Chapter 4
Best Practice in Mathematics

THE WAY IT USED TO BE

We often ask adults to write their math autobiographies, and their stories would make a grown man cry. They struggled through a labyrinth of incomprehensible symbols and rules, memorizing facts and procedures. They remember their panic when called upon to go to the chalkboard to compute $2\frac{1}{2}$ divided by $\frac{5}{12}$. "Ours is not to reason why; we just invert and multiply." Many of these adults are now parents, and they not-so-subtly send a message to their children: "Math is hard. I never could understand it. Gee whiz, I can't even balance my checkbook."

Is mathematics so inherently difficult that only a few who are "wired" for math can understand it? Unfortunately, most people in the United States would say yes. This erroneous view of mathematics has been prevalent for decades. Many come to believe that they are incapable of doing math. As they progress through the grades, fewer and fewer students understand and enjoy math, leaving only a handful to continue with capability and confidence. Most high school students take the minimum number of math classes needed to graduate. By college, only a small percentage of our nation's students elect to major in mathematics. Others take only the minimum courses required, despite the fact that many careers depend upon mathematical knowledge.

It does not have to be this way. We know more than ever before about human cognition and how to help students understand mathematics. Here is how one middle school teacher, Katie George, used "Chocolate Algebra," an activity from Arthur's course on teaching algebra, and implemented it in her classroom at Daniel Wright Junior High School in Lincolnshire, Illinois.

TEACHING MATHEMATICS A BETTER WAY: CHOCOLATE ALGEBRA

Armed with a giant bar of chocolate and a king-sized box of Tootsie Rolls, I prepared my initial attack on linear equations with my seventh-grade students. My objective for the day was simple: introduce linear relationships, one of the cornerstones for beginning algebra students. From the several times I'd done this activity, I was keenly aware that Chocolate Algebra hinges on careful pacing and precise questioning. I planned two 44-minute class periods for this activity.

I began by posing a problem to my class: "If you have $10 to spend on $2 Hershey Bars and $1 Tootsie Rolls, how many ways can you spend your money without receiving change? All chocolate, no change—tax included." I had determined in advance the items to purchase and the dollar amounts to spend so that the tables and graphs would reveal patterns readily. Bringing in props such as large candy bars was very motivating for the students and provided concrete representations.

The students quickly began generating solutions. As expected, most randomly jotted down any combination that popped into their minds. As they shared solutions, it became apparent that we needed an organizational system. With scheming intent, I suggested they each make a simple two-column table (or T-chart) to keep the combinations in order. We decided as a class to label the left-hand column "Number of Hershey Bars" and the right-hand column "Number of Tootsie Rolls."

The combinations elicited from the class were not arranged in any particular order. I asked them, "Did you find all the possible combinations?" To make it easier to answer, we agreed to purchase the most $2 Hershey Bars that we could as a place to start ("the most of the bigger item") and decrease the number of Hershey Bars one at a time. The first row of our table showed that 5 Hershey Bars and 0 Tootsie Rolls were purchased. The second row had 4 Hershey Bars and 2 Tootsie Rolls. The class quickly saw patterns and the table was complete in a matter of minutes.

We spent a lot of time talking about patterns in the tables. By using a different color pen to highlight one pattern, I asked the class to explain to me how the numbers changed in the table. "The left side goes down by one and the right side goes up by two," one student exclaimed. "Why?" I asked. Another student asked, "Will the left side always go down and the right side always go up?" Yet a different student asked, "Will the numbers at the top always make the pattern?" I countered, "Where does the pattern come from?" Many students volunteered that because the Hershey Bar was twice as expensive, there would be a 2:1 relationship. "Oh, the Hershey Bar is exchanged for two Tootsie Rolls!" one student said as a big light bulb appeared over her head. As I expected, the vast majority of the students now understood the tables and could recognize the patterns.

To extend the concept, I had them try to buy $1 Tootsie Rolls and $5 Toblerone Bars with $27 dollars and, finally, $5 Toblerones and $2 Hershey Bars with $37. Again, they made tables and discussed the patterns they saw. These students had worked with equations having one variable but this would be their first classroom experience with two, so I wanted to tread very carefully. I ended by asking, "Is there another way to represent this situation?"

To start the second day, we pulled out our tables again and reviewed the situation from the previous class. Out to the side of the table on the board, I wrote each solution as a coordinate point with parentheses and a comma. "Does this format remind you of anything you have seen before?" I inquired. "Yes, it is for graphing," several students replied. I spent a couple of minutes reviewing the basics of graphing for those who needed a reminder. "Let's see what happens if we use our table as a collection of coordinate points and put them on a graph," I said. They each graphed the points but seemed thoroughly unimpressed. I could see "so what?" written all over their faces.

"Take your pencil and put it on the point (0, 10). Let's say we want to go from this point to (1, 8), the next point down, but our pencil has to stay on the lines like in a video game. Can someone tell me how to move my pen so that it will be on the point (1, 8)?" I prodded. "Move down," someone called out. "How far should I move?" I responded. "Move down two and over one," a student directed. Almost instantaneously, most of them realized what was going on. "That's the pattern in the table! Cool!" As if I had performed a magic trick, my pre-algebra classes delighted in watching the pattern from the table reappear in the graphs. We moved down two and over one until our pencils were on the point (5, 0). "But our table said up two and down one. It is backward," Jackie whined. "What happens if we go in the other direction? Can we go back to point (0, 10) using a different path?" I inquired, knowing that they were not quite ready for slope but hoping that this would lay a nice foundation for them.

After playing around with the graph and the table for a few minutes, the class had a good initial understanding of the connection between the table and the graph. In fact, they could see that there was a relationship between the number of Hershey Bars and the number of Tootsie Rolls. The word *cool* was completely overused in my classroom over the course of those two days.

Looking at the tables, I hinted that perhaps we could make an equation from this information. Given the look on their faces, I knew that my suggestion had been a huge leap from a candy-purchasing example to the mysterious world of mathematics. How could we bridge the expanse? "Let's look at our first table. There is a lot going on here. What numbers are always staying the same?" Without much hesitation, the kids recognized that the price of the candy ($2 and $1) and our budget ($10) always stayed the same. On the board, I wrote down those numbers ($2 $1 $10) with space between them for the symbols and variables that I was hoping they would produce.

"Let's look at the table again. What is changing? What is different in every row or every situation?" I asked. A bit more hesitation than the first question, but they recognized that the number we were buying was changing. "What can we do to show that a number is changing or that we don't know what the number will be?" I inquired. "Use a variable," Sunny said. "*X*," Jackie contributed.

"Think about the relationship between the price of the candy, the quantity that we purchased, and our total budget. How can we add variables to the numbers I wrote down to show that the amounts change?" I asked. Most of the students wanted to use *H* for the number of Hershey Bars and *T* for the number of Tootsie Rolls.

"Where should we put the *H* and the *T*?" I pushed. The students decided on the variables and their placement to come up with the following equation: $\$2\ H + \$1\ T = \$10$. "How do we know that this will work? Let's look back at the table," I directed. Derek explained that as we made the table we had multiplied the number of Hershey Bars by 2 and the number of Tootsie Rolls by 1 and we had to be certain that these two amounts added up to 10. He explained that we got our equation right from the table.

At the end of the lesson, I asked them to think about what we had done and what they had learned. Many mentioned the use of a table and "the most of the bigger." Others noticed that we were using variables but not solving for them the way that we had in the past. A few students thought we were just buying candy. A few were not sure how we got our equation. This was an introductory activity, requiring follow-up and extensions. For homework, the kids needed to come up with their own problem with a budget and two items to buy. They should make a table, a graph, and try to write an equation. Most students delighted in creating their own problem and created a table with ease. Quite a few were able to make a graph to go along with their table, but the biggest challenge was in writing an equation. As this is the most abstract part of the activity, it did not come as a surprise that equation writing was the biggest challenge.

Chocolate Algebra has many, many layers, easily extended or modified; I just change the objects and their costs. It is a fabulous springboard for a unit on linear equations. It can be used in various formats throughout a school year as a way to expand on concepts over time. All my students, regardless of their level, had a revelation at some point.

A LOOK AT THE STANDARDS DOCUMENTS

In 1989 the National Council of Teachers of Mathematics (NCTM) released their landmark document *Curriculum and Evaluation Standards for School Mathematics*. It was followed by *Professional Standards for Teaching Mathematics* (1991), *Assessment Standards for School Mathematics* (1995), and twenty-two addenda booklets that addressed mathematical topics at various grade levels. Taken collectively, the NCTM standards and their related materials offer a significantly broadened view of the nature of mathematics, what it means to know mathematics, how students can learn mathematics, and what kinds of teaching practices best foster this learning.

The influence of the original standards has been substantial. The National Science Foundation (NSF) funded the development of a dozen new curriculum programs that embodied the standards. Most commercial publishers of mathematics textbooks in various ways incorporated these standards in their programs. However, the NCTM standards also stimulated a backlash, dubbed by the media the Math Wars, not unlike the Reading Wars we talked about in Chapter 2. When planning to revise the standards, the NCTM solicited input from a wide range of sources, including many of its most vocal critics. The council published a revision of the standards that integrated the three previous documents into one, *Principles and Standards for School Mathematics* (2000). It did not entirely satisfy the critics.

The *2000 NCTM Standards* offer a vision for mathematics based on six major principles

1. **Equity** (maintaining high expectations and support for *all* students).

2. **Curriculum** (articulating coherent, important mathematics across the grades).
3. **Teaching** (challenging and supporting students in building new knowledge).
4. **Learning** (helping students build an understanding of mathematics by actively creating meaning by connecting new knowledge with their prior knowledge).
5. **Assessment** (supporting the learning of important mathematics through formative and summative assessment of what students actually understand).
6. **Technology** (expanding the mathematics that can be taught and enhancing student learning).

In *NCTM 2000* these principles are applied to the **ten standards** for grades K-12. Five **content** standards address the familiar branches of mathematics, and five **process** standards describe the interrelated aspects of cognition that build understanding of concepts.

The ten standards are explained in a global fashion for grades pre-K–12. Then each standard is examined in detail in four grade-level bands (pre-K–2, 3–5, 6–8, and 9–12). **Expectations** of what students should understand, know, and be able to do for each of the five content standards for each grade level are provided in a ten-page appendix. These expectations are a great resource to those developing curriculum frameworks.

Content Standards	Process Standards
Number and Operations	Problem Solving
Algebra	Reasoning and Proof
Geometry	Communication
Measurement	Connections
Data Analysis and Probability	Representations

NCTM 2000 presents additional concepts in the five content standards that were not in the curriculum of prior generations. Also, many concepts are represented and connected in new and exciting ways. The five process standards are drawn from extensive research on human cognition and mathematics. It is our job as teachers to help students learn how to use these processes appropriately to develop the mathematical knowledge described in the content standards.

QUALITIES OF BEST PRACTICE IN TEACHING MATHEMATICS

Teachers should help *all* students understand that mathematics is a dynamic, coherent, interconnected set of ideas. Unfortunately, few teachers, let alone students, have experienced mathematics this way. Most students and adults see mathematics as a collection of unrelated topics, theorems, procedures, and facts. Study after study for the past twenty-five years has found the mathematics curriculum of the United States to be narrowly focused on procedures and facts, not concepts, and highly repetitive, with significant overlap and review from year to year—sometimes covering a topic in the same superficial manner for four or five years in a row.

In order to see mathematics as a coherent whole, one must realize that although numbers and computation are an important part of mathematics, they are only one part. *Mathematics is the science of patterns.* Mathematical concepts describe patterns and relationships. A concept is an abstract idea that explains and organizes information. Mathematicians look for relationships among ideas and try to see patterns in these relationships. Every branch of mathematics (e.g., geometry, probability) has its own patterns. Expert mathematicians use abstract, symbolic notation to describe the patterns they conceive.

The *2000 NCTM Standards* call for the creation of a mathematics curriculum for all students that includes familiar strands but also addresses big ideas, such as patterns, dimension, quantity, uncertainty, shape, and change. These big ideas anchor the important concepts of mathematics as well as terminology, definitions, notation, and skills. Teachers can promote coherence by emphasizing big ideas and helping students see the connections among concepts.

Yes, but . . . is this kind of mathematics teaching really possible?

Sure. Many other countries do mathematics this way. Comparisons of U.S. mathematics curricula with the five top-scoring countries (respectively, Singapore, Korea, Chinese Taipei, Hong Kong, and Japan) in the Third International Mathematics and Science Study (TIMSS) revealed that they focused more on reasoning and understanding concepts, while the U.S. schools stuck more to procedures and facts. The curricula of high-scoring countries had more in-depth study of fewer topics each year (e.g., ten in Japan) compared to the U.S., which had superficial coverage of thirty to thirty-five topics. These high-scoring countries included significant amounts of algebra and geometry in grades six through eight, with the expectation that all students would learn these topics. This expectation contrasted sharply with the finding that 80 percent of U.S. eighth graders study almost exclusively arithmetic topics, with little coverage of algebra and virtually no geometry. The TIMSS authors said the U.S. mathematics and science curricula are "a mile wide and an inch deep" (Schmidt, McKnight, and Raizen 1998, 1).

The goal of teaching mathematics is to help all students understand concepts and use them powerfully. Students should develop true understanding of mathematical concepts and procedures. They must come to see and believe that mathematics makes sense, that it is understandable and useful to them. They can become more confident in their own use of mathematics. Teachers and students must come to recognize that mathematical thinking is part of everyone's mental ability, and not confined to just a gifted few.

Research in cognitive psychology over the past twenty-five years has consistently shown that understanding increases the ability to learn, remember, and use mathematics (Bransford, Brown, and Cocking 2000). When students learn with understanding, they are able to use their new knowledge flexibly, making connections to new situations. Furthermore, developing a deep, connected understanding of mathematics promotes the learning of computational skills.

Yes, but . . . do you really believe that all or even most students can understand math?

Many more students are capable of learning and understanding more mathematics than previous generations ever thought possible. Conceptual understanding does not come from a teacher *telling* students what a concept is. Concepts are *built* by each person; understanding is created. Students have to explore many examples and talk about what they see and think, as well as hear explanations from the teacher.

In Japan, teachers' primary concern is helping students understand mathematical concepts. The additional time gained by in-depth attention to fewer ideas allows the teachers to help students examine mathematical relationships in depth. The TIMSS research found that in Japan more than half (54 percent) of the problems that students worked on emphasized making connections among many mathematical concepts, versus an anemic 17 percent in the United States. What were American teachers doing? More than two-thirds of their problems emphasized procedural skills. When challenging problems were addressed, the Japanese teachers *required* students to discuss solutions to make connections; *none* of the American teachers in the study did so. In fact, a third of time the teachers *just gave the answer.* It is not surprising to find that the average amount of time spent on a problem was fifteen minutes in Japan and five minutes in the United States (Hiebert et al. 2003).

Teachers in Japan focused on developing new concepts and solving problems that reveal concepts; they spent 60 percent of class time on new content (compared to less than 25 percent by the Americans). Instead of working with new content, the American teachers spent over half the class time on review (versus less than one-fourth of the time by the Japanese). An astonishing 28 percent of U.S. classes were devoted *entirely* to review (versus only 5 percent of Japanese classes). American teachers focused much more heavily on memorizing, although interviews revealed that many American teachers thought they were teaching for understanding. American students were practicing skills while Asian students were thinking. Clearly, the Japanese teachers believed their students could understand, and they did. In contrast, American traditionalists want us to go "back to basics." *We never left.*

Five intertwined processes build mathematical understanding. Teaching for conceptual understanding means helping students build a web of interconnected ideas. Teachers provide experiences for students in which they actively engage in these key processes:

- making connections
- creating representations
- using reasoning and developing proofs
- communicating ideas
- problem solving

Teachers help students *make connections* to their prior mathematical knowledge, between related mathematical concepts, and between concepts and procedures. They help students build bridges between situations or contexts that may appear different but are examples of the same concept. They help students realize the connections between different representations of a problem, which is especially important in moving from concrete to more abstract representations.

A skillful teacher is always juggling examples and explanations. For students to see patterns or to develop true conceptual understanding, they will need many more examples than are provided in the textbook. Presentation of an explanation, no matter how brilliantly worded, will not connect ideas unless students have had ample opportunities to wrestle with examples. An explanation must have something to which it connects.

Making connections requires *reasoning.* Teachers should provide experiences so that students can make and investigate mathematical conjectures, select and use various types of reasoning (inductive pattern finding, deductive logic), and develop and evaluate mathematical arguments and proofs. Reasoning mathematically is essentially a habit; it is developed by use in a variety of contexts. When students believe that mathematics is supposed to make sense, that patterns can be uncovered, and that they can justify the results of their investigations, they are more willing to develop the habit of reasoning.

Problem solving is an excellent vehicle for developing understanding. Traditionally, problem solving has been seen as an application of skills *after* mastery. But the *2000 NCTM Standards* show that problem solving is a means to build mathematical knowledge. Teachers should choose worthwhile mathematical problems or tasks for students to work on. How to solve these problems should not be obvious; students should have to think. The best problems are authentic, challenging, intriguing, mathematically rich, and perhaps counterintuitive. With help from the teacher to apply and adapt good problem-solving strategies, the students can attack problems and develop understanding. Teachers need to help students develop *metacognition*—being aware of their own problem-solving processes, monitoring their progress, and reflecting on their own thinking.

Teachers need to ensure that students gain experience with a variety of strategies and are able to decide when to use each one. With the most powerful strategies, students *create their own representations.* The common strategies of looking for a pattern and using logical reasoning are overarching and are essential to doing mathematics. Students must be encouraged to look for patterns and to use logical reasoning in *every* problem. But at a more specific level, students should develop capability with five critical strategies that are based on creating representations:

- Discuss the problem in small groups (language representations).
- Use manipulatives (concrete, physical representations and tactile sense).
- Act it out (representations of sequential actions and bodily kinesthetic sense).
- Draw a picture, diagram, or graph (visual, pictorial representations).
- Make a list or table (symbolic representations).

These representations build understanding of the problem (and often find a solution) because in creating them, students are developing different *mental models* of the problem or phenomena. In worthwhile tasks, students may use several of these representations, moving from one to another to figure out more about the problem. Later they might draw on supplementary strategies (e.g., guess and check; work backwards, simplify problem), but these cannot be used effectively unless one understands the problem. As students become more mathematically sophisticated, they are able to use more abstract and symbolic strategies (e.g., use proportional reasoning, apply a formula).

Students often need help from the teacher to move back and forth between representations, seeing how they are related and how each reveals something different. Recall how Katie George asked questions designed to help students see the connections between the actual chocolate items and the numbers in the table, between a row in the table and a point on the coordinate graph, and between the symbols "(4, 2)" and the situation in real life. Flexibility of translating between representations and realizing the value of each are good indicators of true understanding.

In mathematics, students should be encouraged and helped to *communicate* their ideas by using a full range of language representations—speaking, writing, reading, and listening. Communication and reflection go hand in hand. Even though symbols are used to represent the most abstract aspects of mathematics, the symbols represent ideas that are developed and expressed through language. Oral language—discussing, verbalizing thoughts, "talking mathematics" for most students, most of the time, greatly facilitates their understanding. Of course, teachers must build a safe environment in their classroom where students believe they can freely express their ideas without negative consequences for mistakes.

Math journals provide another opportunity for students to use language to express and justify their reasoning and ideas. They can describe how they solved a problem, why they used a particular approach or strategy, what assumptions they made, and so forth. When they have to explain a mathematics concept in their own words, students have to think and rethink what is really important. With feedback from the teacher, they begin to move from the specifics of each activity to more general and abstract conceptions, expressed more precisely in mathematical language. Eventually, children's mathematical language, oral and written, becomes a powerful tool for thinking, helping them create *models*—mental maps used to organize their world, solve problems, and explore relationships.

All students should understand and be able to use number concepts, operations, and computational procedures. *NCTM 2000* defines the term *computational fluency* as "having and using efficient and accurate methods for computing" (NCTM 2000, 32). "Developing fluency requires a balance and connection between conceptual understanding and computational proficiency. . . . [S]tudents must become fluent in arithmetic computation—they must have efficient and accurate methods that are supported by an understanding of numbers and operations" (NCTM 2000, 35). Five critically important processes that lead to understanding, proficiency, and fluency need to be developed in many different contexts to gain generalized understanding. They are explained on page 119. When students have many successful experiences using five processes, remembering math facts becomes a simple matter.

Yes, but . . . aren't these ideas controversial?

There is definitely a controversy. Groups of parents, along with a number of mathematicians and scientists, formed a group called Mathematically Correct (MC), and they have spoken out against the NCTM standards. With the help of the Internet, they organized opposition across the United States. In letters to the editor and over the Web, they have posted horror stories of bad math teaching, attributed to the standards. Their website has a hundred "papers"—an amazing collection of half-truths, misconceptions, and rhetoric. The MC folks refer to themselves as traditionalists and criticize NCTM for:

- having students derive math facts and rely on calculators instead of memorizing basic math facts
- having students invent procedures instead of learning traditional algorithms
- focusing on problem solving; cooperative, small groups; and discovery instead of direct instruction
- promoting a curriculum that is "soft and fuzzy" and dumbed-down, with too much fun and games in place of "rigor"

The MC supporters are saying the same things that traditionalists have said for most of this century. From the MC website we read: "'Understanding' is a complex, poorly understood process that involves linking multiple stored 'chunks' of knowledge. We have no idea how this magical process occurs."

This quote would be news to the phalanx of cognitive psychologists whose illumination of human "understanding" is described in Bransford, Brown, and Cocking (2000). The MC traditionalists appear to be unaware of the research on cognition showing how concepts are more easily developed, reflected upon, and understood from rich experiences than from facts and procedures that are memorized, but not understood.

This controversy has a long history. "Drill does not develop meaning. Repetition does not lead to understanding," wrote William Brownell in 1935 (10). "[Algebra] presents mechanical processes and therefore forces the student to rely on memorization rather than understanding. . . . On the whole, the traditional curriculum does not pay much attention to understanding," wrote mathematician Morris Kline in 1973 (4–5). A group called Mathematically Sane has launched a website to counter the traditionalist critics; visit http://www.mathematicallysane.com to see a more comprehensive rebuttal.