Chapter 3

Mortgage Loan Foundations: The Time Value of Money

Financing the purchase of real estate usually involves borrowing on a long- or short-term basis. Because large amounts are usually borrowed in relation to the prices paid for real estate, financing costs are usually significant in amount and weigh heavily in the decision to buy property. Individuals involved in real estate finance must understand how these costs are computed and how various provisions in loan agreements affect financing costs and mortgage payments. Familiarity with the mathematics of compound interest is essential in understanding mortgage payment calculations, how loan provisions affect financing costs, and how borrowing decisions affect investment returns. It is also important for investment analysis calculations that we examine later in this text. This chapter provides an introduction to the mathematics of finance, sometimes referred to as the "time value of money," or TVM. It forms a basis for concepts discussed in financing single-family properties and income-producing properties, and in funding construction and development projects.

Compound Interest

Understanding the process of compounding in finance requires the knowledge of only a few basic formulas. At the root of these formulas is the most elementary relationship, **compound interest**. For example, if an individual makes a bank deposit of $10,000 that is compounded at an annual interest rate of 6 percent, what will be the value of the deposit at the end of one year? In examining this problem, one should be aware that any compounding problem has four basic components:

1. An initial deposit, or present value of an investment of money.
2. An interest rate.
3. Time.
4. Value at some specified future period.

In our problem, the deposit is $10,000, interest is to be earned at an annual rate of 6 percent, time is one year, and value at the end of the year is what we would like to know. We have four components, three of which are known and one that is unknown.

Compound or Future Value

In the preceding problem, we would like to determine what value will exist at the end of one year if a single deposit or payment of $10,000 is made at the beginning of the year and the deposit balance earns a 6 percent rate of interest annually. To find the solution, we must introduce some terminology:

PV = **present value,** or principal at the beginning of the year

i = the interest rate

I = dollar amount of interest earned during the year

FV = principal at the end of n years, or **future value**

n = number of years

In this problem, then, $PV = \$10{,}000$, $i = 6$ percent, n = one year, and FV, or the value after one year, is what we would like to know.

The value after one year can be determined by examining the following relationship:

$$FV = PV + I_1$$

or the future value, FV, at the end of one year equals the deposit made at the beginning of the year, PV, plus the dollar amount of interest, I_1, earned in the first period. Because $PV = \$10{,}000$, we can find FV by determining I_1. Since we are compounding annually, FV is easily determined to be $10,600, which is shown in Exhibit 3–1.

Multiple Periods

To find the value at the end of two years, we continue the compounding process by taking the value at the end of one year, $10,600, making it the deposit at the beginning of the second year, and compounding again. This is shown in Exhibit 3–2.

EXHIBIT 3–1
Compound Interest Calculation for One Year

$$I_1 = PV \times i$$
$$= \$10{,}000(.06)$$
$$= \$600$$

Future value at the end of one year (n = 1 year) is determined as

$$FV = PV + I_1$$
$$= \$10{,}000 + \$600$$
$$= \$10{,}600$$

or

$$FV = PV(1 + i)$$
$$= \$10{,}000(1 + .06)$$
$$= \$10{,}600$$

EXHIBIT 3–2
Compound Interest Calculation for Two Years

$$\$10{,}600(.06) = I_2$$
$$\$636 = I_2$$

and value at the end of two years, or $n = 2$ years, is now

$$\$10{,}600 + I_2 = FV$$
$$\$10{,}600 + \$636 = \$11{,}236$$

Exhibit 3–2 shows that a total future value of $11,236 has been accumulated at the end of the second year. Note that in the second year, interest is earned not only on the original deposit of $10,000, but also on the interest ($600) that was earned during the first year. *The concept of earning interest on interest is an essential idea that must be understood in the compounding process and is the cornerstone of all financial tables and concepts in the mathematics of finance.*

From the computation in Exhibit 3–2, it should be pointed out that the value at the end of year 2 could have been determined directly from *PV* as follows:

$$\begin{aligned} FV &= PV(1 + i)(1 + i) \\ &= PV(1 + i)^2 \end{aligned}$$

In our problem, then, when $n = 2$ years

$$\begin{aligned} FV &= PV(1 + i)^2 \\ &= \$10{,}000(1 + .06)^2 \\ &= \$10{,}000(1.123600) \\ &= \$11{,}236 \end{aligned}$$

From this computation, the $11,236 value at the end of two years is identical to the result that we obtained in Exhibit 3–2. Being able to compute *FV directly* from *PV* is a very important relationship because it means that the future value, or value of any deposit or payment left to compound for any number of periods, can be determined from *PV* by simple multiplication. Therefore, if we want to determine the future value of a deposit made today that is left to compound for any number of years, we can find the solution with the general formula for compound interest, which is

$$FV = PV(1 + i)^n$$

By substituting the appropriate values for *PV*, *i*, and *n*, we can determine *FV* for any desired number of years.[1]

Other Compounding Intervals

In the preceding section, the discussion of compounding applies to cases where funds were compounded only once per year. Many savings accounts, bonds, mortgages, and other investments provide for monthly, quarterly, or semiannual compounding. Because we will be covering mortgage loans extensively in a later chapter, which involve monthly compounding almost exclusively, it is very important that we consider the other compounding intervals.

[1] At this point, the reader may realize that these problems can be solved with a financial calculator or computer software. We will provide illustrations using notation and keystroke sequences that are consistent with the use of a financial calculator to solve many of the problems in this and other chapters in this book. The reader may choose to change this approach by using computer software in lieu of a financial calculator.

When compounding periods other than annual are considered, a simple modification can be made to the general formula for compound interest. To change the general formula

$$FV = PV(1 + i)^n$$

where

n = years

i = annual interest rate

PV = deposit

for any compounding period, we divide the annual interest rate (i) by the desired number of compounding intervals *within* one year. We then increase the number of time periods (n) by multiplying by the desired number of compounding intervals *within* one year. For example, let m be the number of intervals *within* one year in which compounding is to occur, and let n be the number of years in the general formula. Then, we have

$$FV = PV\left[1 + \frac{i}{m}\right]^{n \cdot m}$$

Therefore, if interest is to be earned on the \$10,000 deposit at an annual rate of 6 percent, *compounded monthly,* to determine the future value at the end of one year, where $m = 12$, we have

$$\begin{aligned} FV &= \$10{,}000\left[1 + \frac{.06}{12}\right]^{1 \cdot 12} \\ &= \$10{,}000(1.061678) \\ &= \$10{,}616.78 \end{aligned}$$

If we compare the results of monthly compounding with those from compounding annually, we can immediately see the benefits of monthly compounding. If our initial deposit is compounded monthly, we would have \$10,616.78 at the end of the year, compared with \$10,600.00 when annual compounding is used.

Another way of looking at this result is to compute an **effective annual yield (*EAY*)** on both investments. This is done by assuming that \$10,000 is deposited at the beginning of the year and that all proceeds are withdrawn at the end of the year. For the deposit that is *compounded monthly,* we obtain

$$\begin{aligned} EAY &= \frac{FV - PV}{PV} \\ &= \frac{\$10{,}616.78 - \$10{,}000.00}{\$10{,}000} \\ &= 6.1678\% \end{aligned}$$

The result can be compared with the effective annual yield obtained when *annual compounding* is used, or

$$\begin{aligned} EAY &= \frac{\$10{,}600 - \$10{,}000}{\$10{,}000} \\ &= 6\% \end{aligned}$$

From this comparison, we can conclude that given the same nominal annual rate of interest, or 6 percent, the effective annual yield is higher when monthly compounding is used.

This comparison should immediately illustrate the difference between computing interest at a *nominal* annual rate of interest and computing interest at the same nominal annual rate of interest, *compounded monthly*. Both deposits are compounded at the same nominal annual rate of interest (6%); however, one is compounded 12 times at a monthly rate of (.06/12), or .005, on the ending monthly balance, while the other is compounded only once, at the end of the year at the rate of .06. It is customary in the United States to use a nominal rate of interest in contracts, savings accounts, mortgage notes, and other transactions. How payments will be made or interest accumulated (i.e., annually, monthly, daily) is then specified in the agreement. It is up to the parties involved in the transaction to ascertain the effective annual yield.

From the above analysis, one result should be very clear. Whenever the nominal annual interest rates offered on two investments are equal, the investment with the more frequent compounding interval within the year will always result in a higher effective annual yield. In our example, we could say that a 6 percent annual rate of interest compounded monthly provides an effective annual yield of 6.168 percent.

Other investments offer semiannual, quarterly, and daily compounding. In these cases, the basic formula for compound interest is modified as follows:

Nominal Annual Rate (%)	Compounding Interval		Modified Formula	Effective Annual Yield* (%)
6	Annually	$m = 1$	$FV = PV(1 + i)^{n \cdot 1}$	6.00
6	Semiannually	$m = 2$	$FV = PV\left[1 + \frac{i}{2}\right]^{n \cdot 2}$	6.09
6	Quarterly	$m = 4$	$FV = PV\left[1 + \frac{i}{4}\right]^{n \cdot 4}$	6.14
6	Monthly	$m = 12$	$FV = PV\left[1 + \frac{i}{12}\right]^{n \cdot 12}$	6.17
6	Daily	$m = 365$	$FV = PV\left[1 + \frac{i}{365}\right]^{n \cdot 365}$	6.18

*Also known as annual percentage yield (APY).

For example, if a deposit of $10,000 is made and an annual rate of 6 percent compounded daily is to be earned, we have:

$$FV = \$10{,}000\left[1 + \frac{.06}{365}\right]^{1 \cdot 365}$$
$$= \$10{,}000(1.061831)$$
$$= \$10{,}618.31$$

and the effective annual yield would be $\frac{\$10{,}618.31 - \$10{,}000}{\$10{,}000} = 6.1831$ percent. If the money was left on deposit for two years, the exponent would change to 2×365, and FV at the end of two years would be $11,274.86.

Many banks and savings institutions disclose what is referred to as the **APY (annual percentage yield)** on CDs, checking accounts, and so on. Conceptually, the effective annual yield shown here and the APY are generally the same. However, federal regulations require banks to include certain fees and penalties when applicable. Such fees could make the APY differ from the effective annual yield in our examples.

Throughout this book, we will follow the convention of using nominal rates of interest in all problems, examples, and exhibits. Hence, the term *interest rate* means a *nominal*, annual rate of interest. This means that when comparing two alternatives with *different* compounding intervals, the nominal interest rate should not be used as the basis for comparisons. In these cases, the concept of effective annual yield should be used when developing solutions.

Calculating Compound Interest Factors

Finding a solution to a compounding problem involving many periods is very awkward because of the amount of multiplication required. Calculators that are programmed with compound interest functions eliminate much of the detail of financial calculations. Another approach for finding solutions to compound interest problems can be used by calculating interest factors that can be used to solve many problems. *We will illustrate how these factors are calculated so the reader will understand the link between the mathematics of finance and calculators that have been programmed to provide solutions more efficiently. We also do this so that in the event that problems with many parts and multiple inputs must be solved, the reader may break the problem down and solve it in steps by using the necessary factors.*

To become familiar with the factors for various interest rates, recall that, in the problem discussed earlier, we wanted to determine the future value of a \$10,000 deposit compounded at an annual rate of 6 percent after one year. Looking at the 6 percent column in Exhibit 3–3 corresponding to the row for one year, we find the interest factor 1.060000. When multiplied by \$10,000, this interest factor gives us the solution to our problem.

$$FV = \$10,000(1.060000)$$
$$= \$10,600$$

The interest factor for the future value of \$1, at 6 percent for one year, is 1.060000—the same result had we computed $(1 + .06)^1$, or 1.06 from the general formula for compound interest. In other words,

$$(1 + .06)^1 = 1.06$$

Calculators can be used to determine interest factors in Exhibit 3–3 for many combinations of interest rates and time periods. These factors allow us to find a solution to any compounding problem as long as we know the deposit (*PV*), the interest rate (*i*), and the number of periods (*n*) over which annual compounding is to occur. For example, by using keystrokes on a calculator, we can calculate the factor for 6 percent interest and one year as follows:

EXHIBIT 3–3
Interest Factors for an Amount of \$1 at Compound Interest for Various Interest Rates and Compounding Periods

	Rate			
Year	6%	10%	15%	20%
1	1.060000	1.100000	1.150000	1.200000
2	1.123600	1.210000	1.322500	1.440000
3	1.191016	1.331000	1.520875	1.728000
4	1.262477	1.464100	1.749006	2.073600
5	1.338226	1.610510	2.011357	2.488320

$$PV = \$1$$
$$i = 6\%$$
$$n = 1$$
$$PMT = 0$$
$$\text{Solve for } FV = 1.06$$

Similarly, if we wanted the factor for 10 percent interest and four years, we would solve for FV as follows:

$$PV = \$1$$
$$i = 10\%$$
$$n = 4$$
$$PMT = 0$$
$$\text{Solve for } FV = 1.464100$$

Question: What is the future value of $5,000 deposited for four years compounded at an annual rate of 10 percent?

Solution: $FV = \$5{,}000(1.464100)$
$= \$7{,}320.50$

As was the case with the interest factors for annual compounding, interest factors for *monthly* compounding for selected interest rates and years have been computed from the modified formula $PV(1 + i/12)^{n \cdot 12}$ and are compiled in Exhibit 3–4. To familiarize the student with these interest factors, in Exhibit 3–4 selected interest rates and periods have been chosen and factors have been calculated.

EXHIBIT 3–4
Interest Factors for an Amount of $1 at Compound Interest for Various Interest Rates and Compounding Periods

	Rate		
Month	6%	8%	
1	1.005000	1.006670	
2	1.010025	1.013378	
3	1.015075	1.020134	
4	1.020151	1.026935	
5	1.025251	1.033781	
6	1.030378	1.040673	
7	1.035529	1.047610	
8	1.040707	1.054595	
9	1.045911	1.061625	
10	1.051140	1.068703	
11	1.056396	1.075827	
12	1.061678	1.083000	
Year			**Month**
1	1.061678	1.083000	12
2	1.127160	1.172888	24
3	1.196681	1.270237	36
4	1.270489	1.375666	48

In our earlier problem, we wanted to determine the future value of a $10,000 deposit that earned interest at an annual rate of 6 percent, compounded *monthly.* This can be easily determined by choosing the factor for 6 percent and 12 months, or one year, in Exhibit 3–4. That factor is 1.061678. Hence, to determine the value of the deposit at the end of 12 months, or one year, we have

$$FV = \$10{,}000(1.061678)$$
$$= \$10{,}616.78$$

In other words, the interest factor for a 6 percent rate of interest compounded *monthly* for one year is 1.061678, which is the same result that we would obtain if we expanded $(1 + .06/12)^{1 \cdot 12}$ by multiplying, or

$$\left[1 + \frac{.06}{12}\right]^{1 \cdot 12} = 1.061678$$

Question: What is the future value of a single $5,000 deposit earning 8 percent interest, *compounded monthly,* at the end of two years?

Solution: $$FV = \$5{,}000\left[1 + \frac{.08}{12}\right]^{2 \cdot 12}$$
$$= \$5{,}000(1.172888)$$
$$= \$5{,}864.44$$

Using Financial Functions: Calculators and Spreadsheets

Finding a solution to a compounding problem involving many periods may be greatly simplified with the use of a calculator. Calculators programmed with compound interest functions eliminate the need for financial tables for many problem situations. Spreadsheets such as Excel are programmed with functions that allow users to input financial variables and solve. We will present alternative solutions to most of the time value of money problems in the remainder of this text using a general function that corresponds to the format of a financial calculator. Refer to your specific calculator manual to confirm whether all operations are similar. Unless specified otherwise, the solutions assume that payments are made at the *end* of each period, and that money spent is (−) and money received is (+). When solving problems, we will use the following format:

n = number of years, unless stated otherwise

i = interest rate per year, unless stated otherwise

PV = present value

PMT = payments

FV = future value

Solutions will follow the format above, with the unknown variable being solved listed last. Most calculations are carried out to at least six decimal places, then final monetary amounts are rounded back to two decimal places.

For example, in the problem discussed earlier, we wanted to determine the future value of a $10,000 deposit compounded at an annual rate of 6 percent after one year.

Financial Functions

Concept Box 3.1

With the advent of technology, calculations involving the TVM and related concepts in mortgage lending, valuation, and investments have been greatly simplified. Many TVM and related concepts involve equations with exponents, which can make derivations and solving for solutions cumbersome. Professionals in the field of finance have come to use either, or both, *financial calculators* and *computer software* (Excel) that are programmed and allow computations to be made much quicker and more efficiently.

What follows is a glossary of functions that are used frequently in the chapters that follow. The reader should become familiar with these functions and the notation.

PMT (n, i, PV, FV)—Function to calculate level payments (annuity) that occur over a specified time period that have a specified present value and may include a final payment in the future.

PV (n, i, PMT, FV)—Function to calculate the present value of level payments (annuity) and/or an amount to be received at a single point in time in the future.

FV (n, i, PV, PMT)—Function to calculate the future value of an annuity and/or level payments (annuity) that occur over a specified time period.

i (n, PV, PMT, FV)—Function to calculate the interest rate (also an IRR) that would be achieved on an investment made in the present that will have level payments over time and/or a future value at the end of the specified investment period (*n*).

n (i, PV, PMT, FV)—Function to calculate the number of periods (*n*) required to repay a loan (*PV*) at a specified rate of interest (*i*) for various future values (zero, non zero).

Notes:

1. Values for variables in these functions must be adjusted for frequency of receipts and disbursements as well as compounding and discounting intervals within a year.
2. In the most basic applications, these variables are assumed to be constant with respect to time. In those problems requiring that some variables (e.g., *PMT*) be allowed to vary from period to period, we will provide the proper approach to modify functions and solve for solutions as needed.
3. It will become apparent to the reader after solving TVM and related problems that a recurring pattern occurs. This pattern usually involves one equation containing *five* variables: *i, n, PV, FV,* and *PMT.* Most problems provide values (inputs) for *four* of these variables, leaving one unknown for which a solution is being sought. (See glossary of functions above where the variable listed to the left of the parenthesis is the unknown.) This notation should enable the reader to identify the unknown variable more readily.

Solution:

$n = 1$ year

$i = 6\%$

$PMT = 0$

$PV = -\$10{,}000$

Solve for *FV* $= \$10{,}600$

Function:
FV (n, i, PV, PMT)

(Note that this solution is the same as what would have been obtained had we calculated the factor 1.060000 shown in Exhibit 3–3. However, when a calculator is used, the intermediate step in which interest factors are calculated is eliminated.)

The same problem compounded monthly would follow the format listed below. Note: Several calculators include a function for a number of periods. In such cases, the number

Web App

What rate can you earn on your savings today? Go to a website like **www.bankrate.com** that has interest rates on certificates of deposit (CDs) and see what the current rate is for a five-year CD. Assuming you have $10,000 today, how much will that accumulate to after five years at the rate you found on the website?

of periods and interest are typically stated on an annual basis, and the number of periods is entered separately.

Solution:

n = 12 (1 year × 12 periods per year)
i = .5% (6%/12 periods per year)
PMT = 0
PV = −$10,000
Solve for ***FV*** = $10,616.78

Function:
FV (i, n, PV, PMT)

(Note that this solution is the same as what would have been obtained had we calculated the factor 1.061678 shown in Exhibit 3–4.)

The same problem *compounded daily* would look like this.

Solution:

n = 365 (1 year × 365 periods per year)
i = .0164(6%/365 periods per year)
PMT = 0
PV = −$10,000
Solve for ***FV*** = $10,618.31

Function:
FV (i, n, PV, PMT)

Question: What is the future value of $5,000 deposited for four years *compounded annually* at a rate of 10 percent?

Solution:

n = 4 years
i = 10%
PMT = 0
PV = −$5,000
Solve for ***FV*** = $7,320.50

Function:
FV (PV, i, n, PMT)

Question: What is the future value of a single $5,000 deposit earning 8 percent interest, *compounded monthly,* at the end of two years?

Solution:

n = 24 (2 years × 12 periods)
i = .666%(8%/12 periods)
PMT = 0
PV = −$5,000
Solve for ***FV*** = $5,864.44

Function:
FV (PV, i, n, PMT)

Present Value

In the preceding section, we were concerned with determining value at some time in the *future;* that is, we considered the case where a deposit had been made and compounded into the future to yield some unknown future value.

In this section, we are interested in the problem of knowing the future cash receipts for an investment and of determining how much should be paid for the investment at *present.* The concept of present value is based on the idea that money has time value. Time value simply means that if an investor is offered the choice between receiving $1 today or receiving $1 in the future, the proper choice will always be to receive the $1 today because this $1 can be invested in some opportunity that will earn interest, which is always preferable to receiving only $1 in the future. In this sense, money is said to have *time value.*

When determining how much should be paid *today* for an investment that is expected to produce income in the *future,* we must apply an adjustment called **discounting** to income received in the future to reflect the time value of money. The concept of present value lays the cornerstone for calculating mortgage payments, determining the true cost of mortgage loans, and finding the value of an income property, all of which are very important concepts in real estate finance.

A Graphic Illustration of Present Value

An example of how discounting becomes an important concept in financing can be seen from the following problem. Suppose an individual is considering an investment that promises a cash return of $10,600 at the end of one year. The investor believes this investment should yield an annual rate of 6 percent. The question is how much should the investor pay *today* if $10,600 is to be received at the *end* of the year and the investor requires a 6 percent return compounded annually on the amount invested?

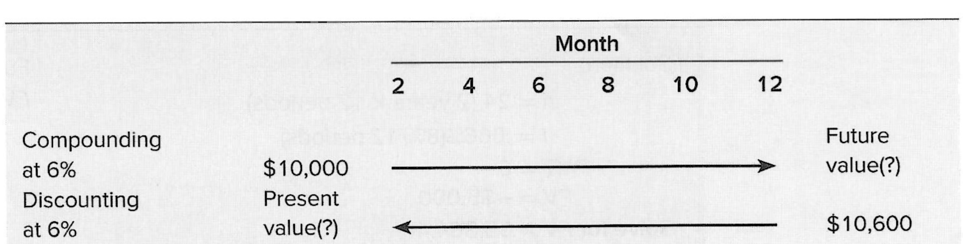
The problem can be seen more clearly by comparing it with the problem of finding the compound value of $1 discussed in the first part of this chapter. In that discussion, we were concerned with finding the future value of a $10,000 deposit compounded monthly at 6 percent for one year. This comparison is depicted in Exhibit 3–5.

In Exhibit 3–5 note that with compounding, we are concerned with determining the *future value* of an investment. With discounting, we are concerned with just the opposite concept; that is, what *present value* or *price* should be paid *today* for a particular investment, assuming a desired rate of interest is to be earned?

Because we know from the preceding section that $10,000 compounded annually at a rate of 6 percent results in a future value of $10,600 at the end of one year, $10,000 is the present value of this investment. However, had we not done the compounding problem in the preceding section, how would we know that $10,000 equals the present value of the investment? Let us again examine the compounding problem considered in the previous section. To determine future value, recall the general equation for compound interest:

$$FV = PV(1 + i)^n$$

EXHIBIT 3–5
Comparison of Future Value and Present Value



In our present value problem, *PV* becomes the *unknown* because *FV*, or the future value to be received at the end of one year, $n = 1$ year, is *known* to be \$10,600. Because the interest rate (i) is also known to be 6 percent, *PV* is the only value that is not known. *PV*, the present value or amount we should pay for the investment today, can be easily determined by rearranging terms in the above compounding formula as follows:

$$FV = PV(1 + i)^n$$

$$PV = FV\frac{1}{(1 + i)^n}$$

In our problem, then, we can determine *PV* directly by substituting the known values into the above expression as follows:

$$\begin{aligned} PV &= FV\frac{1}{(1 + i)^n} \\ &= \$10{,}600\frac{1}{(1 + .06)^1} \\ &= \$10{,}600\frac{1}{1.06} \\ &= \$10{,}600 \times (.943396) \\ &= \$10{,}000 \end{aligned}$$

Note that the procedure used in solving for the present value is simply to multiply the future value, *FV*, by 1 divided by $(1 + i)^n$. We know from the section on compounding that in our problem $(1 + i)^n$ is $(1 + .06)^1$, which equals 1.06. Dividing 1 by 1.06 yields .943396. This result is important in present value analysis because it shows the relationship between future value and present value.

Because we see from Exhibit 3–5 that the discounting process is the opposite of compounding, to find the present value of any investment is simply to compound in a "reverse sense." This is done in our problem by taking the reciprocal of the interest factor for the compound value of \$1 at 6 percent, $1 \div 1.06$, or .943396, and multiplying it by the future value of the investment to find its present value. We can now say that \$10,600 received at the end of one year, when discounted by 6 percent, has a present value of \$10,000. Alternatively, if we are offered an investment that promises to yield \$10,600 after one year and we want to earn a 6 percent annual return, we should not pay more than \$10,000 for the investment (it is on the \$10,000 present value that we earn the 6 percent interest).

Calculating Present Value Interest Factors

Because the discounting process is the reverse of compounding, and the interest factor for discounting $1 \div (1 + i)^n$ is simply the reciprocal of the interest factor for compounding, a series of present value interest factors have been developed. Exhibit 3–6 contains a sample of factors to be used when discounting.

In our problem, we want to know how much should be paid for an investment with a future value of \$10,600 to be received at the end of one year if the investor demands an annual return of 6 percent. The solution can be found by calculating or selecting .943396 from the 6 percent column in Exhibit 3–6. The \$10,600 future value can now be multiplied by .943396, resulting in a present value (*PV*) of \$10,000. To help the reader understand

EXHIBIT 3–6
Interest Factors for the Present Value Reversion of $1 for Various Interest Rates and Time Periods

	Rate		
Year	6%	10%	15%
1	.943396	.909091	.869565
2	.889996	.826446	.756144
3	.839619	.751315	.657516
4	.792094	.683013	.571753
5	.747258	.620921	.497177
6	.704961	.564474	.432328
7	.665057	.513158	.375937

these concepts, this present value factor also may be determined using a financial calculator as follows:

Solution:

n = 1 year
i = 6%
PMT = $0
FV = $1
Solve for PV = .943396

Function:
PV (n, i, PMT, FV)

Question: How much should an investor pay today for a real estate investment that will return $20,000 at the end of three years, assuming the investor desires an annual return of 15 percent interest on the amount invested?

Solution:

$$PV = \$20{,}000 \times \frac{1}{(1+.15)^3}$$
$$= \$20{,}000(.657516)$$
$$= \$13{,}150.32$$

The investor should pay no more than $13,150.32 today for the investment promising a return of $20,000 after three years if a 15 percent return on investment is desired.[2]

Expanding the Use of Calculators for Finding Present Values

As was the case with compounding, financial calculators allow us to calculate present value solutions *directly,* as they have been programmed to calculate factors internally and then complete the required operations to present a final answer. Our problem can be solved with a calculator as follows:

[2] An accepted convention in finance is that when one refers to a percentage return on investment, a nominal annual interest rate is assumed. If solutions are computed based on different compounding intervals within a year, such as monthly, the solution should be designated as an *annual rate of interest compounded monthly.* The latter solution may then be converted, if desired, to an effective annual yield, as shown previously.

Solution:	Function:
n = 3 years	PV (FV, PMT, n, i)
i = 15%	
PMT = 0	
FV = \$20,000	
Solve for PV = −\$13,150.32	

Because we can use the discounting process to find the present value of a future value when *annual* compounding is assumed, we can also apply the same methodology assuming *monthly* discounting. For example, in our illustration involving monthly compounding, the future value of \$10,000 at an annual rate of interest of 6 percent compounded monthly was \$10,616.80. An important question an investor should consider is how much should be paid today for the future value of \$10,616.80 received at the end of one year, assuming that a 6 percent return compounded *monthly* is required?

We could answer this question by finding the reciprocal of the formula used to compound monthly, $1 \div (1 + i/12)^{1 \cdot 12}$, and multiply that result by the future value of \$10,616.80 to find the present value (PV). We may calculate this factor with a calculator as

Solution:	Function:
FV = \$1	PV (FV, PMT, n, i)
PMT = 0	
n = 12 months	
i = 6% ÷ 12	
Solve for PV = .941905	

Many factors have been calculated and included in the table shown in Exhibit 3–7.

In our problem, we want to determine the present value of \$10,616.80 received at the end of one year, assuming a desired rate of return of 6 percent, compounded monthly. By going to the 6 percent column and the row corresponding to one year (12 months) and selecting the interest factor .941905, we can now multiply \$10,616.80 × (.941905) = \$10,000 and see that \$10,000 is the maximum amount one should pay today for the investment.

Alternatively, if the reader is comfortable with the derivation of these interest factors and the discounting process, a solution may be found more directly with a calculator as follows:

Solution:	Function:
n = 1 year × 12 periods = 12	PV (n, i, PMT, FV)
i = 6% ÷ 12 periods = .5%	
PMT = 0	
FV = 10,616.80	
Solve for PV = −\$10,000	

EXHIBIT 3–7
Interest Factors for the Present Value Reversion of $1 for Various Interest Rates and Time Periods

	Rate			
Month	6%	8%	9%	
1	.995025	.993377	.992556	
2	.990075	.986799	.985167	
3	.985149	.980264	.977833	
4	.980248	.973772	.970554	
5	.975371	.967323	.963329	
6	.970518	.960917	.956158	
7	.965690	.954553	.949040	
8	.960885	.948232	.941975	
9	.956105	.941952	.934963	
10	.951348	.935714	.928003	
11	.946615	.929517	.921095	
12	.941905	.923361	.914238	
Year				Month
1	.941905	.923361	.914238	12
2	.887186	.852596	.835831	24
3	.835645	.787255	.764149	36
4	.787098	.726921	.698614	48

Question: How much should an investor pay to receive $12,000 three years (36 months) from now, assuming that the investor desires an annual return of 9 percent compounded *monthly?*

Solution: $PV = \$12{,}000(.764149)$
$= \$9{,}169.79$

Solution:	Function:
$n = 3 \times 12 = 36$	$PV\ (n, i, PMT, FV)$
$i = 9\% \div 12 = .75\%$	
$PMT = 0$	
$FV = 12{,}000$	
Solve for $PV = -\$9{,}169.79$	

The investor should pay no more than $9,169.79 for the investment, or the present value (*PV*) of the investment is $9,169.79.

Compound or Future Value of an Annuity

The first section of this chapter dealt with finding the compound or future value of a *single deposit* or payment made only once, at the beginning of a period. An equally relevant consideration involves a series of equal deposits or payments made at equal intervals. For example, assume deposits of $1,000 are made at the *end* of each year for a period of five years and interest is compounded at an annual rate of 5 percent. What is the future value at the end of the period for a series of deposits plus all compound interest? In this case, the problem involves equal payments (P) or deposits made at equal time intervals. This series of deposits or payments is defined as an **annuity**. Because we know how to find the answer to a problem where only one deposit is made, it is logical and correct to assume that the same basic compounding process applies when dealing with annuities. However, that process is only a partial solution to the problem because we are dealing with a series of deposits that occur annually.

To compute the sum of all deposits made in each succeeding year and include compound interest on deposits only when it is earned, the general formula for compounded interest must be expanded as follows:

$$FV = P(1+i)^{n-1} + P(1+i)^{n-2} + \cdots + P$$

This may also be written as

$$FV = P \cdot \sum_{t=1}^{n-1} (1+i)^t + P$$

which simply means that we may take the constant payment or annuity P and multiply it by the "sum of" the series $1 + i$ expanded from time $t = 1$ to the period $n - 1$, plus P.[3] Hence, the symbol Σ represents the "sum of" that series and is simply a shortcut notation to be used in place of writing $1 + i$ repetitively.

In this expression, FV is now **future value of an annuity**, or the sum of all deposits, P, compounded at an annual rate, i, for n years. The important thing to note in the expression, however, is that each deposit is assumed to be at the *end* of each year and is compounded through year n. The final deposit does not earn interest because it occurs at the end of the final year. Since we are dealing in our example with a series of \$1,000 deposits made over a five-year period, the first \$1,000 deposit is compounded for four periods ($n - 1$), the \$1,000 deposit made at the beginning of the second year is compounded for three periods ($n - 2$), and so on, until the last deposit, P, is reached. The last deposit is not compounded because it is deposited at the end of the fifth year.[4]

To compute the value of these deposits, we could construct a solution like that shown in Exhibit 3–8. Note that each \$1,000 deposit is compounded from the end of the year in which the deposit was made to the end of the next year. In other words, as shown in our expanded formula above, the deposit at the end of year 1 is compounded for four years, the deposit made at the beginning of the second year is compounded for three years, and so on. By carrying this process out one year at a time, we determined the solution, or \$5,525.63, when the compounded amounts in the extreme right-hand column are added.

Although the future value of \$1,000 per period can be determined in the manner shown in Exhibit 3–8, careful examination of the compounding process reveals another, easier way to find the solution. Note that the \$1,000 deposit occurs annually and never changes; it is constant. When the deposits are constant, it is possible to sum all of the individual interest factors (*IF*s) as 5.525631. By multiplying \$1,000 by 5.525631, a solution of \$5,525.63 is obtained, as shown at the bottom of the right-hand column in Exhibit 3–8.

[3] The formula shown here is the formula for an **ordinary annuity,** which assumes that all deposits are made at the *end* of each year. The final P in the expression means that the last payment is not compounded.

[4] The reader should be aware that this formulation is used for *ordinary annuities* or when payments or receipts occur at the end of a period. This is different from the formula for an **annuity due,** which assumes that deposits are made at the *beginning* of a period.

EXHIBIT 3–8
Interest Factors for the Future Value of an Annuity of $1,000 per Year Compounded at 5 Percent Annually

Year	Deposit		Interest Factor	Future Value
1	$1,000	×	1.215506	= $1,215.51*
2	1,000	×	1.157625	= 1,157.63*
3	1,000	×	1.102500	= 1,102.50
4	1,000	×	1.050000	= 1,050.00
5	1,000	×	1.000000	= 1,000.00
	Also 1,000	×	5.525631	= $5,525.63*

*Rounded.

Use of Compound Interest Factors for Annuities

Because the *IF*s in Exhibit 3–8 can be *added* when annuities are being considered, a series of new interest factors have been calculated for various interest rates. A sample of these factors has been compiled in Exhibit 3–9.

In the problem at hand, to determine the future value of $1,000 deposited annually at 5 percent for five years, note that if we go to the 5 percent column in Exhibit 3–9 and obtain the *IF* that corresponds to five years, we can find the solution to our problem as follows:

$$FV = \$1{,}000(5.525631)$$
$$= \$5{,}525.63$$

This amount corresponds to the solution obtained from the long series of multiplications carried out in Exhibit 3–8.

As we have explained, for those readers who understand the compounding process and the equations discussed in the sections above, financial calculators programmed to perform and store the necessary steps for compounding annuities may be used to obtain an answer more directly as follows:

Solution:	Function:
$n = 5$	$FV\ (n, i, PV, PMT)$
$i = 5\%$	
$PV = 0$	
$PMT = -\$1{,}000$	
Solve for $FV = \$5{,}525.63$	

EXHIBIT 3–9
Interest Factors for the Accumulation of $1 per Period for Various Interest Rates and Time Periods

	Rate		
Year	5%	6%	10%
1	1.000000	1.000000	1.000000
2	2.050000	2.060000	2.100000
3	3.152500	3.183600	3.310000
4	4.310125	4.374616	4.641000
5	5.525631	5.637093	6.105100
6	6.801913	6.975319	7.715610
7	8.142008	8.393838	9.487171
8	9.549109	9.897468	11.435888

Question: What is the future value of $800 deposited each year for six years, compounded annually at 10 percent interest after six years?

Solution: $FV = \$800(7.715610)$
$= \$6,172.49$

Solution:

$n = 6$
$i = 10\%$
$PV = 0$
$PMT = -\$800$
Solve for $FV = \$6,172.49$

Function:
$FV\,(n, i, PV, PMT)$

The same procedure used for compounding annuities for amounts deposited or paid annually can also be applied to monthly annuities. A very simple modification can be made to the formulation used for annual annuities by substituting *i*/12 in place of *i* and adding the number of compounding periods per year (*m*) in the annual formulation, as follows:

$$FV = P\left[1+\frac{i}{12}\right]^{n\cdot m-1} + P\left[1+\frac{i}{12}\right]^{n\cdot m-2} + \cdots + P$$

or

$$FV = P\cdot\sum_{t=1}^{n\cdot m-1}\left[1+\frac{i}{12}\right]^{t} + P$$

However, in this formulation, $n\cdot m$ represents months. Deposits or payments, *P*, are made monthly and are constant in amount. Hence, the interest factors used to compound each monthly deposit may be added (as they were for annual deposits in Exhibit 3–8), and a new series for compounding monthly annuities can be computed. This has been done for selected interest rates and years.[5]

Question: An investor pays $200 per month into a real estate investment that promises to pay an annual rate of interest of 8 percent compounded *monthly.* If the investor makes consecutive monthly payments for five years, what is the future value at the end of five years?

Solution: $FV = \$200(73.476856)$
$= \$14,695.37$

[5] Like annual compounding, this formulation assumes that deposits are made at the *end* of each month, or that an *ordinary annuity* is being compounded.

Solution: Function: $FV(n, i, PV, PMT)$

$n = 5 \times 12 = 60$ months

$i = 8\%/12 = .666\%$

$PV = 0$

$PMT = -\$200$

Solve for $FV = \$14{,}695.37$

In this case, the value of payments earning interest at an annual rate of 8 percent compounded monthly = $14,695.37.

Present Value of an Annuity

In the preceding section, our primary concern was to determine the future value of an annuity, or constant payments received at equal time intervals. In this section, we consider the **present value of an annuity**, or the series of annual income receipts the investment produces over time. Because an investor may have to consider a series of income payments when deciding whether to invest, this is an important problem. Recall that when dealing with the present value of a single receipt, or ending value, *PV,* we took the basic formula for compounding interest and rearranged it to determine the present value of an investment as follows:

$$FV = PV(1+i)^n$$
$$PV = FV \div (1+i)^n$$
$$PV = FV \cdot \frac{1}{(1+i)^n}$$

To consider the present value of an annuity, we need only consider the sum of individual present value for all payments/receipts. This can be done by modifying the basic present value formula as follows:

$$PV = PMT\frac{1}{(1+i)^1} + PMT\frac{1}{(1+i)^2} + PMT\frac{1}{(1+i)^3} + \cdots + PMT\frac{1}{(1+i)^n}$$

or this can be written as

$$PV = PMT \cdot \sum_{t=1}^{n} \frac{1}{(1+i)^t}$$

Note in this expression that each payment (*PMT*) is discounted for the number of years corresponding to the time when the funds were actually received. In other words, the first payment would occur at the end of the first period and would be discounted only one period, or $PMT \cdot [1 \div (1+i)^1]$. The second receipt would be discounted for two periods, or $PMT \cdot [1 \div (1+i)^2]$, and so on.

Assuming an individual is considering an investment that will provide a series of annual cash receipts of $500 for a period of six years, and the investor desires a 6 percent return, how much should the investor pay for the investment today? We can begin by considering the present value of the $500 receipt in year 1, as shown in Exhibit 3–10. Note that the present value of the $500 receipt is discounted for one year at 6 percent. This is done

EXHIBIT 3–10
Present Value of $500 per Year (discounted at 6 percent annually)

Year	PMT	IF	Present Value
1	$500 ×	.943396	= $ 471.70
2	500 ×	.889996	= 445.00
3	500 ×	.839619	= 419.81
4	500 ×	.792094	= 396.05
5	500 ×	.747258	= 373.63
6	500 ×	.704961	= 352.48
	Also $500 ×	4.917324	= $2,458.66†

†Rounded.

because the income of $500 for the first year is not received until the end of the first period, and our investor only wants to pay an amount today (present value) that will assure a 6 percent return on the amount paid today. Therefore, by discounting this $500 payment by the interest factor for 6 percent, or .943396, the present value is $471.70. Note that the second $500 income payment is received at the end of the second year. Therefore, it should be discounted for *two* years at 6 percent. Its present value is found by multiplying $500 by the interest factor for 6 percent for two years, or .889996, giving a present value of $445. This process can be continued for each receipt for the remaining three years (see Exhibit 3–10). The present value of the entire series of $500 income payments can be found by adding the series of receipts discounted each month in the far right-hand column, which totals $2,458.66.

However, because the $500 series of payments is constant, we may simply sum all interest factors to obtain one interest factor that can be multiplied by $500 to obtain the same present value (see Exhibit 3–10). The sum of all interest factors for 6 percent is 4.917324. When 4.917324 is multiplied by $500, the present value, $2,458.66, found in the lengthy series of multiplications carried out in Exhibit 3–10, is again determined.

Now that the reader has been introduced to the equations and interest factors that result from these equations, a solution can be found directly using a financial calculator as follows:

Solution:	Function:
$n = 6$	$PV(n, i, FV, PMT)$
$i = 6\%$	
$PMT = -\$500$	
$FV = 0$	
Solve for $PV = \$2{,}458.66$	

Use of the Present Value of an Annuity Factors

As we have illustrated, the interest factors in Exhibit 3–10 may be summed, as long as the income payments are equal in amount and received at equal intervals. The sums of *IF*s for various interest rates have been compiled in table form and are listed in Exhibit 3–11. In our problem, we want to determine the present value of $500 received annually for six years, assuming a desired annual rate of return of 6 percent. How much should an investor pay for this total investment today and be assured of earning the desired return? We can solve this problem by computing the solution with a calculator or by looking at Exhibit 3–11, finding the 6 percent column, and looking down the column until we locate the *IF* in the row corresponding to six years, which is 4.917324. Thus,

EXHIBIT 3–11
Interest Factors for the Present Value of an Ordinary Annuity of $1 per Period for Various Interest Rates and Time Periods

	Rate			
Year	5%	6%	10%	15%
1	.952381	.943396	.909091	.869565
2	1.859410	1.833393	1.735537	1.625709
3	2.723248	2.673012	2.486852	2.283225
4	3.545951	3.465106	3.169865	2.854978
5	4.329477	4.212364	3.790787	3.352155
6	5.075692	4.917324	4.355261	3.784483
7	5.786373	5.582381	4.868419	4.160420
8	6.463213	6.209794	5.334926	4.487322

$$PV = \$500(4.917324)$$
$$= \$2{,}458.66$$

This solution corresponds to that obtained in Exhibit 3–10.

Question: An investor has an opportunity to invest in a rental property that will provide net cash returns of $400 per year for three years. The investor believes that an annual return of 10 percent should be earned on this investment. How much should the investor pay for the rental property?

Solution: $PV = \$400(2.486852)$
$= \$994.74$

The investor should pay no more than $994.74 for the investment property. With that amount, a 10 percent return will be earned.

A more direct solution to this problem can be found by using a financial calculator as follows:

Solution:

$n = 3$
$i = 10\%$
$PMT = -\$400$
$FV = 0$
Solve for PV = $994.74

Function:
$PV(n, i, PMT, FV)$

Based on the logic used in discounting annuities paid or received annually, the same procedure can be applied to cash receipts paid or received *monthly.* In this case, the formula used to discount annual annuities is simply modified to reflect monthly receipts or payments, and the discounting interval is changed to reflect monthly compounding:

$$PV = P\left[\frac{1}{1+\frac{i}{12}}\right]^1 + P\left[\frac{1}{1+\frac{i}{12}}\right]^2 + \cdots + P\left[\frac{1}{1+\frac{i}{12}}\right]^{12 \cdot n}$$

where payments (P) occur monthly, the exponents represent months running from 1 through $n \cdot m$, and PV now represents the present value of an annuity received over $n \cdot m$ months.

Like annual discounting, computation of the present value of an annuity can be very cumbersome if one has to expand the above formula for each problem, particularly if the problem involves cash receipts or payments over many months. Hence, a series of interest factors have been computed by expanding the above formula for each monthly interval and adding the resulting interest factors (this was performed with discounting annual annuities in Exhibit 3–10). Like the annual tables, the factors in Exhibit 3–12 are labeled *Present Value of an Ordinary Annuity of $1 per Period* because the period in this case is one month. Hence, if an investor wants to know how much he or she should pay today for an investment that would pay $500 at the end of each month for the next 12 months and earn an annual rate of return of 6 percent compounded monthly on the investment, the investor can easily compute the solution with a calculator or determine it by consulting Exhibit 3–12. Looking to the 6 percent column and dropping down to the row corresponding to 12 months, you find the factor 11.618932. Multiplying $500 by 11.618932 results in $5,809.47, or the amount that the investor should pay today if a 6 percent rate of return compounded monthly is desired.

The calculator solution would be

Solution:	Function:
$n = 12$ months	$PV(n, i, PMT, FV)$
$i = 6\% \div 12 = .005$	
$PMT = \$500$	
$FV = 0$	
Solve for $PV = \$5,809.47$	

EXHIBIT 3–12
Interest Factors for the Present Value of an Ordinary Annuity of $1 per Period for Various Interest Rates and Time Periods

	Rate		
Month	6%	8%	
1	.995025	.993377	
2	1.985099	1.980176	
3	2.970248	2.960440	
4	3.950496	3.934212	
5	4.925866	4.901535	
6	5.896384	5.862452	
7	6.862074	6.817005	
8	7.882959	7.765237	
9	8.779064	8.707189	
10	9.730412	9.642903	
11	10.677027	10.572420	
12	11.618932	11.495782	
Year			**Month**
1	11.618932	11.495782	12
2	22.562866	22.110544	24
3	32.871016	31.911806	36
4	42.580318	40.961913	48

Question: A real estate partnership predicts that it will pay $300 at the end of each month to its partners over the next six months. Assuming the partners desire an 8 percent return compounded monthly on their investment, how much should they pay?

Solution: $PV = \$300(5.862452)$

$= \$1,758.74$

Solution:	Function:
$n = 6$ months	$PV\ (n, i, PMT, FV)$
$i = 8\% \div 12 = .6666\%$	
$PMT = -\$300$	
$FV = 0$	
Solve for $PV = \$1,758.74$	

Accumulation of a Future Sum

The previous two sections have dealt with compounding and discounting single payments on annuities. In some instances, however, it is necessary to determine a series of payments necessary to *accumulate a future sum,* taking into account the fact that such payments will be accumulating interest as they are deposited. For example, assume we have a debt of $20,000 that must be repaid in one lump sum at the end of five years. We would like to make a series of equal annual payments (an annuity) at the end of each of the five years, so that we will have $20,000 at the end of the fifth year from the accumulated deposits plus interest. Assuming that we can earn 10 percent interest per year on those deposits, how much should each annual payment be?

In this case, we are dealing with accumulating a future sum. Recall from Exhibit 3–5 that this means we will be compounding a series of payments, or an annuity, to achieve that future value. Hence, we can work with the procedure for determining future values by compounding as follows:

$$FV = \$20,000$$
$$PMT(6.105100) = \$20,000$$
$$PMT = \$20,000 \div 6.105100$$
$$= \$3,275.95$$

Solution:	Function:
$n = 5$	$PMT\ (n, i, PV, FV)$
$i = 10\%$	
$PV = 0$	
$FV = \$20,000$	
Solve for $PMT = -\$3,275.95$	

This computation merely indicates that when compounded at an annual interest rate of 10 percent, the unknown series of equal payments (*PMT*) will result in the accumulation of $20,000 at the end of five years. Given the interest factor for compounding an annual annuity at 10 percent from Exhibit 3–9, 6.105100, we know that the unknown payment (*PMT*), when multiplied by that factor, will result in $20,000. Hence, by dividing $20,000 by the interest factor for compounding an annual annuity, we can obtain the necessary annual payment of $3,275.95. The result tells us that if we make payments of $3,275.95 at the end of each year for five years, and each of those payments earns interest at an annual rate of 10 percent, a total of $20,000 will be accumulated at the end of five years.

We can see from the above computation that dividing $20,000 by 6.105100 is equivalent to multiplying $20,000 by (1 ÷ 6.105100), or .163797, and the same $3,275.95 solution results. The factor .163797 is referred to in real estate finance as a **sinking-fund factor (SFF)**, which is also used in other applications in real estate. In the case of monthly payments, if we want to know what monthly payments would be necessary to pay off the $20,000 debt at the end of five years, taking into account that each payment will earn an annual rate of 10 percent compounded monthly, we can obtain a calculator solution as follows:

Solution:	Function:
$n = 5 \times 12 = 60$	PMT (n, i, PV, FV)
$i = 10\% \div 12 = .008333$	
$PV = 0$	
$FV = \$20{,}000$	
Solve for $PMT = -\$258.27$	

Required monthly payments would be $258.27.

Determining Yields, or Internal Rates of Return, on Investments

Up to now, this chapter has demonstrated how to determine future values in the case of compounding and present values in the case of discounting. Each topic is important in its own right, but each has also provided tools for determining an equally important component used extensively in real estate financing, that is, calculating rates of return, or **investment yields**. In other words, the concepts illustrated in the compounding and discounting processes can also be used to determine rates of return, or yields, on investments, mortgage loans, and so on. These concepts must be mastered because procedures used here will form the basis for much of what follows in succeeding chapters.

We have concentrated previously on determining the future value of an investment made today when compounded at some given rate of interest, or the present investment value of a stream of cash returns received in the future when discounted at a given rate of interest. In this section, we are concerned with problems where we know what an investment will cost today and what the future stream of cash returns will be, but we do not know what **yield**, or **rate of return** (compounded), will be earned if the investment is made.

Investments with Single Receipts

In many cases, investors and lenders are concerned with the problem of what rate of compound interest, or investment yield, will be earned if an investment is undertaken. To illustrate the investment yield concept, assume an investor has an opportunity today to buy an unimproved one-acre lot for $5,639. The lot is expected to appreciate in value and to be worth \$15,000 after seven years. What rate of interest (or investment yield) would be earned on the \$5,639 investment in the property if it were made today, held for seven years, and sold for \$15,000? Note from our previous discussion of compounding that the \$15,000 also represents the receipt of a future cash flow or future value (FV).

To solve for the unknown rate, we can formulate the problem as follows:

$$PV = FV \cdot \frac{1}{(1+i)^n}$$

$$\$5{,}639 = \$15{,}000 \cdot \frac{1}{(1+i)^7}$$

We want to know the annual rate of compound interest, i, that, when substituted into the above equation, will make the \$15,000 future receipt, or future value, equal to the \$5,639 investment outlay, or present value, today.

One approach is to rearrange the above equation as follows:

$$\frac{1}{(1+i)^7} = \frac{5{,}639}{15{,}000}$$

$$(1+i)^7 = 15{,}000/5{,}639$$

$$1 + i = 1.15$$

$$i = .15 \text{ or } 15\%$$

Another approach is to use trial-and-error to solve for i. A value for i is estimated; then the equation is solved to ascertain whether the future value, or \$15,000, when discounted to present value, PV, will equal \$5,639. When the correct value for i is found, the solution for present value should yield \$5,639.

How do we begin the search for i? One way is to simply guess a solution. Let us try 10 percent. Mathematically we ask, if

$$PV = \$15{,}000 \frac{1}{(1+.10)^7}$$

is $PV = \$5{,}639$?

Solving for PV, we have

$$PV = (\$15{,}000)(.513158)$$

$$= \$7{,}697$$

We note that \$7,697 or PV is much *greater* than the desired PV, or \$5,639. This means that the yield, or rate of compound interest earned on the investment, is *greater* than 10 percent. Hence, we must continue the discounting process by increasing i.

Our next "trial" will be 15 percent. Substituting, we have

$$PV = \$15{,}000 \frac{1}{(1+.15)^7}$$

$$= \$15{,}000(.375937)$$

$$= \$5{,}639.06$$

This time PV equals \$5,639. This "guess" was correct. From this result, we have determined that the yield or internal rate of return, i, earned on the investment is equal to 15 percent.

We have, in essence, "removed" interest compounded at the rate of 15 percent for seven years from the $15,000 receipt of cash (the future value), leaving the initial deposit, or present value, of $5,639.

When trying to find the yield in which only one future value is involved, we can use an approach where the interest factor in the financial tables is first determined as follows:

$$\$5{,}639 = \$15{,}000\frac{1}{(1+i)^7}$$

$$\$5{,}639.06 \div \$15{,}000 =$$

$$.375937 =$$

The above calculations show that the interest factor is .375937, but we still do not know the *interest rate*. However, we do know that the time period over which the investment is to appreciate in value is seven years. Because our solution is .375937 and the term of investment is seven years, the interest tables in Exhibit 3–6 allow us to easily find the correct interest rate. Since the *FV* or cash return of $15,000 is a single receipt, we need only locate a factor for the present value reversion of $1 equal to .375937 in the row corresponding to seven years for some interest rate. We begin the search for the interest rate by choosing an arbitrary interest rate, say 6 percent. The 6 percent column in Exhibit 3–6 shows the factor in this column for seven years is .665057, which is larger than .375937. Moving to the 10 percent column, the factor for seven years is .513158, which is lower than the factor at 5 percent but comes closer to the solution that we are looking for. If we continue this trial-and-error process, the 15 percent column indicates that the factor for seven years is .375937; therefore, the interest rate we desire is 15 percent. We know this is the correct answer because $15,000 (.375937) = $5,639.06.

A more efficient approach to finding the yield or internal rate of return is to use a financial calculator. For calculators that have the capability to solve for yields such as required by the problem at hand, we have

Solution:	Function:
$n = 7$	$i\ (n, PV, PMT, FV)$
$PV = -\$5{,}639$	
$PMT = 0$	
$FV = \$15{,}000$	
Solve for i = 15%	

What does this interest rate, or yield, mean? It means that the $5,639 investment made today, held for seven years, and sold for $15,000, is equivalent to investing $5,639 today and letting it compound annually at an interest rate of 15 percent (note the correspondence between the terms *interest rate* and *yield*).[6] This fact can be determined with the following computation:

$$FV = \$5{,}639(2.660020)$$
$$= \$15{,}000$$

[6] We are now using the terms *yield* and *internal rate of return* for *i*, instead of the interest rate. It is generally accepted practice to use the terms yield or internal rate of return when evaluating *investments*. The term *interest rate* is generally used when *loan terms* are being quoted by lenders. The two concepts are very similar, but the reader should become accustomed to these differences in usage.

This calculation simply shows that $5,639 compounded annually at an interest rate of 15 percent for seven years is $15,000. Hence, making this investment is equivalent to earning a rate of return of 15 percent. This rate of return is usually referred to as the *investment yield* or the **internal rate of return.**

Solution:	Function:
$n = 7$	$FV(n, i, PV, PMT)$
$i = 15\%$	
$PV = -\$5{,}639$	
$PMT = 0$	
Solve for $FV = \$15{,}000$	

The internal rate of return integrates the concepts of compounding and present value. It represents a way of measuring a return on investment, expressed as a compound rate of interest, over the entire investment period. For example, if an investor is faced with making an investment in an income-producing venture, regardless of how the cash returns are patterned, the internal rate of return provides a guide or comparison for the investor. It tells the investor what the equivalent compound interest rate will be on the investment being considered. In the example of the unimproved one-acre lot, the 15 percent yield or internal rate of return is equivalent to making a deposit of $5,639 and allowing it to compound annually at an interest rate of 15 percent for seven years. After seven years, the investor would receive $15,000, which includes the original investment of $5,639 plus all compound interest. With the internal rate of return known, the investor can make an easier judgment about what investment to make. If the 15 percent return is adequate, it will be made; if not, the investor should reject it.[7]

The concepts of the internal rate of return or yield, present value, and compounding are indispensable tools that are continually used in real estate finance and investment. The reader should not venture beyond this section without a firm grasp of the concepts that have been explained. These concepts form the basis for the remainder of this chapter and the chapters that follow.

Yields on Investment Annuities

The concepts illustrated for a single receipt of cash (when the unimproved lot was sold) also apply to situations where a *series* of cash receipts is involved. Consequently, a yield or internal rate of return also can be computed on these types of investments.

Suppose an investor has the opportunity to make an investment in real estate costing $3,170 that would provide him with cash income of $1,000 at the end of *each year* for four years. What investment yield, or internal rate of return, would the investor earn on the $3,170? In this case, we have a series of receipts that we wish to discount by an unknown rate to make the present value of the $1,000 annuity equal the original investment of $3,170. We need to find a solution for *i* in this problem, or the rate of interest that will make the present value of the $1,000 four-year annuity equal to $3,170. Recalling the notation for the present value of an annuity, we have

$$PV = PMT \cdot \sum_{t=1}^{n} \frac{1}{(1+i)^t}$$

[7] When comparing different investments, the investor must also consider any differences in risk. This topic is discussed in later chapters.

Substituting gives

$$\$3{,}170 = \$1{,}000 \sum_{t=1}^{n} \frac{1}{(1+i)^t}$$

Using our shorthand notation, we can express our problem as follows:

$$\$3{,}170 \div \$1{,}000 = \sum_{t=1}^{n} \frac{1}{(1+i)_t^t}$$

$$3.170000 = \sum_{t=1}^{n} \frac{1}{(1+i)_t^t}$$

Solution:

Function:

i (*n*, *PV*, *PMT*, *FV*)

$n = 4$

$PV = -\$3{,}170$

$PMT = \$1{,}000$

$FV = 0$

Solve for i **= 10%**

This procedure is similar to solving for the yield, or internal rate of return, on single receipts discussed in the preceding section, except that we are now dealing with an annuity. Using the same procedure as before, we solve for the interest factor for a four-year period that will correspond to some interest rate. To determine what the interest rate is, search the factors in Exhibit 3–11 (p. 62) in the four-year row until you find a column containing a factor very close to 3.1700. A careful search reveals that the factor will be found in the 10 percent column (the reader should verify this). Hence, based on this procedure, we have determined that the investment yield or internal rate of return (*IRR*) on the \$3,170 invested is 10 percent. A more in-depth analysis of what the internal rate of return means is presented in Exhibit 3–13.

An important component of this analysis is to first determine if the investment outlay of \$3,170 will be recovered from total cash inflows. In Exhibit 3–13 we can see that cash inflows (cash received in years 1–4) total \$4,000. Therefore, in addition to recovering the investment of \$3,170, \$830 in additional cash flow will be received and we can say that the investment will be profitable, or that the investment yield (*IRR*) must be positive.

When the internal rate of return is computed, two additional characteristics are present (see Exhibit 3–13). One is the *recovery of capital* in each period, and the other is *interest earned* in each period. In other words, when the *IRR* is computed based on the \$3,170 investment and the \$1,000 received each year, *implicit* in the cash flows recovered each year during the four-year period is the *full recovery* of the \$3,170 investment *plus* interest. Note also that the total interest/profit that will be received equals \$830 or (\$4,000 – \$3,170). Our goal is to simplify this investment result by finding a rate of compound interest for this investment. This result can be compared with compound rates of interest in other investments that may have different investment and cash flow patterns. Comparing interest rates (*IRR*s) on various investments is easier than comparing cash flows. We also benefit because the time value of money (*TVM*) is taken into account. Hence, the 10 percent investment yield is really a rate of compound interest earned on an outstanding investment balance, after each capital recovery is taken into account, from year to year. Of the total \$4,000 received during the four-year period, total interest earned

EXHIBIT 3–13
Illustration of the Internal Rate of Return (IRR) and Components of Cash Receipts

	Year			
	1	2	3	4
Investment (balance)	$3,170	$2,487	$1,736	$ 910
IRR at 10%	317	249*	174*	91*
Cash received	$1,000	$1,000	$1,000	$1,000
Less: Cash yield at 10%	317	249	174	90*
Recovery of investment	$ 683	$ 751	$ 826	$ 910
Investment (beginning of year)	$3,170	$2,487	$1,736	$ 910
Less: Recovery of investment	683	751	826	910
Investment (end of year)	$2,487	$1,736	$ 910	$ 0

*Rounded.

is $830 and capital recovery is $3,170. This is also equivalent to earning a 10 percent annual rate of compound interest on our investment.

Monthly Annuities: Investment Yields

A similar application for investment yields can be made in cases where *monthly* cash annuities will be received as a return on investment. For example, assume that an investor makes an investment of $51,593 and will receive $400 at the end of each month for the next 20 years (240 months). What annual rate of return, *compounded monthly,* would be earned on the $51,593?

Solution:

$n = 20 \times 12 = 240$

$PV = -\$51,593$

$PMT = \$400$

$FV = 0$

Solve for i

$i\,(\text{monthly}) = 0.5833\%$

$i\,(\text{annualized}) = .005833 \times 12, \text{ or } 7\%$

Function:
$i\,(n, PV, PMT, FV)$

As was the case with finding the *IRR* for investments with annual receipts, we find that the *IRR* is 7 percent compounded monthly on the $51,593 investment. Both the recovery of $51,593 and the $44,407 in interest were *embedded* in the stream of $400 monthly cash receipts ($96,000 in total receipts) over the 20-year period.

Equivalent Nominal Annual Rate (ENAR): Extensions

Earlier in this chapter, we dealt with the problem of determining equivalent annual yields in cases where more than one compounding interval exists within a year. In our example, we showed the effective annual yield for a $10,000 investment compounded annually, monthly, and daily to be 6, 6.17, and 6.18 percent, respectively. In many situations, we may already know the *effective annual yield (EAY)* and would like to know the *nominal annual rate of interest compounded monthly* (or for any period less than one year). For example, we considered a problem in which compounding occurred monthly based on a nominal annual rate of interest of 6 percent. Because compounding occurred in monthly intervals, the effective annual interest rate (6.17%) was larger than the nominal rate (6%).

Additional Financial Functions

Concept Box 3.2

In Exhibit 3–13, we introduced the IRR concept in the context of an investment with cash inflows of $1,000 per year, or a level annuity. In many situations investors must evaluate investments with cash inflows that *vary* from period to period. Investors in these situations also must know the IRR. However, when cash inflows vary, solving for the IRR is slightly more complicated. To illustrate, assume that we have an investment with these characteristics:

PV	Cash Flows (*CF*)			
Investment	Yr. 1	Yr. 2	Yr. 3	Yr. 4
$8,182	$1,000	$2,000	$3,000	$4,000

To solve for the IRR (*i*), we have

$$PV = \frac{CF_1}{(1+i)^1} + \frac{CF_2}{(1+i)^2} + \frac{CF_3}{(1+i)^3} + \frac{CF_4}{(1+i)^4}$$

Substituting

$$\$8{,}182 = \frac{\$1{,}000}{(1+i)^1} + \frac{\$2{,}000}{(1+i)^2} + \frac{\$3{,}000}{(1+i)^3} + \frac{\$4{,}000}{(1+i)^4}$$

Summing the cash inflows shows that a *total* of $10,000 will be received over four years; therefore, we know that because this total exceeds the *PV* of $8,182, the investment will be profitable and (*i*) will also be positive. (At this point, the reader should verify that the current value for (*i*) needed to make the right side of the equation equal to the *PV* of $8,182 is 7 percent.) However, as the reader may realize, solving for (*i*) is very cumbersome.

Introduction of the CF_j Function. Many financial calculators and software programs have been designed to more efficiently solve for IRRs when cash flows vary from period to period. These programs use a "trial and error" or "iterative" approach to solve for (*i*). This approach begins by first choosing a very large value for (*i*), substituting it in the above expression, and then solving for *PV*. The difference between the value *calculated* for *PV* and $8,182 is determined to ascertain how close this difference is to zero. The process continues by selecting values for (*i*) and recalculating *PV*. By using successively higher, then lower values for (*i*), *PV*s are calculated until an (*i*) is found such that the difference between the calculated *PV* and $8,182 is equal to zero.

With advances in electronic technology, these many calculations can be carried out very quickly by using the CF_j and n_j functions. This involves entering cash flows (CF_j) per period (n_j), then solving for (*i*). In our example, we would have

Solution:

Enter CF	Enter n_j
CF_j = −$8,182	$n_j = 1$
CF_j = $1,000	$n_j = 1$
CF_j = $2,000	$n_j = 1$
CF_j = $3,000	$n_j = 1$
CF_j = $4,000	$n_j = 1$

Solve for (*i*) = 7%

When each cash flow is for a single period, we will use the following notation for the IRR function:

IRR (CF_1, CF_2,…,CF_n)

For the above example, we would have IRR(−8182, 1000, 2000, 3000, 4000) = 7%.

Assuming that we wanted to know what the nominal annual rate of interest, compounded *monthly,* would have to be to provide a desired *EAY* of 6 percent, we can employ the following formula, where *ENAR* is the **equivalent nominal annual rate,** compounded monthly:

$$ENAR = [(1 + EAY)^{1/m} - 1] \cdot m$$

In our problem, we would have

$$\begin{aligned} ENAR &= [(1 + .06)^{1/m} - 1] \cdot 12 \\ &= [(1 + .06)^{.083333} - 1] \cdot 12 \\ &= [1.004868 - 1] \cdot 12 \\ &= .0584106 \text{ or } 5.84106\% \text{ (rounded)} \end{aligned}$$

To illustrate this concept, if we have investment A, which will provide an *effective annual yield* of 6 percent, and we are considering investment B, which will provide interest compounded monthly, we would want to know what the equivalent nominal annual rate (*ENAR*) of interest, *compounded monthly,* would have to be on investment B to provide the *same* effective annual yield of 6 percent. That rate would be an annual rate of 5.84106 percent, *compounded monthly.*

$$\begin{aligned} FV &= \$1\left[1 + \frac{0.584106}{12}\right]^{12} \\ &= \$1.06 \text{ (rounded)} \end{aligned}$$

From our example, we know that the *EAY* is (\$1.06 – \$1.00) ÷ \$1.00, or 6 percent. Hence, we now know that an investment of equal risk, with returns compounded *monthly,* must have an annual nominal rate of interest of at least 5.84106 percent to provide us with an equivalent, effective annual yield of 6 percent. Obviously, this application can be modified for any investment with different compounding periods by altering m in the above formula.

Solution:

n = 12 months

i = 5.84106 ÷ 12 = .486755%

PV = –\$1

PMT = 0

Solve for FV = \$1.06

Function:

FV (n, i PV, PMT)

Solving for Annual Yields with Partial Periods: An Extension

Many investments produce monthly cash flows but call for investment returns to be reported as an effective annual rate. However, many investments may be sold within a year (say, after five months into a calendar year). How can monthly cash flows within a year be expressed as an annual rate of interest? Consider the following example.

An investment is made in the amount of \$8,000 and is expected to be owned for two years. The contract calls for investment returns to be reported as an effective annual rate. However, the investment is sold early and monthly cash flows of \$500 are received for 17 months. What is the equivalent *annual* return on the investment? This can be determined as follows:

STEP 1:

Solution: Solve for monthly interest rate
Function: $i\,(n, PMT, PV, FV)$

$n = 17$ months
$PMT = \$500$
$PV = -\$8{,}000$
$FV = 0$

Solve for i = .682083%

STEP 2: The monthly interest rate of .682083% can now be used to determine the effective *annual rate* as follows:

Solution:
Function: $FV\,(PV, i, PMT, n)$

$PV = -\$1$
$i = .682083\%$
$PMT = 0$
$n = 12$

Solve for FV = 1.084991

STEP 3: The effective annual rate of interest would be:
$(FV/PV) - 1 = 1.084991 - 1.000 = .084991$ or 8.5% (rounded)

Therefore, this investment over a 17-month period has produced an equivalent *effective annual yield* (*EAY*) of 8.5 percent. (Note that this yield is also equivalent to a nominal annual rate, *compounded monthly,* which would be .682083 × 12 = 8.19 percent.)

Note on the XIRR

An alternative way of calculating an effective annual rate is to calculate what is referred to as the *XIRR*. The *XIRR* has been increasingly used in performance measurement because it is included in spreadsheet programs like Excel and can handle cash flow patterns that include partial years, as in the previous example, as well as cash flow patterns that are irregular and can occur any day of the year.

The *XIRR* is the rate that solves the following equation

$$\sum P_j/(1 + \text{rate})^{d_j/365} = 0$$

where P_j is the jth payment that is received dj days after the starting date of the investment. Excel allows you to actually input calendar dates for each payment (cash flow), which is then converted to the number of days. The left side of the above equation simply discounts the cash flows to a present value using the effective annual rate as the discount rate. We need to find the rate that makes the *PV* equal to zero.

Using the previous example of $500 occurring monthly for 17 months, we assume that cash flows are now received on about the 30th of each month over the 17 months, as shown in Exhibit 3–14. Column 2 has the days as a fraction of 365. Recall that we calculated an effective annual rate of 8.5 percent in the previous example. We now prove that the *XIRR* calculation gives the same result by using 8.5 percent to see if the result is a zero present value.

Column 3 of Exhibit 3–14 uses the effective annual rate in the above *XIRR* formula because we want to show that this is the rate that makes the present value equal to $0, which it does, as can be seen by the sum of the final column. Each row in this final column indicates the present value of the payment received on that day of the year.

In the above example the cash flows occurred monthly. But cash flows can occur on any day of the year. In Excel, the actual date (e.g., February 12, 2012) would be used in the *XIRR* function to specify that a cash flow occurred on that date.

EXHIBIT 3–14
Proof of *XIRR*

		Rate 8.5%			
		(1)	(2)	(3)	(4)
Month	Days	Days/365	*P*	$(1 + \text{rate})^{(\text{days}/365)}$	*P*/Column 3
0	0	0.000000	−8,000	1.000000	−$8,000.00
1	31	0.084932	500	1.006953	496.55
2	61	0.167123	500	1.013727	493.23
3	92	0.252055	500	1.020775	489.82
4	122	0.334247	500	1.027643	486.55
5	152	0.416438	500	1.034557	483.30
6	182	0.498630	500	1.041517	480.07
7	213	0.583562	500	1.048758	476.75
8	243	0.665753	500	1.055814	473.57
9	274	0.750685	500	1.063155	470.30
10	305	0.835616	500	1.070547	467.05
11	336	0.920548	500	1.077990	463.83
12	366	1.002740	500	1.085243	460.73
13	396	1.084932	500	1.092544	457.65
14	425	1.164384	500	1.099648	454.69
15	455	1.246575	500	1.107046	451.65
16	485	1.328767	500	1.114494	448.63
17	515	1.410959	500	1.121992	445.64
				Sum	.00

Conclusion

This chapter introduced and illustrated the mathematics of compound interest in financial analysis. Although this may be a review for many readers, a thorough understanding of this topic is essential in real estate finance. The concepts and techniques introduced in this chapter are used throughout the remainder of this text to solve a variety of problems encountered in real estate finance. In the following two chapters, we apply the mathematics of finance to the calculation of mortgage payments and the effective cost of various alternative mortgage instruments. Later, we apply the mathematics of finance to the analysis of income property investments. This chapter illustrated the use of financial tables, interest factors, and how these tables and factors can be found using a financial calculator. These tables were included to help the reader to understand the process of compounding, discounting, and finding internal rates of return by using interest factors. The tables are not necessary to solve any of the problems in the remainder of the book. In fact, an alternative calculator solution is also provided for many of the problems. The interest factor solutions are shown only so that the readers can see the mathematics behind the calculator solutions. As we move toward more advanced material, it is assumed that readers can obtain the solutions using a financial calculator or by using a spreadsheet program on a personal computer.

Key Terms

Useful Websites

www.bai.org—Good source of current market rates on various financial instruments as well as discussion of ways to properly compare investments with different payment patterns.

www.interest.com—This site provides the current average mortgage rates, a mortgage calculator, and basic information on home buying.

www.bankrate.com—Source of interest rates for CDs and other investments.

Questions

1. What is the essential concept in understanding compound interest?
2. How are the interest factors (*IFs*) in Exhibit 3–3 developed? How may financial calculators be used to calculate interest factors in Exhibit 3–3?
3. What general rule can be developed concerning maximum values and compounding intervals within a year?
4. What does the time value of money (*TVM*) mean?
5. How does discounting, as used in determining present value, relate to compounding, as used in determining future value? How would present value ever be used?
6. What are the interest factors in Exhibit 3–9 and how are they developed? How can financial calculators be used to calculate interest factors in Exhibit 3–9?
7. What is an annuity? How is it defined? What is the difference between an *ordinary annuity* and an *annuity due*?
8. How must one discount a series of uneven receipts to find present value (*PV*)?
9. What is the sinking-fund factor? How and why is it used?
10. What is an internal rate of return? How is it used? How does it relate to the concept of compound interest?

Problems

1. Jim makes a deposit of $12,000 in a bank account. The deposit is to earn interest *compounded annually* at the rate of 6 percent for seven years.
   a. How much will Jim have on deposit at the end of seven years? (*Hint:* What is future value?)
   b. Assuming the deposit earned a 9 percent rate of interest *compounded quarterly,* how much would he have at the end of seven years?
   c. In comparing (*a*) and (*b*), what are the respective *effective annual yields?* (*Hint:* Consider the future value of each deposit after one year only.) Which alternative is better?
2. Would you prefer making a $25,000 investment that will earn interest at the rate of 6 percent *compounded monthly* or making the same $25,000 investment at 7 percent *compounded annually?* (*Hint:* Consider one year only.)
3. Jones can deposit $5,000 at the end of each six-month period for the next 12 years and earn interest at an annual rate of 8 percent, *compounded semiannually.* What will the value of the investment be after 12 years? If the deposits were made at the beginning of each year, what would the value of the investment be after 12 years?
4. Suppose you deposit $1,250 at the end of each quarter in an account that will earn interest at an annual rate of 10 percent *compounded quarterly.* How much will you have at the end of four years?
5. Suppose you deposit $2,500 at the end of year 1, nothing at the end of year 2, $750 at the end of year 3, and $1,300 at the end of year 4. Assuming that these amounts will be *compounded at an annual rate* of 15 percent, how much will you have on deposit at the end of five years?
6. Suppose you have the opportunity to make an investment in a real estate venture that expects to pay investors $750 at the end of each month for the next eight years. You believe that a reasonable return on your investment should be an annual rate of 15 percent *compounded monthly.*
   a. How much should you pay for the investment?
   b. What will be the total sum of cash you will receive over the next eight years?
   c. What do we call the difference between (*a*) and (*b*)?

7. An investor is considering an investment that will pay $2,150 at the end of each year for the next 10 years. He expects to earn a return of 12 percent on his investment, *compounded annually.* How much should he pay today for the investment? How much should he pay if the investment returns are received at the *beginning of each year?*

8. An investor can make an investment in a real estate development and receive an expected cash return of $45,000 at the end of six years. Based on a careful study of other investment alternatives, she believes that a 9 percent *annual return compounded quarterly* is a reasonable return to earn on this investment. How much should she pay for it today?

9. Walt is evaluating an investment that will provide the following returns at the end of each of the following years: year 1, $12,500; year 2, $10,000; year 3, $7,500; year 4, $5,000; year 5, $2,500; year 6, $0; and year 7, $12,500. Walt believes that he should earn 12 percent compounded annually on this investment. How much should he pay for this investment? What if he expects to earn an annual return of 9 percent *compounded monthly?* How much should he pay?

10. John is considering the purchase of a lot. He can buy the lot today and expects the price to rise to $15,000 at the end of 10 years. He believes that he should earn an investment yield of 8 percent *compounded annually* on his investment. The asking price for the lot is $7,000. Should he buy it? What is the internal rate of return *compounded annually* on the investment if John purchases the property for $7,000 and is able to sell it 10 years later for $15,000?

11. The Dallas Development Corporation is considering the purchase of an apartment project for $100,000. They estimate that they will receive $15,000 at the end of each year for the next 10 years. At the end of the 10th year, the apartment project will be worth nothing. If Dallas purchases the project, what will be its internal rate of return, *compounded annually?* If the company insists on an 8 percent return *compounded annually* on its investment, is this a good investment?

12. A corporation is considering the purchase of an interest in a real estate syndication at a price of $75,000. In return, the syndication promises to pay $1,000 at the end of each month for the next 25 years (300 months). If purchased, what is the expected internal rate of return, *compounded monthly?* How much total cash would be received on the investment? How much is profit and how much is return of capital?

13. An investment in a real estate venture will provide returns at the end of the next four years as follows: year 1, $5,500; year 2, $7,500; year 3, $9,500; and year 4, $12,500. An investor wants to earn a 12 percent return *compounded annually* on her investment. How much should she pay for the investment? Assuming that the investor wanted to earn an annual rate of 12 percent *compounded monthly,* how much would she pay for this investment? Why are these two amounts different?

14. A pension fund is making an investment of $100,000 today and expects to receive $1,600 at the end of each month for the next five years. At the end of the fifth year, the capital investment of $100,000 will be returned. What is the internal rate of return *compounded annually* on this investment?

15. A loan of $60,000 is due 10 years from today. The borrower wants to make annual payments at the end of each year into a sinking fund that will earn *compound interest* at an *annual rate* of 10 percent. What will the annual payments have to be? Suppose that the monthly payments earn 10 percent interest, *compounded monthly.* What would the annual payments have to be?

16. An investor has the opportunity to make an investment that will provide an effective *annual* yield of 12 percent. She is considering two other investments of equal risk that will provide compound interest *monthly* and *quarterly,* respectively. What must the equivalent nominal annual rate (*ENAR*) be for each of these two investments to ensure that an *equivalent annual yield* of 12 percent is earned?

17. An investment producing cash flows in the amount of $1,200 per month is undertaken for a period of 28 months. The investor pays $24,000 for the investment and the contract stipulates that investment returns must be reported on a basis equivalent with *annual compounding.* Given that the investment is sold after 28 months, what would be the equivalent *annual compound* rate

of interest reported to the investor? What would be the annual rate *compounded monthly* for this investment?

18. An investment is expected to produce the following annual year-end cash flows:

year 1: $5,000	year 4: $5,000
year 2: $1,000	year 5: $6,000
year 3: $0	year 6: $863.65

The investment will cost $13,000 today.

a. Will this investment be profitable?

b. What will be the *IRR* (*compounded annually*) on this investment?

c. **Prove** your answer in (*b*) by showing how much of each year's cash flow is *recovery of* the $13,000 investment and how much of the cash flow is *return on* investment. (*Hint:* See Exhibit 3–13 and Concept Box 3.2.)

that the demand for mortgage loans is a **derived demand,** or is determined by the demand for real estate.

When supplying funds to the mortgage market, lenders also consider returns and the associated risk of loss on alternative investments in relation to returns available on mortgages. Hence, the mortgage market should also be thought of as part of a larger capital market, where lenders and investors evaluate returns available on mortgages and all competing forms of investment, such as bonds, stocks, and other alternatives, and the relative risks associated with each. Should lenders believe that a greater return can be earned by making more mortgage loans (after taking into account the costs and the risk of loss) than would be the case if they invested in corporate bonds or business loans, more funds would be allocated to mortgage loans, and vice versa. Hence, lender decisions to allocate funds to mortgages are also made relative to returns and risk on alternative loans and investment opportunities.

The Real Rate of Interest: Underlying Considerations

When discussing market interest rates on mortgages, we should keep in mind that these interest rates are based on a number of considerations. We pointed out earlier that the supply of funds allocated to mortgage lending in the economy is, in part, determined by the returns and risks on all possible forms of debt and investment opportunities.

One fundamental relationship that is common to investments requiring use of funds in the economy is that they earn at least the **real rate of interest.**[1] This is the minimum rate of interest that must be earned by savers to induce them to divert the use of resources (funds) from present consumption to future consumption. To convince individuals to make this diversion, income in future periods must be expected to increase sufficiently from interest earnings to divert current income from consumption to savings. If expected returns earned on those savings are high enough to provide enough future consumption, adequate amounts of current savings will occur.

Interest Rates and Inflation Expectations

In addition to the real rate of interest, a concern that all investors have when making investment decisions is how *inflation* will affect investment returns. The rate of inflation is of particular importance to investors and lenders making or purchasing loans made at fixed rates of interest over long periods of time. Hence, when deciding whether to make such commitments, lenders and investors must be convinced that interest rate commitments are sufficiently high to compensate for any expected loss in purchasing power during the period that the investment or loan is outstanding; otherwise, an inadequate real return will be earned. Therefore, a consensus of what lenders and investors expect inflation to be during the time that their loans and investments are outstanding is also incorporated into interest rates at the time investments and loans are made.

To illustrate the relationship between the **nominal interest rate,** or the contract interest rate agreed on by borrowers and lenders, and real rates of interest, suppose that a $10,000 loan is made at a nominal or contract rate of 10 percent with all principal and interest due at the end of one year. At the end of the year, the lender would receive $11,000, or $10,000 plus $10,000 times (.10). If the rate of inflation during that year was 6 percent, then the $11,000 received at the end of the year would be worth about $10,377 ($11,000 ÷ 1.06). Thus, although the nominal rate of interest is 10 percent, the *real* rate on the mortgage is

[1] If the reader can visualize an investment portfolio containing investments in all productive activities in the economy based on the weight that any particular activity has to the total value of all productive activity in the economy, the rate of current earnings on such a portfolio would be equivalent to the real rate of interest. Such a rate would also be the rate required by economic units to save rather than consume from the current income.

just under 4 percent ($377 ÷ $10,000 = 3.77%). Therefore, we conclude that if the lender wanted a 4 percent real rate of interest, the lender would have to charge a nominal rate of approximately 10 percent to compensate for the expected change in price levels due to inflation.[2]

We can summarize by saying that the nominal interest rate on any investment is partially determined by the real interest rate *plus a premium* for the expected rate of inflation. In our example, the real rate of 4 percent plus an inflation premium of 6 percent equals 10 percent. Note that this premium is based on the rate of inflation *expected* at the time that the loan is made. The possibility that inflation will be more or less than expected is one of many risks that lenders and investors must also consider.

We should also point out that the nominal interest rate is usually expressed as an *annual* rate of interest. However, depending on the type of loan, the nominal rate could be an annual rate compounded daily, monthly, quarterly, annually, or continuously. We will explore the effects of compounding, accrued interest, and payment patterns in more detail throughout this chapter.

Interest Rates and Risk

In addition to expected inflation, lenders and investors are also concerned about various *risks* undertaken when making loans and investments. Lenders and investors are concerned about whether interest rates and returns available on various loans and investments compensate adequately for risk. Alternatively, will a particular loan or investment provide an adequate risk-adjusted return?

Many types of risk could be discussed for various investments, but they are beyond the scope of this book. Consequently, we will focus on risks affecting mortgage loans. Many of these risks are, however, present to greater and lesser degrees in other loans and investments.

Default Risk

One major concern of lenders when making mortgage loans is the risk that borrowers will default on obligations to repay interest and principal. This is referred to as **default risk,** and it varies with the nature of the loan and the creditworthiness of individual borrowers. The possibility that default may occur means that lenders must charge a premium, or higher rate of interest, to offset possible loan losses. Default risk relates to the likelihood that a borrower's income may fall after a loan is made, thereby jeopardizing the receipt of future mortgage payments. Similarly, a property's value could fall below the loan balance at some future time, which could result in a borrower defaulting on payments and a loss to the lender.

Interest Rate Risk

An additional complication in lending and investing arises from the uncertainty in today's world about the future supply of savings, demand for housing, and future levels

[2] Actually, the nominal rate of interest should be (1.06 × 1.04) − 1, or 10.24 percent, if a real rate of 4 percent is desired. For convenience throughout this text, we will *add* the real rate and premium for expected inflation as an approximation to the nominal interest rate. We should point out that the relationship of expected inflation and interest rates has long been a subject of much research. While we show a very simple, additive relationship in our discussion, there may be interaction between real interest rates and inflation. The specific relationship between the two is not known exactly. Hence, the student should treat this discussion at a conceptual or general level of interpretation.

of inflation. Thus, interest rates at a given point in time can only reflect the market consensus of what these factors are expected to be. Investors and lenders also incur the risk that the interest rate charged on a particular loan may be insufficient, should economic conditions change drastically *after* a loan is made. The magnitude of these changes may have warranted a higher interest rate when the loan was made. The uncertainty about what interest rate to charge when a loan is made can be referred to as **interest rate risk.**

For example, **anticipated inflation** may have been 6 percent at the time our $10,000 loan was made. But if *actual* inflation turns out to be 8 percent, this means the interest rate that should have been charged is 12 percent. In this case, we say that the anticipated rate of inflation at the time the loan was made was 6 percent. However, because **unanticipated inflation** of 2 percent occurred, the lender will lose $200 in purchasing power (2% of $10,000) because the rate of interest was too low. This does not mean that lenders did not charge the "correct" interest rate *at the time the loan was made.* At that time, the inflation was expected to be 6 percent. Therefore, to be competitive, a 10 percent interest rate had to be charged. However, the additional 2 percent was unanticipated by all lenders in the market. It is unanticipated inflation that constitutes a major component of interest rate risk to all lenders.

The possibility that too low an interest rate was charged at the time the loan was made is a major source of risk to the lender. Hence, a premium for this risk must also be charged or reflected in the market rate of interest. Interest rate risk affects all loans, particularly those that are made with fixed interest rates, that is, where the interest rate is set for a lengthy period of time when the loan is made. Being averse to risk, lenders must charge a premium to incur this risk.

Prepayment Risk

Some mortgage loans allow borrowers to prepay loans before the maturity date without a penalty. This, in effect, gives borrowers the *option* to prepay the loan, refinance, or pay off the loan balance if a property is sold. If loans are prepaid when interest rates fall, lenders must forgo the opportunity to earn interest income that would have been earned at the original contract rate. As funds from the prepaid loans are reinvested by lenders, a lower rate of interest will be earned. When interest rates increase, however, the loan is not as likely to be prepaid. The risk that the loan will be prepaid when interest rates fall below the loan contract rate is referred to as **prepayment risk.**

Other Risks

There are additional risks that lenders and investors consider that may vary by type of loan or investment. For example, the *liquidity* or *marketability* of loans and investments will also affect the size of the premium that must be earned. Securities that can be easily sold and resold in well-established markets will require lower premiums than those that are more difficult to sell. This is called **liquidity risk.**

Legislative risk is another risk associated with mortgage lending that also may result in a premium. It can refer to changes in the regulatory environment in which markets operate; for example, regulations affecting the tax status of mortgages, rent controls, state and federal laws affecting interest rates, and so on, are all possibilities that lenders face after making loans for specified periods of time. Lenders must assess the likelihood that such events may occur and be certain that they are compensated for undertaking these risks when loans are made.

A Summary of Factors Important in Mortgage Loan Pricing

We can now see that the interest rate charged on a particular mortgage loan will depend on the real interest rate, anticipated inflation, interest rate risk, default risk, prepayment risk, and other risks. These relationships can be summarized in general as follows:

$$i = r + p + f$$

In other words, when pricing or setting the rate of interest (i) on a mortgage loan, the lender must charge a premium (p) sufficiently high to compensate for default and other risks and a premium (f) that reflects anticipated inflation to earn a real rate of interest (r) that is competitive with real returns available on other investment opportunities in the economy. If lenders systematically *underestimate* any of the components in the above equation, they will suffer real economic losses.

Pricing decisions by lenders are rendered complex because mortgage loans are made at fixed interest rates for long periods of time. For example, if we assume that a mortgage loan is to be made with a one-year maturity, the interest rate charged at origination should be based on what the lender expects each of the components discussed above to be during the coming year. More specifically,

$$i_1 = r_1 + p_1 + f_1$$

or the mortgage interest rate (i) at origination (time t) would be based on the lender's expectations of what the real rate of interest, the rate of inflation, and risk premiums (for risks taken in conjunction with making the mortgage loan over and above the level of risk reflected in the real rate of interest) should be for the term of the loan.

Understanding Fixed Interest Rate Mortgage (FRM) Loan Terms

As previously discussed in Chapter 2, there are many terms and options in mortgage loan agreements that are very important. We begin our analysis of fixed interest rate loans with a discussion of some of the most elementary terms:

- Loan amount
- Loan maturity date
- Interest rate
- Periodic payments

The *loan amount* identifies the amount borrowed and what the borrower is legally required to repay. The *loan maturity date* is the date by which the loan must be fully repaid. While these terms are relatively easy to understand, when analyzing and comparing various loan alternatives, the interest rate and its affect on periodic payments can be more complex and will be discussed further.

When dealing with fixed interest rate loans, in addition to the loan amount and maturity, lenders generally quote what is referred to as a *nominal annual rate of interest.* To elaborate, say a 30-year loan is made for $60,000 and the interest rate is 12 percent. That rate of interest is referred to as the *nominal rate because no reference is made as to how interest is to be calculated or how frequent payments will be.* If, in this case, *interest* is to be calculated *monthly* and *payments* are to be made *monthly,* one interpretation of the 12 percent *nominal interest rate* would be an annual rate of 12 percent interest *compounded monthly.* Recall from Chapter 3 that another way to interpret an annual rate of interest, compounded monthly, is to *calculate its equivalent annual rate of interest.* That is, a rate of interest *compounded annually* that would be equivalent to a loan with an annual rate of interest *compounded monthly.* This can be done in our example as follows:

STEP 1:

Solution: Find the *FV* of an amount that earns interest at an annual rate *compounded monthly:*

$PV = -\$60,000$

$i = 12\%/12 = 1\%$ or .01

$n = 12$

Solve for *FV* = $67,609.50

Function: *FV* (*PV, i, n*)

STEP 2: Find the equivalent annual interest rate *compounded annually:*

$FV = \$67,609.50$

$n = 1$

$PV = \$60,000.00$

Solve for *i* = 12.6825%

In Step 1, using our approach outlined in Chapter 3, we consider a $60,000 deposit made today and compounded monthly for 12 months at 12 percent, leaving an *FV* of $67,609.50. We then solve for the *annual* compound rate equivalent in Step 2 by changing the compounding period to 1. This produces $i = 12.6825$ percent. In other words, a loan quoted with a 12 percent annual rate of interest compounded *monthly* is equivalent to a loan with an annual rate of 12.6825 percent compounded *annually.* The difference in these rates is due to the fact that when interest on a loan is compounded monthly and paid monthly, the incremental value is greater than receiving cash flows annually.[3] This is true even though the interest rate may be quoted as 12 percent in both cases. So, a loan with an annual rate of 12 percent compounded *monthly* is worth the equivalent of a loan with an annual rate of 12.6825 percent compounded *annually.* The latter can be thought of as an *effective* or *equivalent* annual rate of interest. Alternately, a loan with an annual rate of 12.6825 percent compounded *annually* is equivalent to a 12 percent annual rate compounded *monthly.*

So why do lenders quote *nominal* rates of interest (12% in our case)? If interest is to be compounded monthly (or for other periods), why not quote equivalent annual rates (12.6825% in our case)? If the latter approach was used, uniformity would be achieved because interest rates on all loans could be quoted based on an equivalent annual rate *regardless* of how frequently interest is calculated and payments are made (daily, monthly, quarterly, annually). The answer partially lies in the evolution of banking, financial instruments, simplicity when making interest calculations, and a general lack of knowledge (understanding) among people working in finance-related fields. *In the discussion that follows, and throughout this book, we will follow the practice used by lenders and use the nominal rate of interest for all mortgage loan examples. In most cases, interest will be calculated monthly and payments will be made monthly. However, the reader should be aware that interest could be calculated and payments made over very different time periods.*

[3] The reader may recall that we discussed this concept with annual percentage yield (APY) in Chapter 3.

Calculating Payments and Loan Balances—Fixed Interest Rate Loans

The Importance of Accrued Interest and Loan Payments

A very important concept that must be understood when calculating payments or outstanding loan balances for real estate loans is: (1) the relationship between *accrued interest* and *loan payments* for a given period and (2) how any differences between them will affect loan balances. For example, as discussed above, many loans require interest to be *accrued monthly* ($i/12$); that is, if a fixed interest rate mortgage (FRM) loan is made in an amount of \$60,000 (PV) at a 12 percent interest rate and interest (i) is to be accrued monthly, the dollar amount of interest accrued as of the end of the first month would be calculated as

$$\$60{,}000 \times (.12/12) = \$600$$

We should also point out that ($i/12$) is referred to as the **accrual rate**. The amount of interest *accrued* and *owed* to the lender at the end of the month will be \$600.

The borrower and lender may also negotiate payments (PMT). The ratio of these payments to the loan amount is referred to as the **pay rate**. If the borrower and lender agree that payments (PMT) to be made at the end of each month are to be equal to accrued interest, then the monthly pay rate and the monthly accrual rate are the same.[4] This means that dollar payments (PMT) will be \$600, or *exactly equal* to \$600 accrued interest. When the monthly accrual rate and the monthly pay rate are equal, the outstanding loan balance remains unchanged. So, in our example, the loan balance would remain \$60,000 at the end of the month. *We should again stress at this point that, in any given period, the pay rate and accrual rate do not have to be equal.*

Loan Amortization Patterns

In the previous section, we emphasized the relationship between accrued interest and payments on mortgage loans. We used the example of an interest-only loan which indicated that the pay rate and accrual rate were equal. When considering other loan types, we will see that the monthly accrual and pay rates are frequently *not* equal. There are many situations when lenders and borrowers consider *different* loan structures and vary the pay rate and accrual rate. In these cases, loan balances will be affected and will change depending on the difference between the two.

Differences between Accrued Interest and Payments

We now consider situations where the pay rate, and therefore, monthly *payments* are (1) greater than, (2) equal to, or (3) less than monthly accrued interest. We then consider the effect that each case has on loan balances. At this point in our discussion of **constant payment mortgage (CPM)** loans, we will use examples for fixed interest rate loans that are classified in four very general ways:

Type of CPM Loan	Pay Rate	Loan Balance at Maturity
1. Fully amortizing	Greater than accrual rate	Fully repaid
2. Partially amortizing	Greater than accrual rate	Not fully repaid
3. Interest only	Equal to accrual rate	Equal to amount borrowed
4. Negative amortizing	Less than accrual rate	Greater than amount borrowed

[4] We will use *monthly time periods* for accrued interest and payments to limit the number of possible examples. The reader should be aware that accrual and payment periods do not have to be equal.

Notice that the first loan type, which we refer to as **fully amortizing**, means that the pay rate will *exceed* the accrual rate. This means that monthly payments will *exceed accrued interest* by an amount sufficient to pay the accrued interest due each month and *fully repay the loan by the maturity date.*

The second loan type, or the **partially amortizing loan**, refers to the case when the borrower and lender agree that, like the fully amortizing loan, the pay rate will result in a payment that will *exceed accrued interest,* but not by as much as the payment for the fully amortizing loan. Therefore, the loan will *not be fully repaid at maturity.* It will be only *partially repaid.*

The third loan type, or the **interest-only loan,** is sometimes called a **zero amortizing loan.** As we have discussed, in this case the pay rate will equal the accrual rate. Consequently, the loan balance at the end of each month will remain the same as the original loan amount. The *full, original, loan amount will have to be paid at maturity.*

Finally, the fourth loan type, or the **negative amortizing loan,** represents the case where borrowers and lenders agree that the pay rate will be *less than* the accrual rate. As a result, payments will *not* equal the amount of interest due and the loan balance will actually *increase* each month. At maturity, the *loan balance will be greater than the original loan amount.*

We will now illustrate payments for each category of loan. We also should note at this point that each category of loan will have constant, or level, monthly payments. We will discuss other monthly payment patterns later in this chapter.

Fully Amortizing, Constant Payment Mortgage (CPM) Loans

The most common loan payment pattern used in real estate finance during the post-depression era, and one which is still very prevalent today, is the fully amortizing, constant payment mortgage (CPM). (**Amortization** means the process of loan repayment over time.) The CPM loan payment pattern is used most extensively in financing single family residences and, to a lesser extent, income-producing properties such as multifamily apartment complexes and shopping centers. This payment pattern means simply that a level, or constant, monthly payment is calculated on an original loan amount at a fixed rate of interest for a given term. Each monthly payment includes interest and *some* repayment of principal. At the end of the term of the CPM loan, the original loan amount, or **principal**, is completely repaid, or has been fully amortized. The lender has earned and the borrower has paid a fixed rate of interest on the monthly loan balance.

To illustrate how the monthly loan payment calculation is made, we turn to our previous example of a \$60,000 loan made at a 12 percent (nominal) rate of interest for 30 years. What are the constant monthly mortgage payments on this loan, assuming it is to be fully amortized at the end of 30 years? Based on our knowledge of discounting annuities from Chapter 3, the problem involves no more than finding the present value of an annuity and can be formulated as follows:

$$PV = \sum_{t=1}^{n} \left[\frac{PMT_t}{1 + \frac{i}{12}} \right]^t$$

where

PV = present value

PMT = payment

i = fixed nominal, interest rate on mortgage

n = number of months loan will remain outstanding because *PMT* is a constant, this is also equivalent to

$$PV = PMT \times \sum_{t=1}^{n} \frac{1}{\left(1 + \frac{i}{12}\right)^t}$$

and

$$PMT = \frac{PV}{\sum_{t=1}^{n} \frac{1}{\left(1 + \frac{i}{12}\right)^t}}$$

Solution:

$n = 30 \times 12 = 360$

$i = 12\%/12 = 1\%$ or .01

$PV = -\$60{,}000$

$FV = 0$

Solve for *PMT* = $617.17

Function:

PMT (n, i, PV, FV)

In this case, we are interested in solving for *PMT,* or the constant monthly payment (annuity) that will fully repay the loan amount (*PV*) and earn the lender 12 percent interest compounded monthly. The required payment will be $617.17.

Consider the fully amortizing loan pattern illustrated in Exhibit 4–1. The initial, relatively low principal reduction, shown in column 6, results in a high portion of interest charges in the early monthly payments. Note that the ending loan balance after the first six months (column 6) is $59,894.36; thus, only $105.64 has been amortized from the original balance of $60,000 after six months. Interest paid during the same six-month period totals $3,597.38. The explanation for the high interest component in each monthly payment is

EXHIBIT 4–1
Fully Amortizing Loan Pattern

(1) Month	(2) Beginning Loan Balance	(3) Monthly Payment	(4) Interest (.12 ÷ 12)	(5) Amortization*	(6) Ending Loan Balance
1	$60,000.00	$617.17	$600.00	$17.17	$59,982.83
2	59,982.83	617.17	599.83	17.34	59,965.49
3	59,965.49	617.17	599.65	17.52	59,947.97
4	59,947.97	617.17	599.48	17.69	59,930.28
5	59,930.28	617.17	599.30	17.87	59,912.41
6	59,912.41	617.17	599.12	18.05	59,894.36
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.
358	1,815.08	617.17	18.15	599.02	1,216.06
359	1,216.06	617.17	12.16	605.01	611.06
360	611.06	617.17	6.11	611.06	-0-

*Amortization increases each month by the factor $1 + i/12$; that is, 17.17(1.01) = 17.34, and so on.