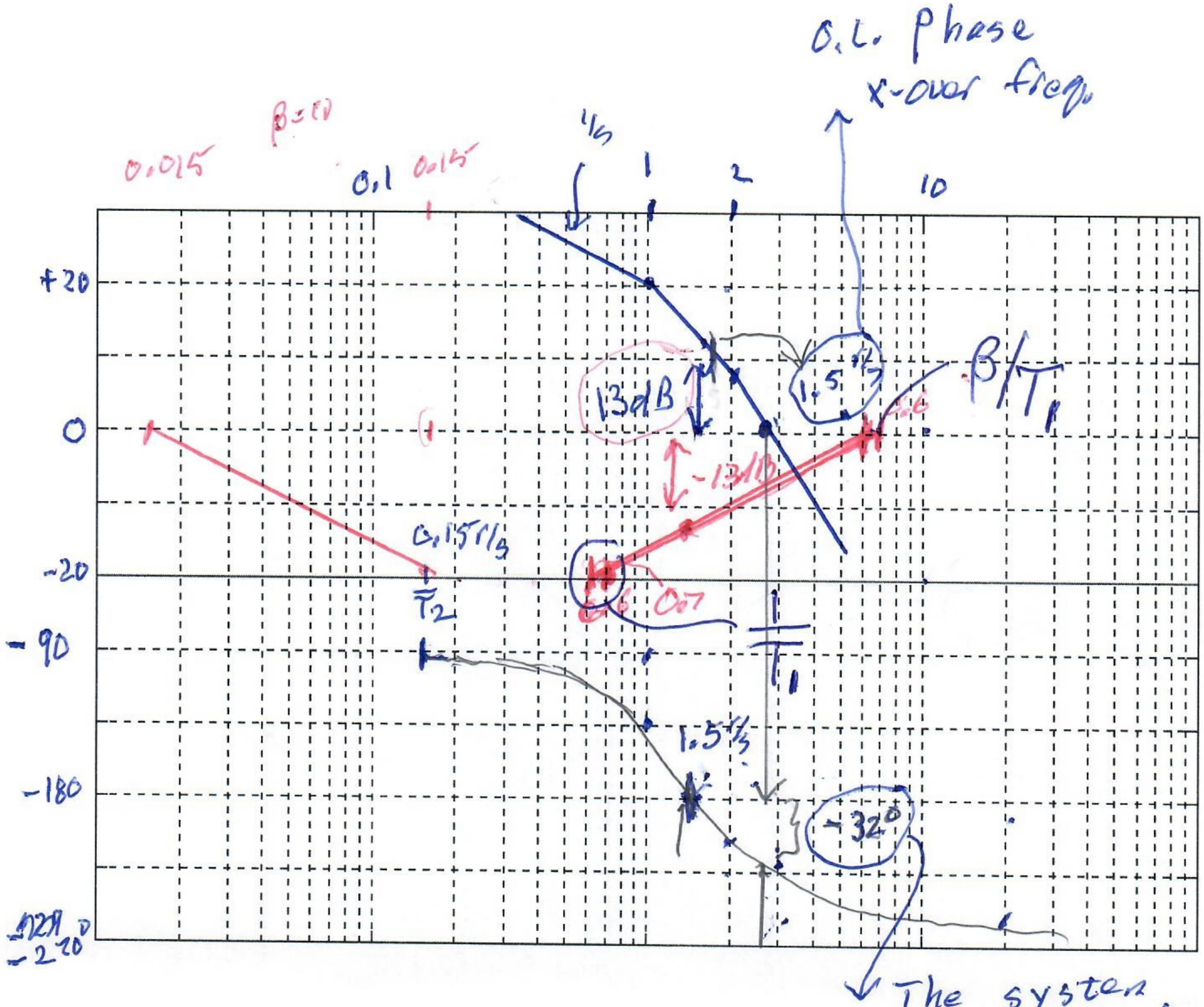
Use Lag to bring the New Gain X-over freq. to the old Phase X-over freq. Then set the lead Compensator such that we get -13 dB ~~at~~ ~~[illegible]~~ to achieve new gain X-over freq.

+5° to comp. for Lag

Now compute β to give 50° of phase Lead.

$$\sin \phi_m = \frac{\beta - 1}{\beta + 1} \Rightarrow \beta = 10 \qquad \phi_m = 55^{\circ}$$

Place the pole zero ~~zero~~ of the lag ~~lead~~ Compensator One decade Back from the new gain X-over freq.



The system is unstable

Lead/Lag Compensation Continued.

We need the zero of the Lag compensater to be at

$$\frac{1}{T_2} = 0.15 \Rightarrow T_2 \simeq 6.7, \quad \beta = 10 \nearrow 55^{\circ} \text{ lead.}$$

Place Lead compensater such that at New Gain X-over freq. which is $\omega_{cNew} = 1.5 \, r/s$.

We have a attenuation of -13dB. Reading from Bode we get

$$\frac{1}{T_1} = 0.7, \quad \frac{\beta}{T_1} = 7.$$

The full Lead Lag Compensater become

$$G_c(s) = \underset{K.}{20} \frac{(s + 0.7)}{(s + 7)} \frac{(s + 0.15)}{s + 0.015}$$

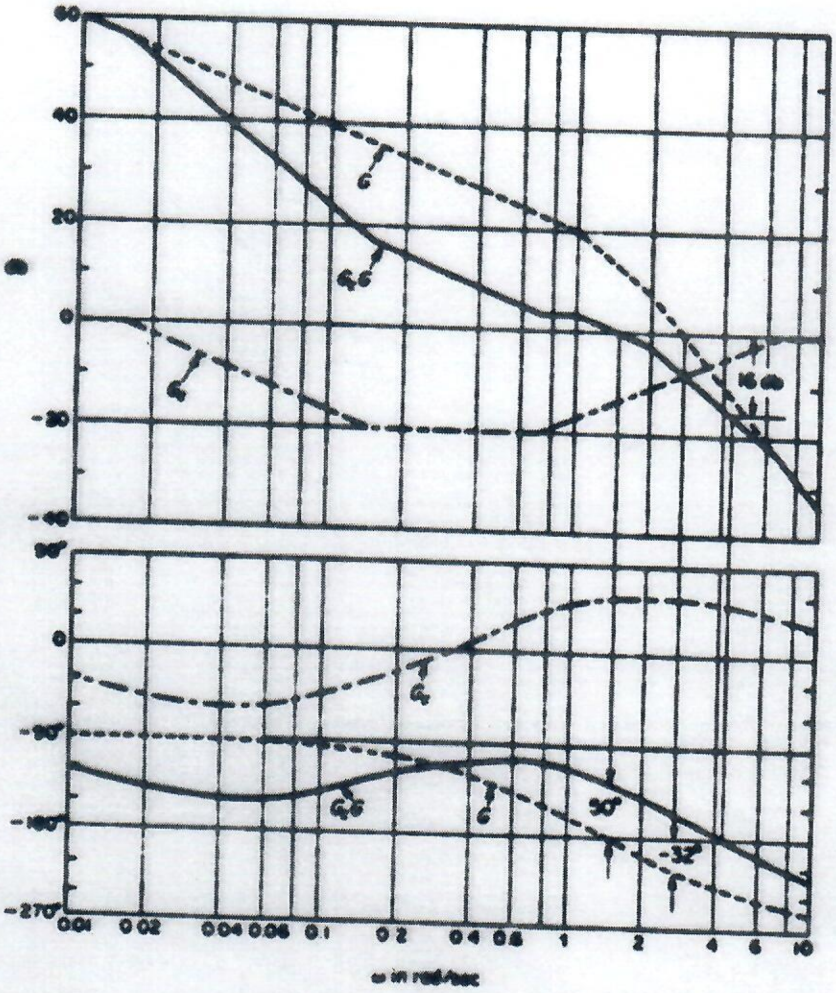


Figure 7–44 Bode diagrams for the uncompensated system, the compensator, and the compensated system. (G: uncompensated system, G_c: compensator, G_cG: compensated system.)

contributes −13 db at ω = 1.5 rad/sec, then the new gain crossover frequency is as desired. From this requirement, it is possible to draw a straight line of slope 20 db/decade, passing through the point (−13 db, 1.5 rad/sec). The intersections of this line and the 0-db line and −20-db line determine the corner frequencies. Thus, the corner frequencies for the lead portion are ω = 0.7 rad/sec and ω = 7 rad/sec. Thus, the transfer function of the lead portion of the lag–lead compensator becomes

$$\frac{s + 0.7}{s + 7} = \frac{1}{10}\left(\frac{1.43s + 1}{0.143s + 1}\right)$$

Combining the transfer functions of the lag and lead portions of the compensator, we obtain the transfer function of the lag–lead compensator. Since we chose $K_c = 1$, we have

$$G_c(s) = \left(\frac{s + 0.7}{s + 7}\right)\left(\frac{s + 0.15}{s + 0.015}\right) = \left(\frac{1.43s + 1}{0.143s + 1}\right)\left(\frac{6.67s + 1}{66.7s + 1}\right)$$

The magnitude and phase-angle curves of the lag–lead compensator just designed are shown in Figure

State Space Methods

$$\dot{\bar{X}} = A\bar{X} + B\textcircled{$\bar{u}$} \nearrow \text{input}$$

$$\bar{Y} = C\bar{X} \quad \text{output equation}$$

$\bar{X} \in R^{n}$

$A \in R^{n\times n}$ $\swarrow$ m inputs

$B \in R^{n\times m}$

$\bar{u} \in R^{m}$

$\bar{Y} \in R^{\textcircled{k}}$ – k outputs

$C \in R^{k\times n}$

We know solution to the unforced scalar; $\dot{x} = -ax \quad x(t) = e^{-at}x(0)$

Writing e^{at} as a series expansion

$e^{at} = 1 + at + (at)^2/2! + \cdots$ Based on this.

Define the state transition matrix

$$e^{At} = I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \cdots$$

$$\frac{de^{At}}{dt} = 0 + A + A^2t + \frac{A^3t^2}{2!} + \cdots$$

$$= Ae^{At}$$

The Matrix unforced response,

$$\bar{X}(t) = e^{At}\bar{X}(0)$$

The forced Response becomes $X(0) = 0$

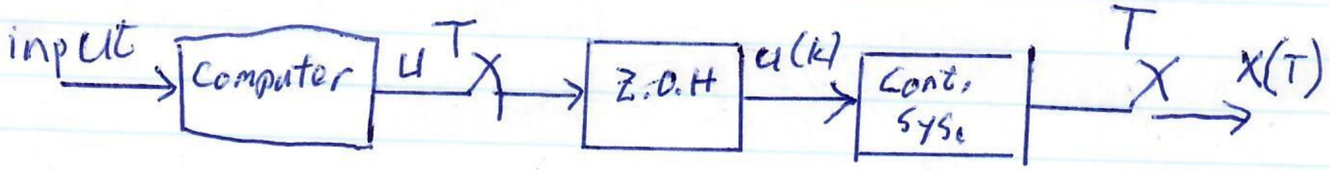
$$\bar{X}(t) = \int_0^t e^{A(t-\lambda)} u(\lambda) B d\lambda \rightarrow \text{convolution Integral.}$$

The complete Solution starting at time $t=0$

$$\bar{X}(t) = e^{At}\bar{X}(0) + \int_0^t e^{A(t-\lambda)} Bu(\lambda) d\lambda$$

If start at τ

$$\bar{X}(t) = e^{A(t-\tau)}\bar{X}(\tau) + \int_\tau^t e^{A(t-\lambda)} Bu(\lambda) d\lambda$$



We can solve for 1 time step.

$$\bar{x}(T) = e^{AT}\bar{x}(0) + \overbrace{\int_0^T e^{A(T-\lambda)}}^{= B_d} d\lambda \, B\bar{u}(0)$$

$$\bar{x}(2T) = e^{AT}\bar{x}(T) + \underbrace{\int_T^{2T} e^{A(2T-\lambda)}}_{= B_d} d\lambda \, B\bar{u}(T)$$

We can write this as a discrete time equation

$$\boxed{\bar{x}(k+1) = A_d\bar{x}(k) + B_d\bar{u}(k)} \quad \text{difference Equation}$$

$$A_d = e^{AT} \qquad B_d = \int_0^T e^{A\lambda} d\lambda B$$

Solving using Laplace Transforms.

$$\dot{\bar{x}}(t) = A\bar{x}(t) + B\bar{u}(t), \qquad y = C\bar{x}(t)$$

Let's take Laplace Transform of Both Sides

$$s\bar{X}(s) - \bar{X}(0) = A\bar{x}(s) + B\bar{u}(s)$$

$$s\bar{X}(s) - A\bar{x}(s) = \bar{X}(0) + B\bar{u}(s)$$

$$(sI - A)\bar{x}(s) = \bar{X}(0) + B\bar{u}(s)$$

The Laplace Trasform of

$$\bar{x}(s) = (sI-A)^{-1}\bar{x}(0) + (sI-A)^{-1}B\bar{u}(s)$$

$$\bar{y}(s) = C\bar{x}(s)$$

Let's look at the initial condition response

$$\mathcal{L}^{-1}\{\bar{x}(s) = \bar{x}(t) = \mathcal{L}^{-1}\{(sI-A)^{-1}\}\bar{x}(0)$$

We now have a closed form Solution to the state transition matrix (We know $\bar{x}(t) = e^{At}\bar{x}(0)$)

$$\underline{e^{At} = \mathcal{L}^{-1}\{(sI-A)^{-1}\}}$$

Let's look at the forced Response & the Transfer function Matrix, $\bar{x}(0)=0$

$$\bar{y}(s) = C\bar{x}(s) = \underbrace{C(sI-A)^{-1}B}_{H(s)}\bar{u}(s)$$

We define the transfer function matrix as

$$H(s) = \overset{k\times n}{C}\underbrace{(sI-A)^{-1}}_{n\times n}\overset{n\times m}{B} \leftarrow$$

This relate a T.F from each input to each output

$H(s) \in \mathbb{C}^{k\times m}$.

it is a matrix of Transfer function.

How do we compute a matrix inverse

$$M^{-1} = \frac{adj(M)}{\det(\underset{M}{\cancel{sI-A}})} = \frac{adj(M)}{\det(M)}$$

$$(sI-A)^{-1} = \frac{adj(sI-A)}{\det(sI-A)}$$

Then the T. F. Matrix it can be written

$$\boxed{H(s) = \frac{c\ adj(sI-A)B}{\det(sI-A)}}$$

The order of the adj(sI−A) is s^{n-1} or less.

$$\det(sI-A) = s^n + a_{n-1}s^{n-1} + \dots + a_0$$

All the T.F's in H(s) (k×m) all share the same characteristic Eqn. det(sI−A)

There whenever det(sI−A)=0 that is a pole of every T.F. in H(s)

$\det(\lambda_i I - A) = 0 \Rightarrow$ when this hold we call λ_i an eigenvalue.

If $\det(\lambda_i I - A) = 0$ then the rows & columns of $(\lambda_i I - A)$ does not span the state & there is a null space. The vector in the null space.

$$(\lambda_i I - A)v_i = 0$$

v_i ⟶ eigenvector.

If we write our system State Space Matrix form then

$\dot{\bar{X}} = A\bar{x} + B\bar{u}$ ⇒ Then the dynamics modes dynamic modes are the poles of the transfer, and they are equal to the eigenvalues of A!