Chapter 6

Early Operations

IN A NUTSHELL

The main ideas in this chapter are listed below:

1. The four operations—addition, subtraction, multiplication, and division—are related.
2. There are multiple meanings for each operation. Students need exposure, over time, to each of these meanings.
3. Different models should be employed to model the operations since some ideas about the operations are better explained with one model than another.

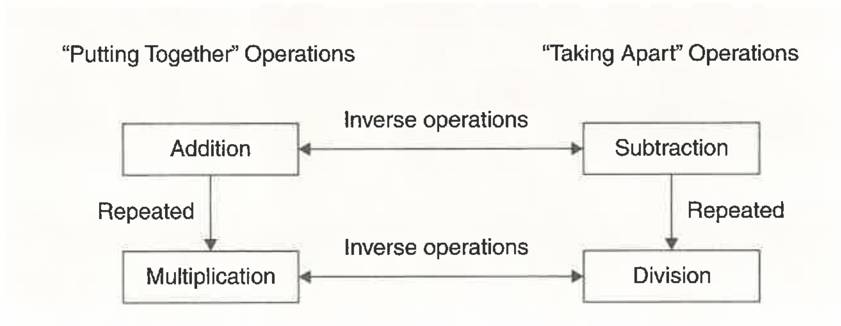
CHAPTER PROBLEM

A division problem includes these words:
- bananas
- most
- 24
- less

What could the problem be?
How could you solve it using addition instead of division?

How the Four Operations Are Related

The four operations that are the focus of K–8 mathematics are addition, subtraction, multiplication, and division. The picture below shows how they are related.



TEACHING TIP

Students might be asked to solve subtraction, multiplication, or division questions using more than one operation to reinforce the relationship between the operations.

Frequently, the same problem can be viewed using any of the four operations. For example, consider this problem:

Rafi had 12 cookies on a plate. If he ate them 3 at a time, how many times could he go back for cookies?

- Using addition: I'll add 3s until I get to 12: $3 + 3 + 3 + 3 = 12$. Solution: 4 times.
- Using subtraction: I'll subtract 3s until there are 0 cookies left: $12 - 3 - 3 - 3 - 3 = 0$. Solution: 4 times.
- Using multiplication: I'll figure out what to multiply by 3 to get 12. $4 \times 3 = 12$. Solution: 4 times.
- Using division: I'll divide 12 by 3: $12 \div 3 = 4$. Solution: 4 times.

Addition and Subtraction

Teachers generally introduce addition and subtraction formally before multiplication and division. Because it is is important to foster an understanding of the connection between addition and subtraction, it is a good idea to teach them together, helping students see that wherever there is an addition situation, there is an implicit subtraction situation, and vice versa.

Meanings of Addition

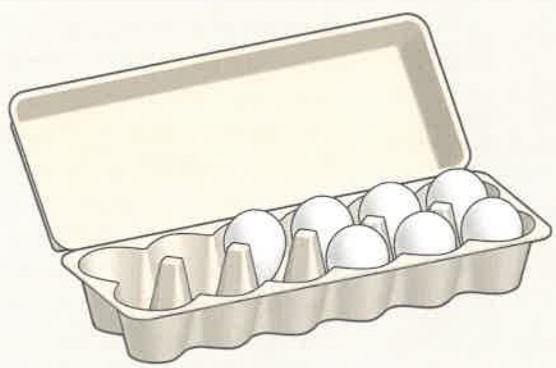
Although addition always relates to the situation of combining things, students find it easier to first consider active situations, where a joining actually occurs, and later more static situations, where a whole is made up of two parts. Sometimes the part that is added is based on a comparison. For example, Ben had 3 more books than Andrea. Andrea had 6 books. How

MEANING	EXAMPLE
Joining is an active addition situation.	A child has 5 marbles. His mom then brings him 3 new ones. 5 marbles + 3 new marbles = 8 marbles
Part-part-whole is a static addition situation where no action takes place.	A child has 8 marbles; 5 of them are blue and 3 are red. No action occurs, but it is still an addition situation. 8 marbles = 5 blue marbles + 3 red marbles

many did Ben have? In this case, the two parts are what Andrea had and the extra that Ben had. In all of these situations, the numbers you add are called *addends* and the result is called a *sum*.

Meanings of Subtraction

Subtraction is a more complex operation than addition. Most simply, it is the opposite of addition, but there are many nuances.

MEANING	EXAMPLE
Taking away is an active separating situation.	 12 take away 7 leaves 5. 12 − 7 = 5
Comparing two quantities involves subtracting one from the other.	 12 is 5 more than 7. 12 − 7 = 5
Missing addend involves finding out how much or how many to add.	 12 is 5 more than 7. 12 − 7 = 5 or 7 + □ = 12

It is important for students to realize that these varying meanings for subtraction are related; in other words, it makes sense to apply the same operation in each situation. Therefore, a teacher must ensure that the connections are explored. One example is described below.

> **TEACHING TIP**
>
> Although the missing addend meaning most directly relates subtraction to addition, the relationship can and should be brought out with takeaway and comparison situations as well.

How Is Taking Away Like Comparing?

Consider the situation modeled and described below. To compare the number in the top row (*the minuend*) with the number in the bottom row (*the subtrahend*), you are finding how many more are in the top row than in the bottom row. You can take away the 3 counters that match in each row and what is left (*the difference*) is how many more were in the top row. So to compare to find how many more, you can take away.

Anne has read 9 books.
B.J. has read 3 books.
How many more books has Anne read than B.J.?

Understanding Addition and Subtraction Situations

The chart that follows, a variation of the chart advocated by Carpenter et al. (1999) and the center of work in Cognitively Guided Instruction, summarizes the many different addition and subtraction situations students will encounter. Some involve actions, like joining or taking away (separating), some involve recognizing the parts that make up a whole, and some involve comparing. Sometimes the whole and one part are known and the other part must be determined; other times the two parts are known and the whole must be determined.

Each situation can be represented by either an addition or a subtraction number sentence, or equation, with one number missing, and each is solved either by adding or by subtracting the two known quantities. When equations are introduced can vary. It is important that students have an understanding of what the equal sign represents before these equations are used. It is good if sometimes the computation part is shown on the left and the answer on the right, and sometimes the other way around, as is done in the chart below. What is interesting is that sometimes an addition sentence can be solved by subtracting and sometimes by adding; the same is true of subtraction sentences.

It is important to ensure that students encounter a wide variety of these structures to help them construct a more complete understanding of addition and subtraction. The different meanings for the operations can be distinguished only through appropriate contexts. Note that the term *separating* is used to mean *take away*.

ACTIVITY 6.1

We normally leave out one number in an addition or a subtraction sentence. Leaving out two numbers is a more open approach and leads to many alternative responses. For example, pose this question:

I have some books to start with. I get some more. Now I have 8. How many did I start with, and how many did I get?

Critical Thinking

ADDITION AND SUBTRACTION SITUATIONS

Joining situations	Result (sum) is unknown *I have 3 books to start with. I get 1 more. How many do I have now?* $3 + 1 = \square$ $a + b = \square$ To solve, you can add $3 + 1$.	Change (addend) is unknown *I have 5 books to start with. I get some more. Now I have 10. How many more did I get?* $5 + \square = 10$ $a + \square = c$ To solve, you can subtract $10 - 5$.	Start (addend) is unknown *I had some books to start with. I got 3 more. Now I have 8. How many did I start with?* $\square + 3 = 8$ $\square + b = c$ To solve, you can subtract $8 - 3$.

ADDITION AND SUBTRACTION SITUATIONS (Continued)

Separating situations	Result (difference) is unknown *I had 6 books. I gave away 3. How many do I have left?* $6 - 3 = \square$ $c - a = \square$ To solve, you can subtract $6 - 3$.	Change (subtrahend) is unknown *I had 6 books. I gave away some. I have 2 left. How many did I give away?* $6 - \square = 2$ $c - \square = b$ To solve, you can subtract $6 - 2$.	Start (minuend) is unknown *I had some books. I gave away 4. I have 6 left. How many did I start with?* $6 = \square - 4$ To solve, you can add $6 + 4$.
Part-part-whole situations	Whole (sum) is unknown *I have 3 small books and 4 big ones. How many books do I have altogether?* $3 + 4 = \square$ $a + b = \square$ To solve, you can add $3 + 4$.	Part (addend) is unknown *I have 7 books altogether. 3 of them are small books. How many are not small books?* $7 = 3 + \square$ $c = a + \square$ To solve, you can subtract $7 - 3$.	Both parts (addends) unknown *I have 7 books. Some are small and some are big. How many of each could I have?* $7 = \square + \square$ To solve, you can find combinations that make 7.
Comparing situations	Difference is unknown *I have 7 big books and 2 small ones. How many more big books than small ones do I have?* $7 - 2 = \square$ $c - a = \square$ To solve, you can subtract $7 - 2$.	Comparing quantity (subtrahend) is unknown *I have 7 big books and some small ones. I have 2 more big books than small ones. How many small books do I have?* $7 - \square = 2$ $c - \square = b$ To solve, you can subtract $7 - 2$.	Referent quantity (minuend) is unknown *I have some big books. I have 3 small books. I have 2 fewer small books than big books. How many big books do I have?* $2 = \square - 3$ $b = \square - a$ To solve, you can add $2 + 3$.

Solving Addition and Subtraction Number Stories

Students must decode and interpret number stories and not just rely on looking for "clue words" to decide what operation to perform. Notice that the word *more* appears both the join result unknown situation in the Addition and Subtraction Situations chart, which involves adding the two given numbers, as well as the join change unknown situation, which can involve subtracting the two given numbers.

Generally, but not always, students find joining and separating (take-away) situations easiest to deal with compared to other addition and subtraction situations. If students are encouraged to model number stories as they are presented, however, even young children can deal with part-part-whole and comparing situations. Students should always be encouraged to check to see if their answers seem reasonable.

ACTIVITY 6.2

Provide students with a "script," for example, an expression like $4 + 6$ or an equation like $12 - 5 = 7$, and ask them, with a partner, to act out a "play" to show that script. They should be encouraged to use a variety of situations that are meaningful to their own experiences, rather than merely copying a familiar form over and over.

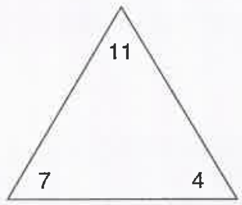
Creative Thinking

Relating Addition and Subtraction

Notice that any addition situation can also be viewed as a subtraction one, and vice versa. In fact, what we call a fact family is a set of number sentences, or equations, that can be used to describe the same situation.

ACTIVITY 6.3

You might provide triangle cards like this one and ask students to describe the four facts in the fact family that go with them. Then ask why sometimes there are only two facts (e.g., with 10, 5, and 5).



Building Number Sense

Fact Family for 2, 3, and 5

$3 + 2 = 5$	3 blue + 2 red = 5 altogether
$2 + 3 = 5$	2 red + 3 blue = 5 altogether
$5 - 2 = 3$	5 altogether; 2 are red, so 3 must be blue
$5 - 3 = 2$	5 altogether; 3 are blue, so 2 must be red

Addition and Subtraction Principles

There are a number of principles about addition and subtraction that students need to learn and discuss. Some of them are restatements of what mathematicians call properties inherent in our number system; others are built from those properties.

The idea is not to memorize the principles by name, but to be so familiar with our number system, that they are used naturally and informally. These principles are meant to become intuitive and second nature. These will lead to computational strategies that are discussed further in Chapter 7.

ADDITION AND SUBTRACTION PRINCIPLES

1. Addition and subtraction "undo" each other. They are related inverse operations. For example, if $4 + 8 = 12$, then $12 - 8 = 4$.
2. You can add numbers in any order (the commutative property). With subtraction, the order in which you subtract the numbers matters.
3. To add three numbers, you can add the first two and then the last one, or the sum of the last two to the first one (the associative property). For example, to add $5 + 4 + 3$, you can add $5 + 4$ to get 9 and then add 3, or you can add $4 + 3$ to get 7 and then calculate $5 + 7$. This property exists since addition is actually defined in terms of how to combine two numbers, so rules for three numbers had to be created.
4. You can add or subtract in parts. For example, $2 + 5 = 2 + 4 + 1$; $8 - 5 = 8 - 3 - 2$.
5. To subtract two numbers, you can add or subtract the same amount to or from both numbers without changing the difference. For example, $12 - 7 = (12 + 1) - (7 + 1)$.
6. When you add or subtract 0 to or from a number, the answer is the number you started with.
7. When you add 1 to a number, the sum is the next counting number. When you subtract 1 from a number, the difference is the counting number that comes before.

TEACHING TIP

Using the formal terms *commutative property* and *associative property* is not essential. Sometimes students make up names that are more meaningful to them. For example, some students call the commutative property the *turn-around property*.

It is equally important not to overwhelm children with too many strategies all at the same time. The principles should come out over time.

Principle 1: Addition and subtraction "undo" each other. They are related, but inverse, operations.

Mathematicians define subtraction as the inverse operation to addition; that is, it undoes what addition accomplishes. Suppose you start with 4 items, and you add 8 more and end up with 12. How do you get back to where you started at 4? You subtract the 8 you added. Similarly, if you start with 12 items and take away 4, you get back to the starting number, 12, by adding the 4 you took away.

This relationship allows students to use an addition table to solve a subtraction problem. They look in the 4 row of the table for the number 12 to determine $12 - 4$ (by looking up at the column header where 12 is found).

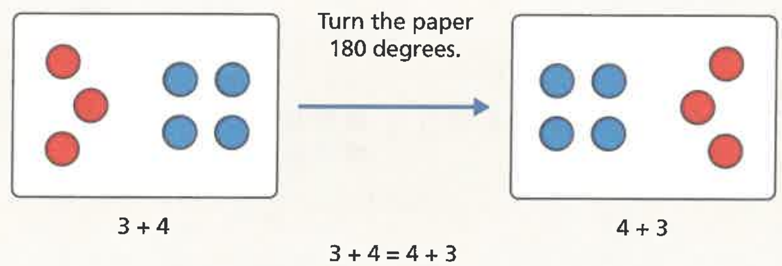
ACTIVITY 6.4

Students might have visuals of a variety of items with price tags with amounts 10¢ or lower. They are told that they started with 10¢, they bought one of the items, and how much is left. They have to decide what was purchased. For example, if 6¢ was left, they have to realize they bought a 4¢ item since $10 = 4 + 6$.

Building Number Sense

Principle 2: You can add numbers in any order (the commutative property). With subtraction, the order in which you subtract the numbers matters.

The commutative property, or order property, can be modeled and explained using counters. Show a group of 3 counters and a group of 4 counters on a piece of paper, and then turn the paper 180 degrees. It is clear that you have exactly the same counters, so $3 + 4 = 4 + 3$.



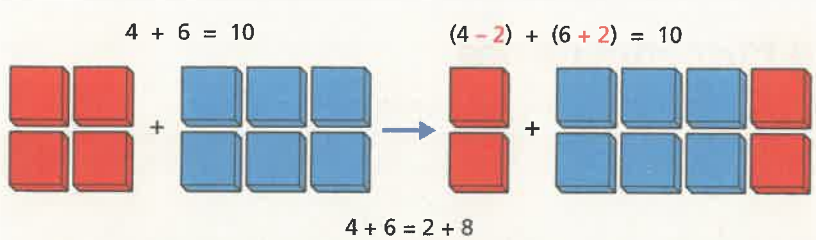
Modeling to explain the commutative principle of addition

This property also shows up in the addition table when students notice that any row is equivalent to a matching column.

However, order does matter when you subtract. For example, if you take 3 cookies away from 7, you have 4 left. You cannot take 7 cookies away from 3, although later students will learn that you can calculate $3 - 7$, using integers.

Principle 3: To add three numbers, you can add the first two and then the last one, or the sum of the last two to the first one (the associative property).

As a consequence of this property, you can take away from one number and add what you took away to the other number without changing the sum. Suppose you are adding 4 counters and 6 counters as shown below; you can move any number of counters from one pile to the other without changing the total. Moving 2 counters from the left pile to the right changes the calculation to $2 + 8$, but does not change the sum. In fact, you have renamed $(2 + 2) + 6$ as $2 + (2 + 6)$.



Modeling to explain the associative principle of addition

Principle 4: You can add or subtract in parts.

Suppose you want to add 5 objects to a group of objects. It really does not matter if you add all 5 objects at once or if you do it in stages. For example, you can add 3 and then add 2; you are still adding 5 objects ($2 + \mathbf{5} = 2 + \mathbf{3} + \mathbf{2}$). The same is true for subtraction. For example, to subtract 5 objects from a group, you can take away 3 and then take away 2 more ($8 - \mathbf{5} = 8 - \mathbf{3} - \mathbf{2}$).

ACTIVITY 6.5

Pose a problem like this one:

Jane had 3 dollars and got 8 more. Kyle had 8 dollars and got some more. Now they both have the same amount. How many did Kyle get? How do you know without adding?

Building Number Sense

ACTIVITY 6.6

Students might respond to this sequence of calculations to practice addition and observe an application of the associative principle.

$9 + 1 =$

$8 + 2 =$

$7 + 3 =$

$6 + 4 =$

$5 + 5 =$

Ask: *How could knowing one of these facts help you figure out the others?*

Building Number Sense

ACTIVITY 6.7

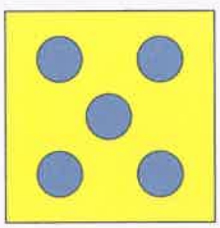
Pose the following question.

- *Choose an amount of cookies on a plate. Choose an amount of cookies that you want to eat. Tell how many would be left.*
- *Now imagine a plate with 2 more cookies. How many cookies would you need to eat this time to have the same number left? Why is that?*

Building Number Sense

ACTIVITY 6.8

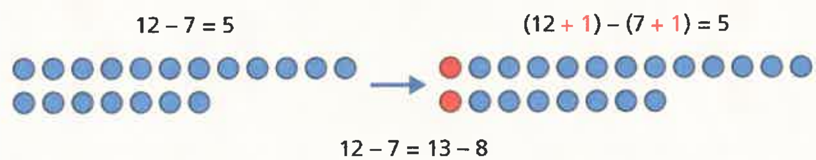
To make sure that all students have success, play a game with a small group of students where you roll a great big die that is easy for all students to see. Then say, "And □ more is?" Vary the amount □ to add, depending on students' readiness. Be sure to include the questions, "And 0 more is?" or "And 1 more is?" Students may then be ready to add larger numbers.



Building Number Sense

Principle 5: To subtract two numbers, you can add or subtract the same amount to or from both numbers without changing the difference.

If a boy is 12 years old and his brother is 7, the older brother is 5 years older than the younger one. What about next year and the year after that? How much older will he be then? He will always be 5 years older. Because the same amount is added to each age each year, the difference remains unchanged. You could model the situation as shown.



Principle 6: When you add or subtract 0 to or from a number, the answer is the number you started with.

One useful model to show this principle is a walk-on number line:

3 + 2 means start at 3 and step 2 spaces forward, ending up on 5.
3 + 0 means start at 3 and step 0 spaces forward, staying at 3.

Principle 7: When you add 1 to a number, the sum is the next counting number. When you subtract 1 from a number, the difference is the counting number that comes before.

Again, a walk-on number line is helpful for demonstrating this principle since adding 1 means to move 1 space forward, for example, $3 + 1 = 4$, and subtracting 1 means to move 1 space back, for example, $3 - 1 = 2$.

Common Errors and Misconceptions

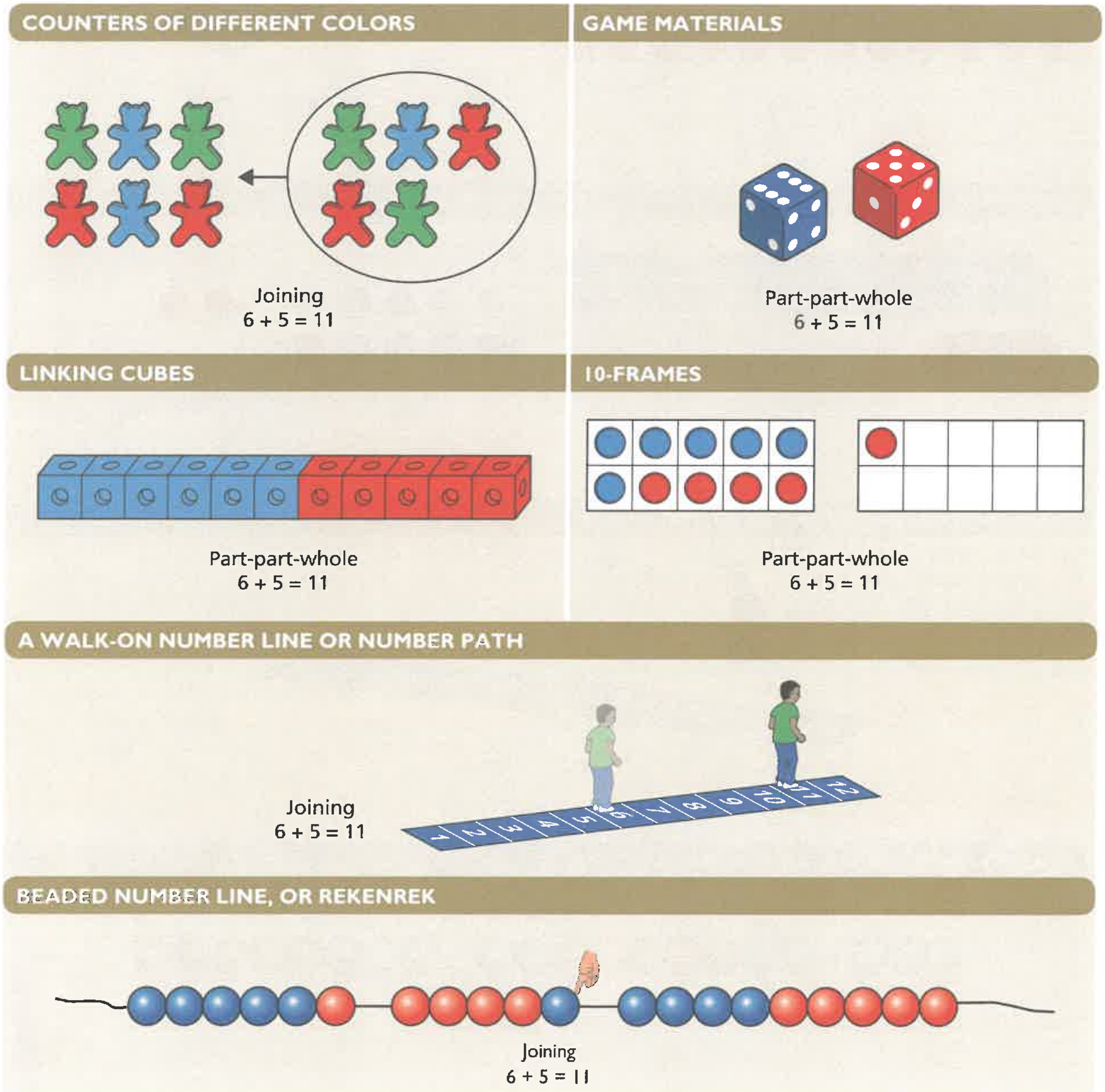
Addition and Subtraction: Common Errors, Misconceptions, and Strategies

COMMON ERROR OR MISCONCEPTION	SUGGESTED STRATEGY
Understanding Open Sentences Students are confused by open sentences with missing addends, and missing subtrahends and minuends. For example, they might interpret $3 + \square = 8$ as $8 + 3 = \square$.	Have students explain what the sentence actually says; for example, "A number is added to 3 and you end up with 8." Then ask how they know the missing number has to be less than 8.
Interpreting Walk-On Number Lines Students have difficulty using walk-on number lines to add or subtract. For example, to show $8 - 3$, some students will start at 8 but include 8 as they count back 3, landing on 6 instead of 5.	Encourage students to stand with both feet on the start number of the number step before taking their first step. This will force their first step onto the next square.

Appropriate Manipulatives

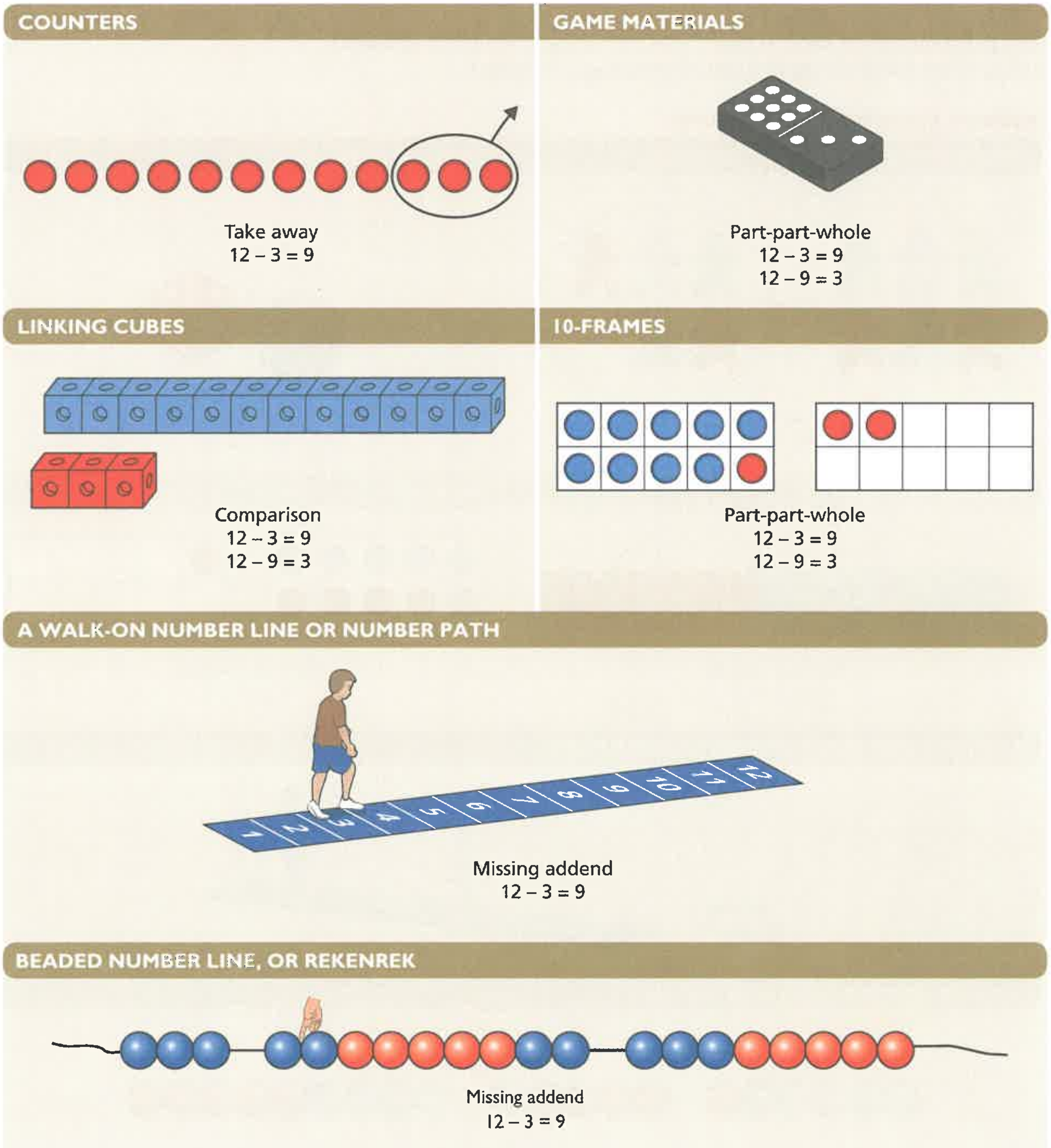
Manipulatives can be used to model both the joining (or active addition) and part-part-whole (or static addition) meanings of addition.

Addition: Examples of Manipulatives



Manipulatives can be used to model take-away, comparison, missing addend, and part-part-whole meanings of subtraction. Sometimes the meaning of subtraction is tied to the physical representation, but other times it depends on the words that are said when looking at the model. Counters of different colors are more useful for missing addend and, particularly, for comparison, than single-colored counters.

Subtraction: Examples of Manipulatives

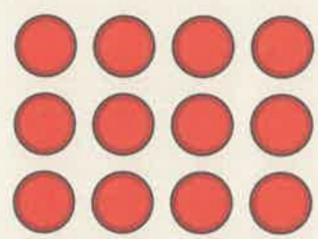
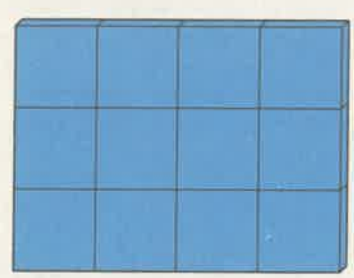


Multiplication and Division

The other two important operations in the elementary grades are multiplication and division. Because it is is important to foster an understanding of the connection between multiplication and division, it is a good idea to teach them together, helping students see that wherever there is a multiplication situation, there is an implicit division situation, and vice versa.

Meanings of Multiplication

The operation of multiplication is used in many situations. The two numbers multiplied are called *factors* and the result is called the *product*.

MEANING	EXAMPLE
Repeated Addition	The first factor, in this case 3, tells how many times to add the second factor, 4. $3 \times 4 = 4 + 4 + 4$
Equal Groups or Sets	3×4 is the total number of objects in 3 sets of 4.  $4 + 4 + 4 = 12$ $3 \times 4 = 12$ This model shows both equal groups and repeated addition.
An Array (An array is not different from an equal group situation; it is simply one that is arranged in a rectangular fashion.)	3×4 is the total number of items in a 3-by-4 array.  $3 \times 4 = 12$ An array of 3 rows with 4 counters in each row has 12 counters altogether.
Area of a Rectangle	3×4 is the area of a 3-by-4 rectangle.  $3 \times 4 = 12$ A rectangle that is 3 units wide by 4 units long has an area of 12 square units.

MEANING	EXAMPLE

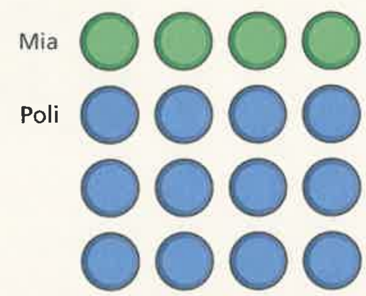
A Rate/Comparison

A rate describes how a quantity is "stretched," whether a continuous measure, like a length, or a discrete quantity, like a set of pencils. 3×4 represents the final amount when a 4-unit measure is increased to 3 times its original value. The result, 12, is compared to the original 4, by thinking of it as 3 times as much.

$3 \times 4 = 12$

A 4-unit long line stretched to 3 times its length is 12 units long.

3×4 describes how many pencils Poli has if Mia has 4 pencils and Poli has 3 times as many.

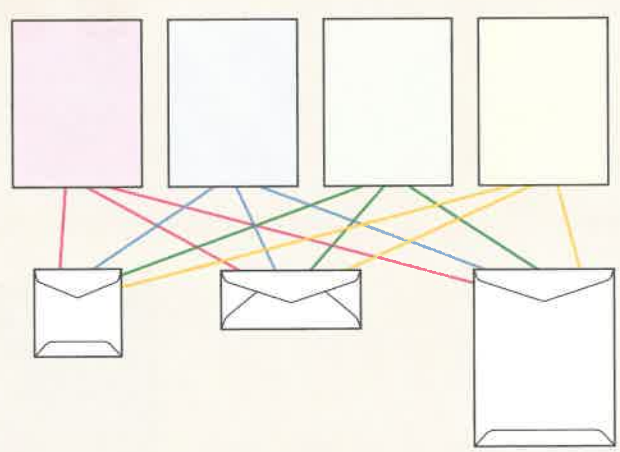


$3 \times 4 = 12$

At a rate of 3 times as many, Poli has 12, if Mia has 4.

Combinations

3×4 is the number of paper–envelope combinations possible, if there are 3 kinds of envelopes and 4 colors of paper.



$3 \times 4 = 12$

3 envelopes and 4 colors of paper make 12 different paper–envelope combinations.

Limitations of Meanings

Although all these meanings make sense when whole numbers are involved, some need to be adjusted when decimal, fraction, or integer numbers are involved. For example, $1\frac{1}{2} \times 3$ can mean to repeatedly add 3, but the second addend is only half of 3. The same adjustment needs to be made for equal sets or arrays.

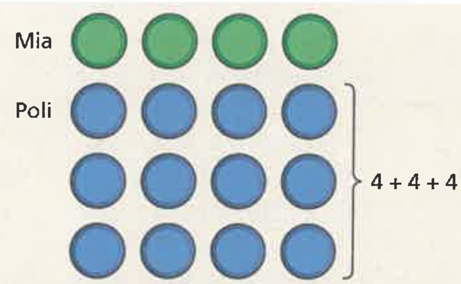
The combinations meaning is difficult to make sense of unless both numbers are whole numbers. This meaning is usually introduced later than some of the others.

Even the rate and area meanings, which make sense for whole numbers, fractions, and decimals, must be adjusted for integers. A definition of a negative rate or a negative dimension is required. This is further discussed in Chapter 13.

How the Meanings of Multiplication Are Equivalent

Simply knowing the meanings will not help students develop a well-rounded understanding of the operation. It is important for students to understand why these various meanings are equivalent and are, therefore, all related to multiplication. One such relationship is shown below.

How Is a Rate the Same as Repeated Addition or an Array? If Mia has 4 items and Poli has 3 times as many items as Mia, then Poli has 12 items (3 items for each 1 item that Mia has). This can be modeled in an array as shown (Poli's array) and can be represented by adding the number in each row using repeated addition: $4 + 4 + 4$.



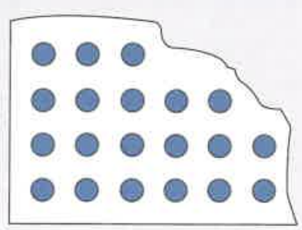
Poli's items form an array.

NUMBER TALK

Draw a picture that shows why 4×8 is double 4×4.

ACTIVITY 6.9

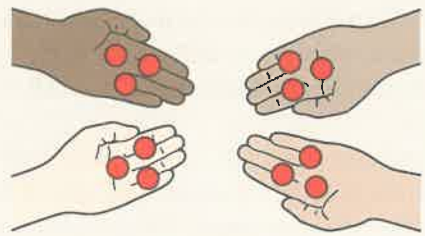
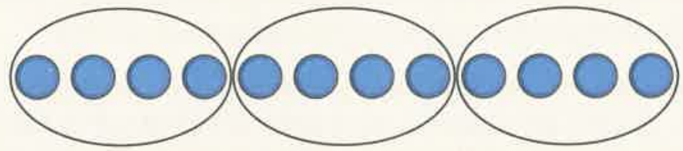
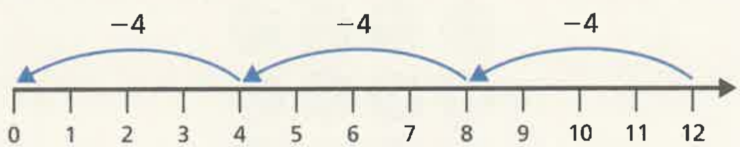
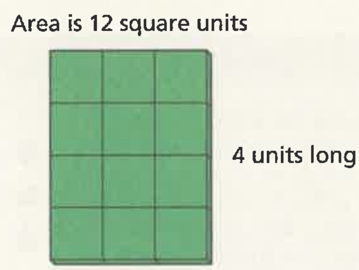
Draw arrays on cards and cut off a corner so that some dots are missing, but the numbers of rows and columns are clear. Present a card, suggesting that a dog bit off the missing corner. Ask how many dots were originally on the card, given that the array was originally complete.



Building Number Sense

Meanings of Division

When one number is divided by another, the number being divided is called the *dividend*, the other number is called the *divisor*, and the result is called the *quotient*.

MEANING	EXAMPLE
Equal Sharing (Partitive)	$12 \div 4 = 3$ is the amount each person gets if 12 items are shared equally among 4 people.  $12 \div 4 = 3$
Equal Grouping (Quotative)	$12 \div 4 = 3$ is the number of equal groups of 4 you can make with 12 items.  $12 \div 4 = 3$
Repeated Subtraction	$12 \div 4 = 3$ is the number of times you can subtract 4 from 12 before you get to 0.  $12 - 4 - 4 - 4 = 0$ $12 \div 4 = 3$
Width or Length of a Rectangle	$12 \div 4 = 3$ is the width of a rectangle with an area of 12 square units and a length of 4 units.  $12 \div 4 = 3$ units wide
Rate	$12 \div 4 = 3$ is the unit rate if 4 units make 12.  4 toy cars for \$12 If 4 toy cars cost \$12, then 1 toy car costs \$3.

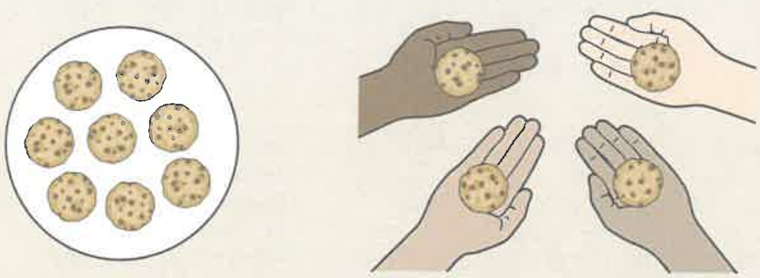
How the Meanings of Division Are Equivalent

It is important for students to understand why these various meanings are equivalent and, therefore, all relate to division. One such relationship is shown below.

How Is Equal Sharing Like Equal Grouping? Suppose you want to share 12 items among 4 people. One of the most natural ways to do this is to give each of the 4 people one item, then give each of them a second item, and then a third item, until all the items are gone. After you have given each of the 4 people his or her first item, you have created a group of 4. After you have given each of the 4 people a second item, you have created another group of 4, and so on. So, even though you are sharing, you are also creating equal groups of 4 for each round of sharing.

TEACHING TIP

Although the rate meaning of division is more likely to be used in middle school, elementary students can be asked what one item costs if the total price of several is known.



Sharing 12 cookies among 4: as you share equally, you create equal groups.

NUMBER TALK

You shared some cookies with friends. Everyone got the same number but one was left over. How many cookies and how many friends might there have been?

Understanding Multiplication and Division Situations

As with addition and subtraction, there are many types of multiplication and division situations. Each situation can be represented by either a multiplication or a division number sentence with one number missing, and each is solved by either multiplying or dividing the two known quantities. Sometimes a multiplication sentence can be solved by dividing and sometimes by multiplying; the same is true of division sentences. It is also possible to be more open-ended and use these same sorts of multiplication/division situations when two quantities are unknown, for example, $\square \div \triangle = 3$.

As with addition and subtraction, it is very important to ensure that students encounter a wide variety of these structures to help them construct a more complete understanding of multiplication and division. The different meanings of multiplication and division can only be distinguished through appropriate contexts.

MULTIPLICATION AND DIVISION SITUATIONS

Equal group situations (including equal groups and arrays)	Whole (product) is unknown *I have 3 boxes with 4 tomatoes in each. How many tomatoes do I have?* $3 \times 4 = \square$ $a \times b = \square$ Multiply to solve.	Size of group (quotient) is unknown *I have 12 tomatoes packed equally in 3 boxes. How many tomatoes are in each box?* $12 \div 3 = \square$ $c \div a = \square$ You can divide to solve.	Number of groups (divisor) is unknown *I have some boxes of tomatoes. I have 12 tomatoes altogether, and there are 4 in each box. How many boxes do I have?* $12 \div \square = 4$ $c \div \square = b$ You can divide to solve.
Area situations	Area (product) is unknown *A rectangle has a length of 4 units and a width of 2 units. What is its area in square units?* $4 \times 2 = \square$ $a \times b = \square$ Multiply to solve.	One dimension (length or width) is unknown *A rectangle has an area of 15 square units. One dimension is 3 units. What is the other dimension?* $15 \div 3 = \square$ $c \div a = \square$ You can divide to solve.	Both dimensions are unknown *A rectangle has an area of 24 square units. What are the possible dimensions if the lengths are whole numbers?* $\square \times \square = 24$ $\square \times \square = c$
Rate situations (comparison)	Greater amount (product) is unknown. *Meg has 5 pennies and I have 3 times as many as she does. How many do I have?* $3 \times 5 = \square$ $a \times b = \square$ Multiply to solve.	Lesser amount (factor) is unknown. *I have 15 pennies. I have 3 times as many pennies as Meg. How many does Meg have?* $15 = 3 \times \square$ $c \div a = \square$ You can divide to solve.	Size of comparison ratio (factor) is unknown *Meg has 5 pennies and I have 15. How many pennies do I have for each of Meg's?* $15 = \square \times 5$ $c \div b = \square$ You can divide to solve.
Combination situations	Total (product) is unknown *I have 3 shirts and 2 pairs of pants. How many shirt–pant outfits do I have?* $3 \times 2 = \square$ $a \times b = \square$ Multiply to solve.	Size of one set (factor) is unknown *I have 8 shirt–pant outfits and 4 shirts. How many pairs of pants do I have?* $4 \times \square = 8$ $c \div a = \square$ You can divide to solve.	

Notice that the sentence $\square \div 4 = 3$ is not shown in the chart. It is equivalent, though, to the first situation, whole is unknown. For example, the problem might be worded as, "I have some tomatoes. I was able to make 3 boxes of 4 tomatoes. How many tomatoes did I have?" and could be solved by multiplying 3×4.

Solving Multiplication and Division Problems

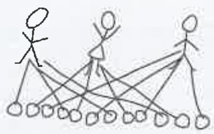
As with addition and subtraction, students should work with a wide variety of structures to help them construct a more complete understanding of multiplication and division situations. Because students find equal group situations easier to understand, combinations and comparison situations are usually encountered after equal group situations have been explored. As with addition and subtraction, the most difficult situations for students are those represented by $\square \times b = c$ and $\square \div a = b$, where the first term is unknown.

Once students are comfortable solving simple multiplication and division number stories, they may want to create their own number stories.

Use 12 counters.
Write a division sentence that describes this situation:

There are 12 cookies to share among 3 people.
How many cookies will each get?

each get 4 cookies



Student Response

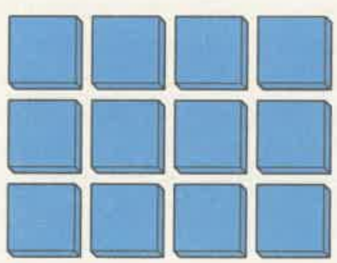
This student solves a division problem using a diagram to explain his or her thinking rather than representing it symbolically. You might follow up by asking how he or she would write this using a sentence of the form [ ] ÷ [ ] = [ ].

Even early on, students should meet and create division problems where there are remainders. For example, a student might encounter a problem where 19 books are being shared equally by 4 students. They have to figure out what it makes sense to do with the remainder.

Relating Multiplication and Division

Notice that any multiplication situation can also be viewed as a division situation, and vice versa. In fact, what teachers call a fact family is a set of number sentences, or equations, that can be used to describe the same situation.

Fact Family for 3, 4, and 12



$4 \times 3 = 12$ 4 columns of 3 squares are 12 squares

$3 \times 4 = 12$ 3 rows of 4 squares are 12 squares

$12 \div 4 = 3$ 12 squares in 4 columns form 3 rows

$12 \div 3 = 4$ 12 squares in 3 rows form 4 columns

Multiplication and Division Principles

Students need to become familiar with and use a number of principles about multiplication and division. As with the addition and subtraction principles, these principles are meant to become intuitive and not memorized; they are simply used so often they become second nature. In the explanations of the principles on the following pages, different meanings of multiplication and different models are used each time. Often, one meaning of an operation is much more helpful than another in explaining a particular idea.

As with addition, some properties of multiplication have been named by mathematicians, in particular, the commutative, associative, and distributive properties. These are identified in the chart.

TEACHING TIP

It is important to emphasize how principles that apply to multiplication sometimes, but not always, apply to division.

MULTIPLICATION AND DIVISION PRINCIPLES

1. Multiplication and division "undo" each other. They are related inverse operations. For example, if $12 \div 3 = 4$, then $3 \times 4 = 12$.
2. You can multiply numbers in any order (the commutative property). With division, the order in which you divide the numbers matters.
3. To multiply two numbers, you can divide one factor and multiply the other by the same amount without changing the product (a variation of the associative property). For example, $8 \times 3 = 4 \times 6$, $(8 \div 2) \times (3 \times 2)$.
4. To divide two numbers, you can multiply both numbers by the same amount without changing the quotient. For example, $15 \div 3 = 30 \div 6$, $(15 \times 2) \div (3 \times 2)$.
5. You can multiply in parts (the distributive property). For example, $5 \times 4 = 3 \times 4 + 2 \times 4$.
6. You can multiply in parts by breaking up the multiplier as a product (a variation of the associative property). For example, $6 \times 5 = 2 \times 3 \times 5$.
7. You can divide in parts by splitting the dividend into parts (the distributive property), but not the divisor. For example, $48 \div 8 = 32 \div 8 + 16 \div 8$. It is not equal to $48 \div 4 + 48 \div 4$.
8. You can divide by breaking up the divisor as a product. For example, $36 \div 6 = 36 \div 3 \div 2$.
9. When you multiply by 0, the product is 0.
10. When you divide 0 by any number but 0, the quotient is 0.
11. You cannot divide by 0.
12. When you multiply or divide a number by 1, the answer is the number you started with.

TEACHING TIP

Encourage students to always think about what a calculation means rather than simply memorizing principles or properties.

Principle 1: Multiplication and division "undo" each other. They are related inverse operations.

Multiplication and division are inverse operations; one "undoes" the other. In fact, division is defined as the "opposite" of multiplication. If you start with 12 items and share them among 3 people, each person gets 4 items. How do you get back to where you started (to 12)? You think of the 3 people, each with 4 items, as a multiplication situation, or 3 groups of 4, which is 12.

This allows a student to use a multiplication table to determine a quotient. He or she solves $12 \div 4$ by looking for 12 in the 4 row and reading up to see the column header, in this case, 3.

ACTIVITY 6.10

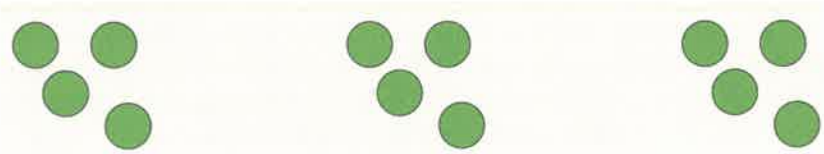
Tell students that they can use more than 20 but fewer than 30 items, and that you want them to make equal groups. If it's available, have students model the groups using an interactive whiteboard, moving items around.

Ask them to write both multiplication and division sentences that describe the equal groups they can make.

Building Number Sense

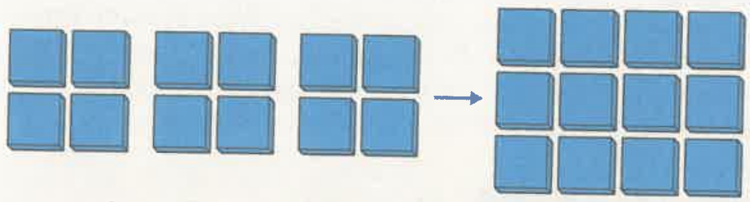
Principle 2: You can multiply numbers in any order (the commutative property). With division, the order in which you divide the numbers matters.

If you model 3 groups of 4 as shown below, it is not clear why it is the same as 4 groups of 3.



But, if you rearrange the same 12 items in an array as shown at the top of the next page, it is obvious why $3 \times 4 = 4 \times 3$. The same model shows both 3 sets of 4 and 4 sets of 3.

This property shows up in the multiplication table when students notice that the row for any number matches the column for that number.



3 groups of 4 = 12

3 rows of 4 = 12
4 columns of 3 = 12

$3 \times 4 = 12$
$4 \times 3 = 12$

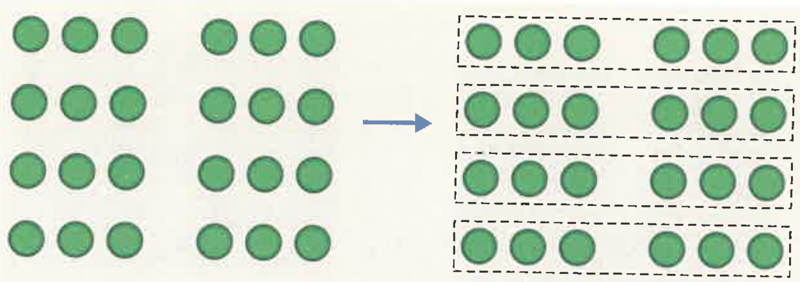
Modeling to explain the commutative property of multiplication

Note that the order does matter when you divide. For example, if you were to divide 12 items into groups of 3, you would have 4 groups of 3. If you were to try to divide 3 items into groups of 12, you would not have even 1 full group.

Principle 3: To multiply two numbers, you can divide one factor and multiply the other by the same amount without changing the product.

The associative property of multiplication suggests that to multiply $a \times b \times c$, you can multiply $a \times b$ first and then multiply by c, or calculate $b \times c$ first and then multiply that product by a. For example, $4 \times 2 \times 3 = 8 \times 3$ or 4×6. The property exists since multiplication is only defined for two numbers, and rules had to be created to allow you to deal with more numbers than that.

Consider the 8 groups of 3 (8×3) shown below. If you pair up groups of 3, you will have 6 in each group or twice as many in each group but only 4 groups, that is, half as many groups (4×6).



8 groups of 3

4 groups of 6

$8 \times 3 = 4 \times 6$

8 groups of 3 can be viewed as 4 groups of 6 (a consequence of the associative property of multiplication).

TEACHING TIP

Some teachers teach the half/double strategy for multiplication (that you can halve one factor if you double the other), a specific application of Principle 3. Similarly, you could multiply either number by any whole number other than 0 and divide the other number by that same whole number and not change the result.

Make sure students understand that Principle 3 does not apply to any other operation.

To multiply 12×5, Chris thinks "6×10." Explain his thinking.

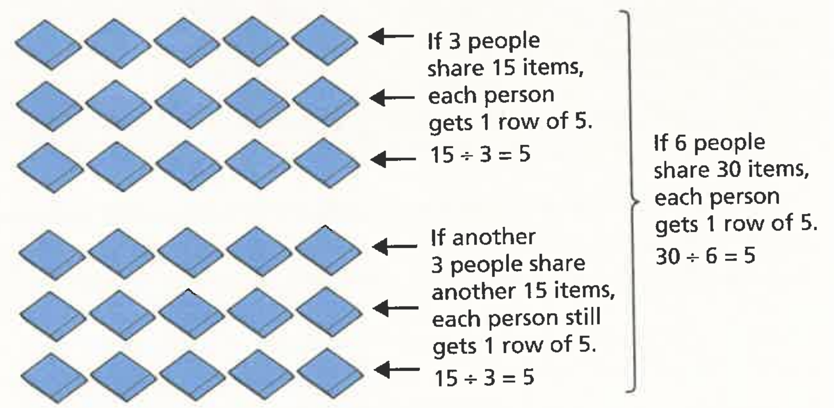
It's easier to multiply by 10 in your head and the answer doesn't change anyway because it's half of 12 but twice 5.

Student Response

This student demonstrates an understanding of the value of using an equivalent form of a product with numbers that are easier to multiply mentally.

Principle 4: To divide two numbers, you can multiply both numbers by the same amount without changing the quotient.

For instance, 15 ÷ 3 asks how much each person gets if 3 people share 15 items equally. It makes sense that if there are twice as many items to be shared by twice as many people, the share size stays the same. It does not matter what the dividend and divisor are multiplied by. As long as they are multiplied by the same amount, the quotient does not change.

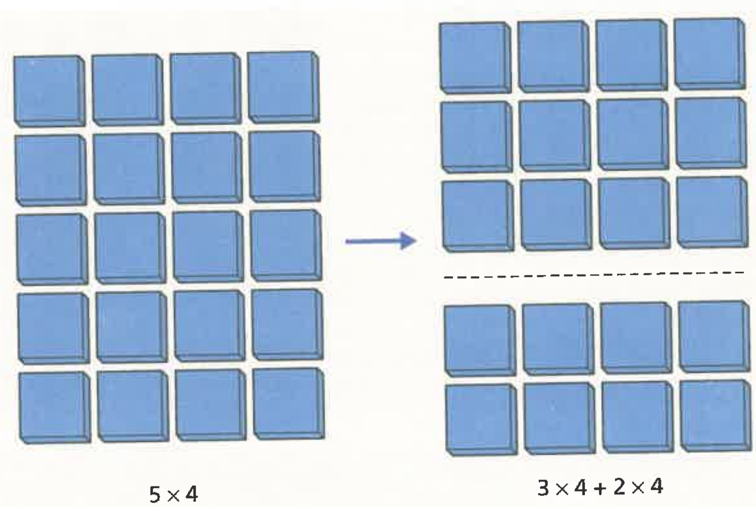


$15 \div 3 = (15 \times 2) \div (3 \times 2)$
$15 \div 3 = 30 \div 6$

If there are twice as many items to be shared by twice as many people, the share size stays the same.

Principle 5: You can multiply in parts (the distributive property).

For instance, you can separate 5 rows of 4 squares into 3 rows of 4 and 2 rows of 4 ($5 \times 4 = 3 \times 4 + 2 \times 4$) without changing the total number.



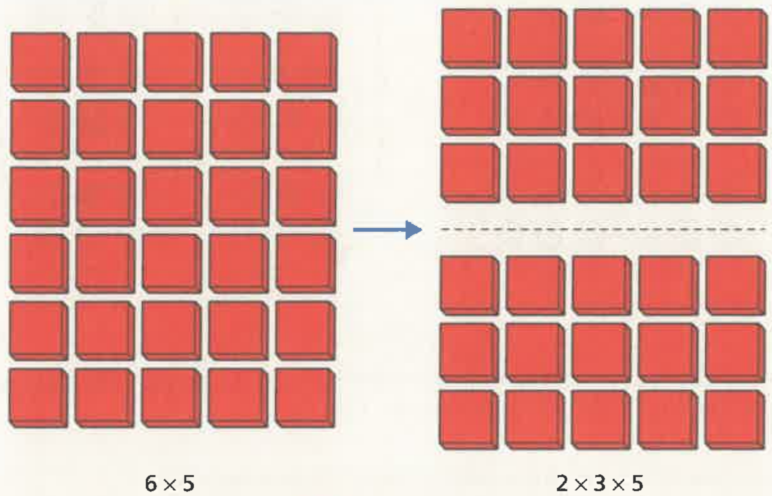
$5 \times 4 = 3 \times 4 + 2 \times 4$

Distributing 5 rows of 4 over 3 rows of 4 and 2 rows of 4

This property shows up in the multiplication table when students notice that if you add matching elements in two rows in the table, you often get another row in the table. For example, if you add any number in the 2 row to the number in the 5 row below it, you get the number in the 7 row below them. The same idea works with columns.

Principle 6: You can multiply in parts by breaking up the multiplier as a product (a variation of the associative property).

An array is a good way to show this principle. For example, you can easily separate 6 rows of 5 squares into 2 groups, each with 3 rows of 5, without changing the total number of squares, as shown below.



$6 \times 5 = 2 \times 3 \times 5$

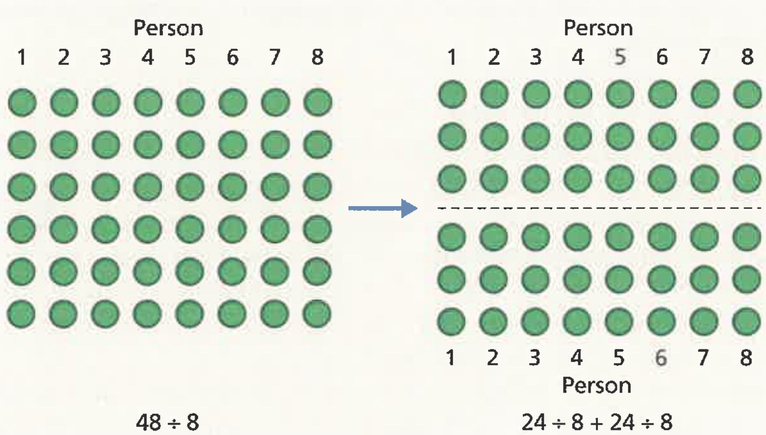
An array of 6 rows of 5 can be viewed as 2 arrays, each with 3 rows of 5.

> **NUMBER TALK**
>
> How might you figure out what 8 x 6 is if you didn't already know?

Principle 7: You can divide in parts by splitting the dividend into parts (the distributive property), but not the divisor.

For example, 48 ÷ 8 asks how many items each person gets when 48 items are shared equally among 8 people. It is possible to "distribute" the first 24 among the 8 people, and then "distribute" the other 24 among the same 8 people.

Notice that 48 ÷ 8 does NOT equal 48 ÷ 6 + 48 ÷ 2. This is because 48 ÷ 8 tells how many groups of 8 are in 48. 48 ÷ 2 tells how many groups of 2 are in 48; that is already a lot more than 48 ÷ 8, even before you add the 48 ÷ 6.

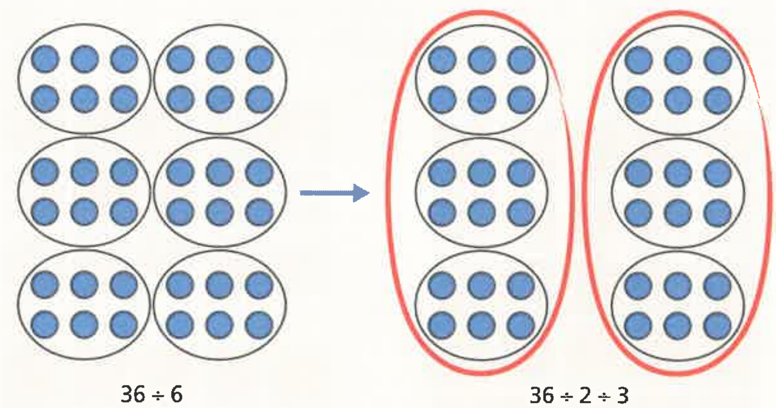


$48 \div 8 = 24 \div 8 + 24 \div 8$

Principle 8: You can divide by breaking up the divisor as a product.

The expression 36 ÷ 6 tells how many items there are in each group if 36 items are shared equally among 6 groups. You could share the 36 items among

6 groups ($36 \div 6$), or you could split the 36 items into 2 groups of 18 and then share within each of the 18s, each into 3 smaller groups of 6 (that is, $36 \div 2 \div 3$).



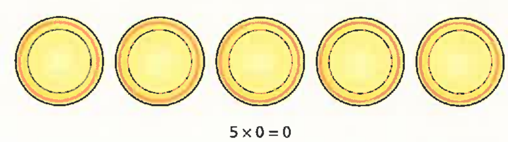
Share 36 into 6 groups by sharing first into the 2 groups circled in red, and then sharing within each red group into 3 groups.

NUMBER TALK

Choose a number greater than 20 to divide by 4. How would you do it?

Principle 9: When you multiply by 0, the product is 0.

To show, say, 5 sets of 0, you might use 5 empty plates. Since there is nothing on any of the plates, the total number of items is 0.



It does not matter how many empty plates there are; any number of plates with 0 items on them results in 0 items altogether.

Principle 10: When you divide 0 by a number other than 0, the quotient is 0.

For example, $0 \div 5$ asks the share size if 5 people share nothing. The amount is clearly nothing.

TEACHING TIP

Rather than initiating a conversation about why we can't divide by 0, wait for students to ask about the situation.

Use only one of the provided explanations, choosing whichever is most similar to situations that the students are familiar with.

Principle 11: You cannot divide by 0.

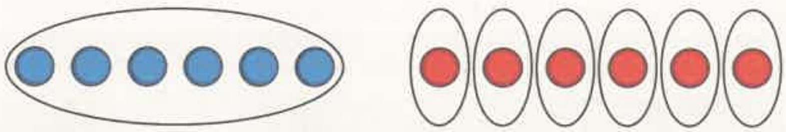
It is more difficult to explain $5 \div 0$ than $0 \div 5$. How can 5 items be shared among 0 groups? Here a repeated subtraction meaning for division might make more sense to explain the principle. To divide, for example, $15 \div 5$, you can subtract 5 three times from 15, until you get to 0, so $15 \div 5 = 3$. So, to divide $5 \div 0$, you must determine how many times you can subtract 0 from 5 before getting down to 0. There is no answer; you will simply never get to 0. Therefore, $5 \div 0$ is undefined.

As for $0 \div 0$, you can subtract 0 any number of times from 0 to get to 0, so the answer could be 1, 2, 3, or any number. Since there are too many answers, $0 \div 0$ is indeterminate.

Another way to describe this principle is to appeal to the relationship between multiplication and division. If the answer were $\square$, that is, $5 \div 0 = \square$, then $0 \times \square = 5$. Since $0 \times$ anything is 0, there are no answers. If $0 \div 0 = \square$, then $0 \times \square = 0$. Here there are an infinite number of answers (any number works). In either case, it makes no mathematical sense to divide by 0.

Principle 12: When you multiply or divide a number by 1, the answer is the number you started with.

The following models explain this principle for multiplication and for division.



1×6 is 1 group of 6, or 6 items altogether.

$6 \div 1$ is 6 items in 1 group, or 6 items in each (of 1) group.

6×1 is 6 groups of 1, or 6 items altogether.

$6 \div 1$ is 6 items in groups of 1, or 6 groups altogether.

It is interesting that, for some students, multiplication by 1 is less comfortable than multiplying by other numbers where they actually see several groups of the same size.

Common Errors and Misconceptions

Multiplication and Division: Common Errors, Misconceptions, and Strategies

COMMON ERROR OR MISCONCEPTION	SUGGESTED STRATEGY
Generalizing About Products Students incorrectly generalize that the product of two numbers is always greater than the sum of the same two numbers. These students are stymied by expressions like 5×1, 5×0, and 2×2.	More often than not, this misconception arises because it is inadvertently promoted. Teachers tell their students that multiplication makes things "bigger," and they forget about the contradictions to this rule. This can be avoided if teachers are aware of the contradictions, and are careful with their generalizations.
Reversing Dividend and Divisor Students reverse the dividend and divisor when recording a division sentence (and may or may not calculate correctly). For example, to represent the number of groups of 3 in 15, they record $3 \div 15$ instead of $15 \div 3$. This common error should not be surprising, though, as one of the ways students are taught to record this division is as $3\overline{)15}$ for $15 \div 3$.	The connection between words and symbols must be carefully crafted. When writing $15 \div 3$, you should be saying aloud, "How many 3s are in 15?" and writing both forms of the symbolic expression at the same time, as shown: $3\overline{)15}$ for $15 \div 3$
Remainders Students struggle with division problems that have remainders. For example: • They might think an expression like $17 \div 4$ is impossible since you cannot make perfect groups of 4. • They think that $17 \div 4$ is 4 since you can make 4 groups of 4, or that $17 \div 4$ is 5 since there are 4 groups of 4 and 1 group of 1 for a total of 5 groups.	Students should meet situations involving remainders early on in their work with division. They need to understand that the result need not always be a whole number. Note that these misconceptions about remainders actually lead to correct answers in certain contexts. For example: • 17 marbles shared equally among 4 children: The answer is 4 because you cannot share the leftover marble. • 17 children transported in cars that hold 4 children each: The answer is 5 because you need a fifth car to transport the "leftover" child.

Appropriate Manipulatives

Manipulatives can be used to model many multiplication meanings. (See "Meanings of Multiplication" on page 119 for other examples of ways to use manipulatives to model different meanings of multiplication.)

Multiplication: Examples of Manipulatives

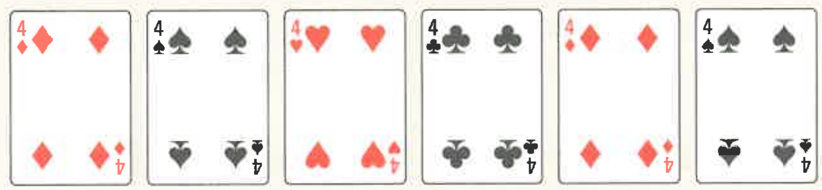
LINKING CUBES

Repeated addition or equal groups

$4 + 4 + 4 + 4 + 4 + 4 = 24$

$6 \times 4 = 24$

GAME MATERIALS

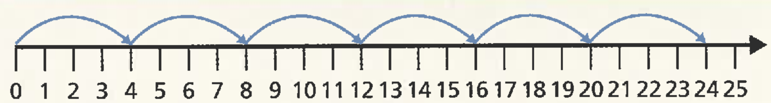


Repeated addition or equal groups

$4 + 4 + 4 + 4 + 4 + 4 = 24$

$6 \times 4 = 24$

NUMBER LINE



Repeated addition or equal groups

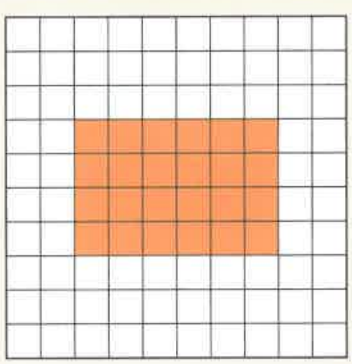
$4 + 4 + 4 + 4 + 4 + 4 = 24$

$6 \times 4 = 24$

Manipulatives can be used to model many division meanings. Some examples are shown below. (See "Meanings of Division" on page 121 for other examples of ways to use manipulatives to model different meanings of division.)

Division: Examples of Manipulatives

A GRID



Length or width of a rectangle

$24 \div 4 = 6$ or $24 \div 6 = 4$

LINKING CUBES

Equal groups

24 split into 4 equal groups results in 6 in each group.

Appropriate Children's Books

Two of Everything: A Chinese Folktale (Hong, 1993)
A magic pot doubles whatever is put into it. The context of the story can be continued to practice doubling. The teacher puts items into a "pot" and students predict how many items will emerge if the pot doubles its contents.

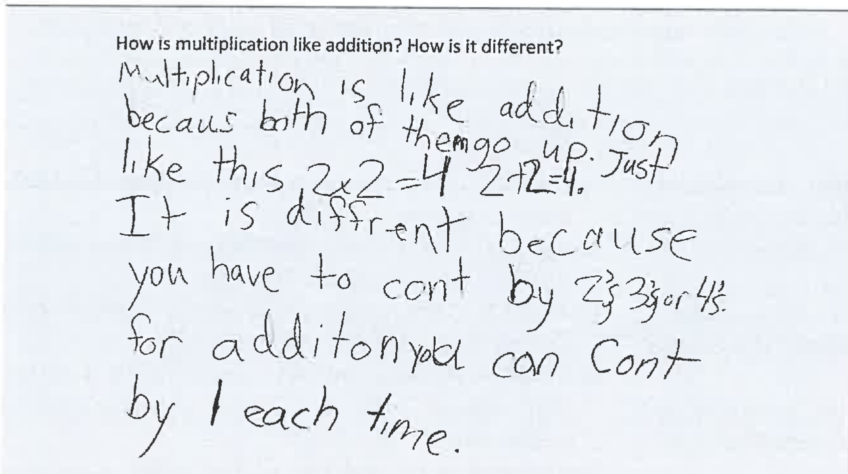
Bean Thirteen (McElligott, 2007)
Students learn that trying to divide up 13 into equal groups is not very easy. Is 13 really that unlucky?

Math-terpieces: The Art of Problem Solving (Tang, 2003)
Addition and subtraction are addressed through artwork. The book is both beautiful and fun.

A Remainder of One (Pinczes, 1995)
Twenty-five bugs march in different-sized groups, where there is usually a remainder of 1. This book suits early division work with remainders.

Assessing Student Understanding

- Ask questions that focus on the meanings of the operations. For example, "What does 4 × 6 mean?" instead of, "How much is 4 × 6?"
- Examine this response. What does it tell you about the student's understanding?


How is multiplication like addition? How is it different?

Multiplication is like addition becaus both of themgo up. Just like this 2x2=4 2+2=4.
It is diffrent because you have to cont by 2's 3's or 4's. for additon you can Cont by 1 each time.

Student Response

This student has answered a question about comparing the operations of addition and multiplication.

- Make sure that students recognize how to apply each operation in situations that represent different meanings; for example, ask how these two situations are alike:

  I have $4 and need $10 to buy a gift. How much more do I need?

  I am 10 years old and my sister is 4. How much older am I?
- Ask students to create problems to match different calculations, for example, 10 − 3 or 16 ÷ 4.

Applying What You've Learned

1. What was your solution to the chapter problem? What is it about your problem that led you to think of the situation as division?
2. How would you help students recognize the relationship between addition and subtraction? Why do you think that your approach might be effective?

3. Teachers often read a subtraction sign saying the words *take away*, for example, 8 − 2 as *eight take away* 2. Would you? Why or why not?
4. Which of the principles for addition and subtraction could be explained effectively using 10-frames? Why?
5. To calculate 14 − 8, a student says, "14, 13, 12, 11, 10, 9, 8, 7" as she holds up 8 fingers, one at a time, and says, "fourteen minus eight is seven." How would you respond to this approach to the calculation?
6. Which of the principles for multiplication and division do you think are most critical for students to understand? Why?
7. Develop an engaging task that would involve students in exploring some aspect of multiplication. What idea about multiplication would they learn? Why do you think your task would be engaging?
8. How would you help students distinguish between the sharing (partitive) and how-many-groups (quotative) meanings of division?

Interact with a K–8 Student:

9. Ask a student to show what each of these means (not what the answer is) with models (if they are not yet in Grade 3, only use parts a) and b)). What meanings of each operation did they choose to use?

a) $6 + 7$ b) $13 - 5$
c) 5×8 d) $16 \div 2$

Discuss with a K–8 Teacher:

10. Ask a teacher whether she or he teaches addition and subtraction at the same time or separately. Ask why she or he takes that approach.

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