9.5 Homework exercises

Determine whether the following series converge or diverge. You should state the name of the test(s) you use and completely justify your reasoning, giving arguments like those in the examples of this text.

1. $\sum_{n=1}^{\infty} \frac{3}{n^4}$

2. $\sum_{n=1}^{\infty} \frac{3}{5n-3}$

3. $\sum_{n=1}^{\infty} \frac{-3}{2n\sqrt{n}}$

4. $\sum_{n=2}^{\infty} e^{-1/n}$

5. $\sum_{n=2}^{\infty} \frac{2n^2+3}{5n^2-4}$

6. $\sum_{n=1}^{\infty} \left(\frac{4}{n^5} + \frac{2}{n}\right)$

7. $\sum_{n=3}^{\infty} \frac{\ln n}{n}$

8. $\sum_{n=1}^{\infty} \ln n$

9. $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+1}$

10. $\sum_{n=2}^{\infty} (n-1)^{-1/2}$

    Hint: Change the starting index to $n = 1$.

11. $\sum_{n=1}^{\infty} 3ne^{-n^2}$

12. $\sum_{n=1}^{\infty} \frac{2n^2+3n-2}{n^2+4n+1}$

13. $\sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right)$

14. $\sum_{n=4}^{\infty} \frac{1}{\ln(2^n)}$

15. $\sum_{n=1}^{\infty} \frac{2+\cos(3n)}{n}$

16. $\sum_{n=0}^{\infty} \frac{1}{e^n+e^{-n}}$

17. $\sum_{n=0}^{\infty} \frac{3+\cos(2n)}{4n^2}$

18. $\sum_{n=0}^{\infty} \frac{3+\cos(2n)}{4\sqrt[3]{n}}$

19. $\sum_{n=2}^{\infty} \frac{3+2^n}{3^n+4}$

20. $\sum_{n=1}^{\infty} \frac{4+\sin(n^2+2n)}{\sqrt[3]{n^5+1}}$

21. $\sum_{n=2}^{\infty} 5^{-n^2-3n}$

22. $\sum_{k=1}^{\infty} \frac{3}{4+\sin^4(2k)}$

23. $\sum_{n=1}^{\infty} \frac{6^n}{6^{2n}+3}$

10.7 Homework exercises

In problems 1-6, determine whether the following series converge or diverge. Completely justify your reasoning:

1. $\sum_{n=1}^{\infty} (-1)^n 8n^{-3/4}$

2. $\sum_{n=2}^{\infty} \frac{(-1)^n n^2}{3^n}$

3. $\sum_{n=4}^{\infty} (-1)^n \frac{5n^4+3}{20n^4+2n^2+n+1}$

4. $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{3n+8}}$

5. $\sum_{n=3}^{\infty} \frac{4\cos(\pi n)}{2^n+3}$

6. $\sum_{n=0}^{\infty} \frac{(-1)^n n^2}{\ln(n+4)}$

In problems 7-22, classify the following statements as true or false:

7. If a negative series converges, then it must converge absolutely.
8. If a series converges conditionally, then its terms can be legally rearranged without affecting the sum.
9. If a series $\sum |a_n|$ diverges, then $\sum a_n$ must also diverge.
10. If a series $\sum |a_n|$ diverges, then $\sum a_n$ cannot converge absolutely.
11. If a series $\sum a_n$ diverges, then $\sum |a_n|$ must also diverge.
12. If a series $\sum a_n$ converges, then $\sum |a_n|$ must also converge.
13. If a series $\sum |a_n|$ converges, then $\sum a_n$ must also converge.
14. It is possible for an alternating series to diverge.
15. It is possible for an alternating series to converge absolutely.
16. It is possible for an alternating series to converge conditionally.
17. It is possible for a positive series to diverge.
18. It is possible for a positive series to converge absolutely.
19. It is possible for a positive series to converge conditionally.
20. If $\lim_{n\to\infty} |a_n| = 1$, then $\sum a_n$ diverges.
21. If $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges.
22. If $\lim_{n\to\infty} |a_n| = 0$, then $\sum a_n$ converges.

In problems 23-34, determine whether the following series converge absolutely, converge conditionally, or diverge. You should state the name of the test(s) you use and completely justify your reasoning, giving arguments like those in the examples of this chapter.

23. $\sum_{n=2}^{\infty} (-1)^n \sin n$

24. $\sum_{n=4}^{\infty} \frac{n+1}{\ln(2n-5)}$

25. $\sum_{k=1}^{\infty} \frac{2\cos(\pi k)}{k^2}$

26. $\sum_{n=0}^{\infty} \frac{(-1)^n}{2\sqrt{n+2}}$

27. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$

28. $\sum_{n=0}^{\infty} \frac{(-1)^n 2n}{n^3+5}$

29. $\sum_{n=0}^{\infty} \frac{(-1)^n 4^{2n}}{7^n n!}$

30. $\sum_{n=1}^{\infty} 4(-1)^{n+1} n^{-3/5}$

31. $\sum_{n=0}^{\infty} \left[\frac{3}{n^5} + \frac{(-1)^n}{n}\right]$

32. $\sum_{n=1}^{\infty} \frac{(-1)^{[\frac{1}{2}n(n+1)]}}{4^n+n^2}$

33. (Challenge) $\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$

34. (Challenge) $1 + \frac{1}{1.1} + \frac{1}{1.11} + \frac{1}{1.111} + \frac{1}{1.1111} + \ldots$

11.4 Homework exercises

In problems 1-21, find the Taylor series representation of the given function.

1. $f(x) = \frac{3}{1-x}$
2. $f(x) = \frac{1}{1-x^2}$
3. $f(x) = \ln(1+x)$
4. $f(x) = \frac{1}{(1-x)^3}$

   Hint: Differentiate $\frac{1}{1-x}$ twice.
5. $f(x) = \frac{1}{1-x^3}$
6. $f(x) = \frac{2}{2+5x}$
7. $f(x) = \frac{-3}{-2-x}$
8. $f(x) = e^{-x}$
9. $f(x) = \cos 4x$
10. $f(x) = \ln(1-2x)$
11. $f(x) = \frac{1}{1+2x}$
12. $f(x) = \cos x^2$
13. $f(x) = \ln(1-x^2)$
14. $f(x) = x \sin 3x$
15. $f(x) = \frac{x}{e^{x^2}}$
16. $f(x) = x \sin x^2 - x^3$
17. $f(x) = e^x + e^{-x}$

    Hint: Find the Taylor series of e^x and e^{-x} independently, and then add them term-by-term.
18. $f(x) = (x^2+2)\cos x$

    Hint: Find the Taylor series of $x^2 \cos x$ and $2\cos x$ independently, and then add them term-by-term.
19. $f(x) = x \arctan x^2$
20. $f(x) = \frac{2}{x^2+1}$
21. $f(x) = \frac{1}{(1-x)^2}$

For problems 22-32, evaluate the following limits without using L'Hopital's Rule:

22. $\lim_{x\to 0} \frac{\sin x}{x}$
23. $\lim_{x\to 0} \frac{\cos x - 1}{x}$
24. $\lim_{x\to 0} \frac{\sin x^3 - x^3}{x^9}$
25. $\lim_{x\to 0} \frac{\arctan x - x}{x^3}$
26. $\lim_{x\to 0} \frac{e^x - 1 - x}{2x^2}$
27. $\lim_{x\to 0} \frac{2e^x - 2 - 2x - x^2}{x^3}$
28. $\lim_{x\to 0} \frac{\arctan 6x^2 - 6x^2}{x^6}$
29. $\lim_{x\to 0} \frac{\sin x^8 - x^8}{x^{20}}$
30. $\lim_{x\to 0} \frac{\sin x^8 - x^8}{x^{24}}$
31. $\lim_{x\to 0} \frac{\sin x^8 - x^8}{x^{30}}$
32. $\lim_{x\to 0} \frac{\arctan x^9 - \ln(x^9+1)}{x^{18}}$

In problems 33-37, approximate each of the following numbers using the second Taylor polynomial of an appropriately chosen function:

33. $\ln .8$

34. $\sin .3$

35. $e^{3/5}$

36. $\arctan \frac{1}{6}$

37. $\sqrt{2}$ *Hint:* Here, the appropriate function is $f(x) = \sqrt{x+1}$. You will have to figure out the second Taylor polynomial of $f(x)$ by computing derivatives of f at zero and using the definition of Taylor series.

In problems 38-42, approximate each of the following numbers using the fourth Taylor polynomial of an appropriately chosen function:

38. $\sin \frac{1}{4}$

39. $\cos .2$

40. $\sqrt{e}$

41. $\arctan \frac{1}{2}$

42. $\sqrt{2}$ *Hint:* As in problem 37, the appropriate function is $f(x) = \sqrt{x+1}$.

43. Approximate $\int_0^{1/2} \cos(4x^2)\, dx$ by replacing the integrand with its fourth Taylor polynomial.

44. Approximate $\int_0^{1/2} \arctan x^2\, dx$ by replacing the integrand with its fourth Taylor polynomial.

45. Approximate $\int_{-1}^{1} e^{-x^3}\, dx$ by replacing the integrand with its fourth Taylor polynomial.

46. Approximate $\int_0^1 \ln(2x^2+1)\, dx$ by replacing the integrand with its fourth Taylor polynomial.

47. Approximate $\int_0^1 x^2 \sin(x^2)\, dx$ by replacing the integrand with its sixth Taylor polynomial.

48. Approximate $\int_0^1 x^8 \sin x\, dx$ by replacing the integrand with its twelfth Taylor polynomial. Describe the integration technique that one would use to find the exact value of this integral. (Isn't using a Taylor polynomial better?)

In problems 49-55, find each higher-order derivative:

49. Find $f^{(6)}(0)$ if $f(x) = \sin x^2$.

50. Find $f^{(36)}(0)$ if $f(x) = \cos x^2$.

51. Find $f^{(100)}(0)$ if $f(x) = 4\ln(2x^2+1)$.