where C_0 is the initial investment, r is the annual percentage rate, and T is the number of years over which the investment runs. The number e is a constant and is approximately equal to 2.718. It is not an unknown like C_0, r, and T.

EXAMPLE 4.14

Continuous Compounding

Linda DeFond invested $1,000 at a continuously compounded rate of 10 percent for one year. What is the value of her wealth at the end of one year?

From Equation 4.9, we have:

$$\$1{,}000 \times e^{.10} = \$1{,}000 \times 1.1052 = \$1{,}105.20$$

This number can easily be read from our Table A.5. One merely sets *r*, the value on the horizontal dimension, to 10 percent and *T*, the value on the vertical dimension, to 1. For this problem, the relevant portion of the table is:

	CONTINUOUSLY COMPOUNDED RATE (*r*)		
PERIOD (*T*)	9%	10%	11%
1	1.0942	1.1052	1.1163
2	1.1972	1.2214	1.2461
3	1.3100	1.3499	1.3910

Note that a continuously compounded rate of 10 percent is equivalent to an annually compounded rate of 10.52 percent. In other words, Linda DeFond would not care whether her bank quoted a continuously compounded rate of 10 percent or a 10.52 percent rate, compounded annually.

EXAMPLE 4.15

Continuous Compounding, Continued

Linda DeFond's brother, Mark, invested $1,000 at a continuously compounded rate of 10 percent for two years.

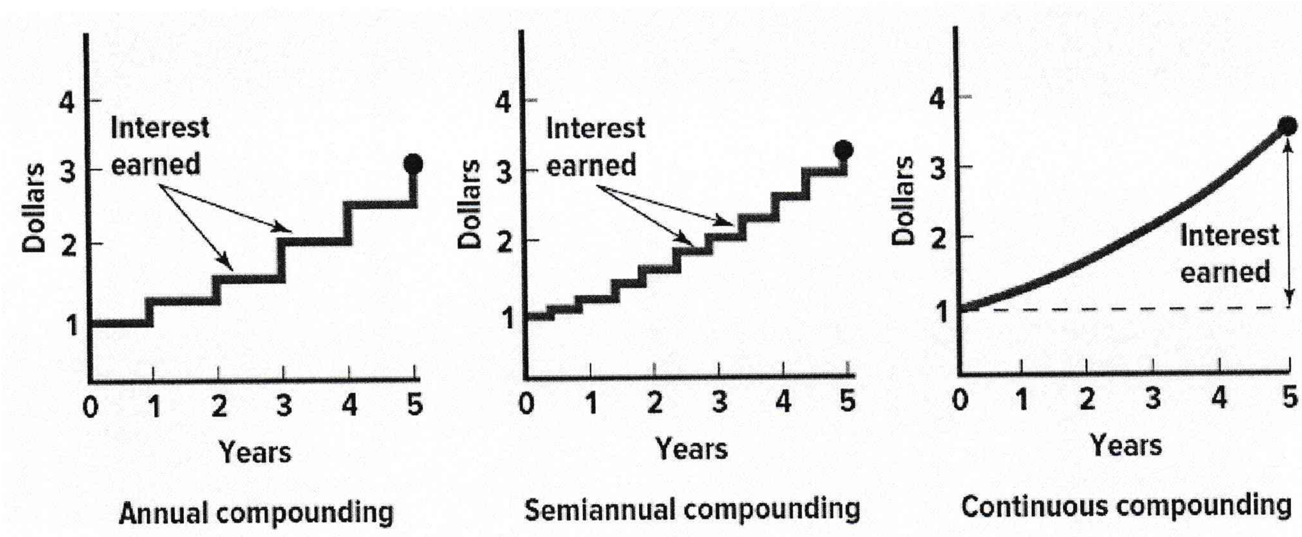
The appropriate equation here is:

$$\$1{,}000 \times e^{.10\times 2} = \$1{,}000 \times e^{.20} = \$1{,}221.40$$

Using the portion of the table of continuously compounded rates reproduced above, we find the value to be 1.2214.

Figure 4.11 illustrates the relationship among annual, semiannual, and continuous compounding. Semiannual compounding gives rise to both a smoother curve and a higher ending value than does annual compounding. Continuous compounding has both the smoothest curve and the highest ending value of all.

FIGURE 4.11
Annual, Semiannual, and Continuous Compounding



EXAMPLE 4.16

Present Value with Continuous Compounding

The Michigan state lottery is going to pay you $1,000 at the end of four years. If the annual continuously compounded rate of interest is 8 percent, what is the present value of this payment?

$$\$1{,}000 \times \frac{1}{e^{.08\times4}} = \$1{,}000 \times \frac{1}{1.3771} = \$726.15$$

4.4 SIMPLIFICATIONS

The first part of this chapter has examined the concepts of future value and present value. Although these concepts allow one to answer a host of problems concerning the time value of money, the human effort involved can frequently be excessive. For example, consider a bank calculating the present value on a 20-year monthly mortgage. Because this mortgage has 240 (= 20 × 12) payments, a lot of time is needed to perform a conceptually simple task.

Because many basic finance problems are potentially so time-consuming, we search out simplifications in this section. We provide simplifying formulas for four classes of cash flow streams:

1. Perpetuity
2. Growing perpetuity
3. Annuity
4. Growing annuity

Perpetuity

A **perpetuity** is a constant stream of cash flows without end. If you are thinking that perpetuities have no relevance to reality, it will surprise you that there is a well-known case of an unending cash flow stream: the British bonds called *consols.* An investor purchasing a consol is entitled to receive yearly interest from the British government forever.

How can the price of a consol be determined? Consider a consol that pays a coupon of C dollars each year and will do so forever. Applying the PV formula gives us:

$$\text{PV} = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots$$

where the dots at the end of the formula stand for the infinite string of terms that continues the formula. Series like the preceding one are called *geometric series.* It is well known that even though they have an infinite number of terms, the whole series has a finite sum because each term is only a fraction of the preceding term. Before turning to our calculus books, though, it is worth going back to our original principles to see if a bit of financial intuition can help us find the PV.

The present value of the consol is the present value of all of its future coupons. In other words, it is an amount of money that, if an investor had it today, would enable him to achieve the same pattern of expenditures that the consol and its coupons would. Suppose that an investor wanted to spend exactly C dollars each year. If he had the consol, he could do this. How much money must he have today to spend the same amount? Clearly he would need exactly enough so that the interest on the money would be C dollars per year. If he had any more, he could spend more than C dollars each year. If he had any less, he would eventually run out of money spending C dollars per year.

The amount that will give the investor *C* dollars each year, and therefore the present value of the consol, is:

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$$PV = \frac{C}{r} \qquad [4.10]$$

To confirm that this is the right answer, notice that if we lend the amount *C/r,* the interest it earns each year will be:

$$\text{Interest} = \frac{C}{r} \times r = C$$

which is exactly the consol payment. To sum up, we have shown that for a consol:

Formula for Present Value of Perpetuity:

$$PV = \frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots \qquad [4.11]$$

$$= \frac{C}{r}$$

It is comforting to know how easily we can use a bit of financial intuition to solve this mathematical problem.

EXAMPLE 4.17

Perpetuities

Consider a perpetuity paying $100 a year. If the relevant interest rate is 8 percent, what is the value of the consol?

Using Equation 4.10, we have:

$$PV = \frac{\$100}{.08} = \$1{,}250$$

Now suppose that interest rates fall to 6 percent. Using [4.10], the value of the perpetuity is:

$$PV = \frac{\$100}{.06} = \$1{,}666.67$$

Note that the value of the perpetuity rises with a drop in the interest rate. Conversely, the value of the perpetuity falls with a rise in the interest rate.

Growing Perpetuity

Imagine an apartment building where cash flows to the landlord after expenses will be $100,000 next year. These cash flows are expected to rise at 5 percent per year. Assuming that this rise will continue indefinitely, the cash flow stream is termed a **growing perpetuity**. The relevant interest rate is 11 percent. Therefore, the appropriate discount rate is 11 percent and the present value of the cash flows can be represented as:

$$PV = \frac{\$100{,}000}{1.11} + \frac{\$100{,}000(1.05)}{(1.11)^2} + \frac{\$100{,}000(1.05)^2}{(1.11)^3} + \cdots$$

$$+ \frac{\$100{,}000(1.05)^{N-1}}{(1.11)^N} + \cdots$$

Algebraically, we can write the formula as:

$$PV = \frac{C}{1+r} + \frac{C \times (1+g)}{(1+r)^2} + \frac{C \times (1+g)^2}{(1+r)^3} + \cdots + \frac{C \times (1+g)^{N-1}}{(1+r)^N} + \cdots$$

where C is the cash flow to be received one period hence, g is the rate of growth per period, expressed as a percentage, and r is the appropriate discount rate.

Fortunately, this formula reduces to the following simplification:

Formula for Present Value of Growing Perpetuity:

$$PV = \frac{C}{r - g} \qquad [4.12]$$

From Formula 4.12, the present value of the cash flows from the apartment building is:

$$\frac{\$100{,}000}{.11 - .05} = \$1{,}666{,}667$$

There are three important points concerning the growing perpetuity formula:

1. *The Numerator.* The numerator in Formula 4.12 is the cash flow one period hence, not at Date 0. Consider the following example:

EXAMPLE 4.18

Paying Dividends

Rothstein Corporation is *just about* to pay a dividend of \$3.00 per share. Investors anticipate that the annual dividend will rise by 6 percent a year forever. The applicable discount rate is 11 percent. What is the price of the stock today?

The numerator in Formula 4.12 is the cash flow to be received next period. Since the growth rate is 6 percent, the dividend next year is \$3.18 (= \$3.00 × 1.06). The price of the stock today is:

$$\$66. = \$3.00 + \frac{\$3.18}{.11 - .06}$$

\$66.	=	\$3.00	+	$\frac{\$3.18}{.11-.06}$
		Imminent dividend		Present value of all dividends beginning a year from now

The price of \$66.60 includes both the dividend to be received immediately and the present value of all dividends beginning a year from now. Formula 4.12 only makes it possible to calculate the present value of all dividends beginning a year from now. Be sure you understand this example; test questions on this subject always seem to trip up a few of our students.

2. *The Discount Rate and the Growth Rate.* The discount rate r must be greater than the growth rate g for the growing perpetuity formula to work. Consider the case in which the growth rate approaches the discount rate in magnitude. Then the denominator in the growing perpetuity formula gets infinitesimally small and the present value grows infinitely large. The present value is in fact undefined when r is less than g.
3. *The Timing Assumption.* Cash generally flows into and out of real-world firms both randomly and nearly continuously. However, Formula 4.12 assumes that cash flows are received and

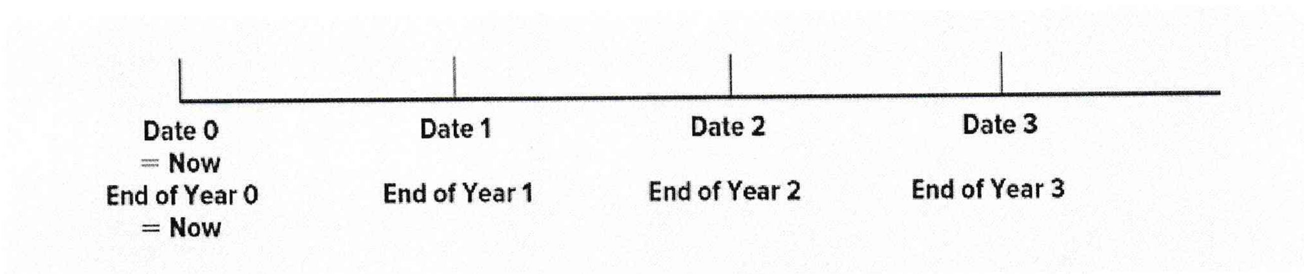
disbursed at regular and discrete points in time. In the example of the apartment, we assumed that the net cash flows of $100,000 only occurred once a year. In reality, rent checks are commonly received every month. Payments for maintenance and other expenses may occur anytime within the year.

The growing perpetuity formula [4.12] can be applied only by assuming a regular and discrete pattern of cash flow. Although this assumption is sensible because the formula saves so much time, the user should never forget that it is an *assumption.* This point will be mentioned again in the chapters ahead.

A few words should be said about terminology. Authors of financial textbooks generally use one of two conventions to refer to time. A minority of financial writers treat cash flows as being received on exact *dates,* for example Date 0, Date 1, and so forth. Under this convention, page 104

Date 0 represents the present time. However, because a year is an interval, not a specific moment in time, the great majority of authors refer to cash flows that occur at the end of a year (or alternatively, the end of a *period).* Under this *end-of-the-year* convention, the end of Year 0 is the present, the end of Year 1 occurs one period hence, and so on. (The beginning of Year 0 has already passed and is not generally referred to.)[5]

The interchangeability of the two conventions can be seen from the following chart:



We strongly believe that the *dates convention* reduces ambiguity. However, we use both conventions because you are likely to see the *end-of-year convention* in later courses. In fact, both conventions may appear in the same example for the sake of practice.

Annuity

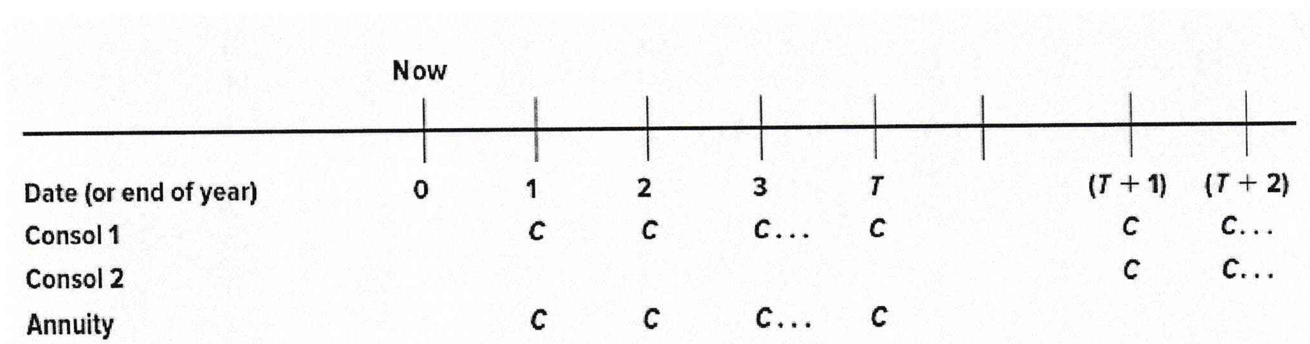
An **annuity** is a level stream of regular payments that lasts for a fixed number of periods. Not surprisingly, annuities are among the most common kinds of financial instruments. The pensions that people receive when they retire are often in the form of an annuity. Leases and mortgages are also often annuities.

To figure out the present value of an annuity we need to evaluate the following equation:

$$\frac{C}{1+r} + \frac{C}{(1+r)^2} + \frac{C}{(1+r)^3} + \cdots + \frac{C}{(1+r)^T}$$

The present value of only receiving the coupons for T periods must be less than the present value of a consol, but how much less? To answer this we have to look at consols a bit more closely.

Consider the following time chart:



Consol 1 is a normal consol with its first payment at Date 1. The first payment of Consol 2 occurs at Date $T + 1$.

The present value of having a cash flow of C at each of T dates is equal to the present value of Consol 1 minus the present value of Consol 2. The present value of Consol 1 is given by:

$$PV = \frac{C}{r} \qquad [4.13]$$

Consol 2 is just a consol with its first payment at Date $T + 1$. From the perpetuity formula, this consol will be worth C/r at Date T.[6] However, we do not want the value at Date T.

We want the value now; in other words, the present value at Date 0. We must discount *C/r* back by *T* periods. Therefore, the present value of Consol 2 is:

$$\text{PV} = \frac{C}{r}\left[\frac{1}{(1+r)^T}\right] \quad [4.14]$$

The present value of having cash flows for *T* years is the present value of a consol with its first payment at Date 1 minus the present value of a consol with its first payment at Date *T* + 1. Thus, the present value of an annuity is Formula 4.13 minus Formula 4.14. This can be written as:

$$\frac{C}{r} - \frac{C}{r}\left[\frac{1}{(1+r)^T}\right]$$

This simplifies to:

Formula for Present Value of Annuity:

$$\text{PV} = C\left[\frac{1}{r} - \frac{1}{r(1+r)^T}\right] \quad [4.15]$$

This can also be written as:

$$\text{PV} = C\left[\frac{1 - \frac{1}{(1+r)^T}}{r}\right]$$

EXAMPLE 4.19

Lottery Valuation

Mark Young has just won the state lottery, paying $50,000 a year for 20 years. He is to receive his first payment a year from now. The state advertises this as the Million Dollar Lottery because $1,000,000 = $50,000 × 20. If the interest rate is 8 percent, what is the true value of the lottery?

Equation 4.15 yields:

$$\text{Present value of Million Dollar Lottery} = \$50{,}000 \times \left[\frac{1 - \frac{1}{(1.08)^{20}}}{.08}\right]$$

$$\begin{aligned} &\text{Periodic payment} && \text{Annuity factor} \\ &= \$50{,}000 && \times\ 9.8181 \\ &= \$490{,}907.37 \end{aligned}$$

Rather than being overjoyed at winning, Mr. Young sues the state for misrepresentation and fraud. His legal brief states that he was promised $1 million but received only $490,907.37.

The term we use to compute the present value of the stream of level payments, *C*, for *T* years is called an **annuity factor**. The annuity factor in the current example is 9.8181. Because the annuity factor is used so often in PV calculations, we have included it in Table A.2 in the back of this book. The table gives the values of these factors for a range of interest rates, *r*, and maturity dates, *T*.

The annuity factor as expressed in the brackets of Formula 4.15 is a complex formula. For simplification, we may from time to time refer to the present value annuity factor as:

$$\text{PVIFA}_{r,T}$$

That is, the above expression stands for the present value of $1 a year for *T* years at an interest rate of *r*.

We can also provide a formula for the future value of an annuity:

$$FV = C\left[\frac{(1+r)^T}{r} - \frac{1}{r}\right] = C\left[\frac{(1+r)^T - 1}{r}\right] \qquad [4.16]$$

As with present value factors for annuities, we have compiled future value factors in Table A.4 in the back of this book. Of course, you can also use a spreadsheet as we illustrate in the nearby *Spreadsheet Techniques* box.

Annuity Present Values

SPREADSHEET TECHNIQUES

Using a spreadsheet to find annuity present values goes like this:

	A	B	C	D	E	F	G
1							
2	**Using a spreadsheet to find annuity present values**						
3							
4	What is the present value of $500 per year for 3 years if the discount rate is 10 percent?						
5	We need to solve for the unknown present value, so we use the formula PV(rate, nper, pmt, fv).						
6							
7	Payment amount per period:	$500					
8	Number of payments:	3					
9	Discount rate:	.1					
10							
11	Annuity present value:	**$1,243.43**					
12							
13	The formula entered in cell B11 is =PV(B9,B8,-B7,0); notice that fv is zero and that						
14	pmt has a negative sign on it. Also notice that rate is entered as a decimal, not a percentage.						
15							
16							
17							

EXAMPLE 4.20

Retirement Investing

Suppose you put $3,000 per year into a Roth IRA. The account pays 6 percent per year. How much will you have when you retire in 30 years?

This question asks for the future value of an annuity of $3,000 per year for 30 years at 6 percent, which we can calculate as follows:

$$FV = C\left[\frac{(1+r)^T - 1}{r}\right] = \$3{,}000 \times \left[\frac{1.06^{30} - 1}{.06}\right]$$
$$= \$3{,}000 \times 79.0582$$
$$= \$237{,}174.56$$

So, you'll have close to a quarter million dollars in the account.

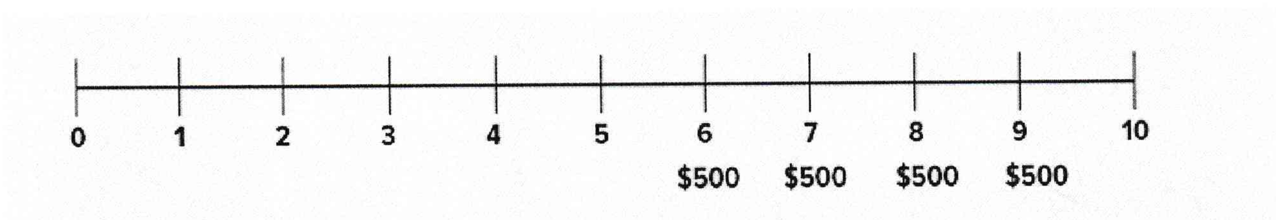
Our experience is that annuity formulas are not hard, but tricky, for the beginning student. We present four tricks below.

TRICK 1: A DELAYED ANNUITY One of the tricks in working with annuities or perpetuities is getting the timing exactly right. This is particularly true when an annuity or perpetuity begins at a date many periods in the future. We have found that even the brightest beginning student can make errors here. Consider the following example:

EXAMPLE 4.21

Delayed Annuities

Danielle Caravello will receive a four-year annuity of $500 per year, beginning at Date 6. If the interest rate is 10 percent, what is the present value of her annuity? This situation can be graphed as:



The analysis involves two steps:

1. Calculate the present value of the annuity using Formula 4.15. This is:

Present Value of Annuity at Date 5:

$$\$500 \times \left[\frac{1 - \frac{1}{(1.10)^4}}{.10}\right] = \$500 \times \text{PVIFA}_{10\%,4}$$
$$= \$500 \times 3.1699$$
$$= \$1,584.93$$

Note that $1,584.93 represents the present value at Date 5.

Students frequently think that $1,584.93 is the present value at Date 6, because the annuity begins at Date 6. However, our formula values the annuity as of one period prior to the first payment. This can be seen in the most typical case where the first payment occurs at Date 1. The equation values the annuity as of Date 0 in that case.

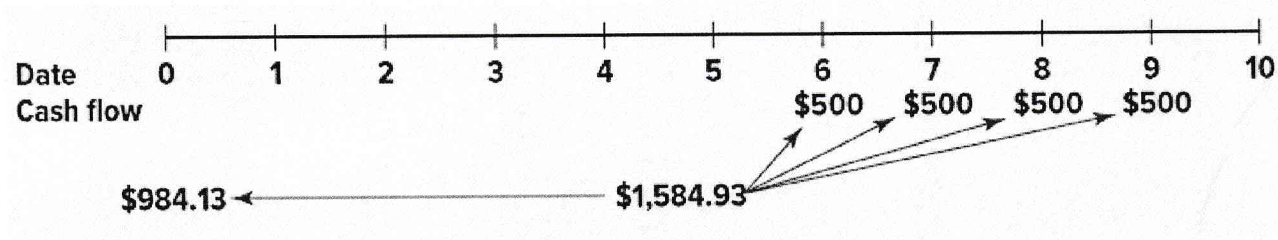
2. Discount the present value of the annuity back to Date 0. That is:

Present Value at Date 0:

$$\frac{\$1,584.93}{(1.10)^5} = \$984.12$$

Again, it is worthwhile mentioning that, because the annuity formula brings Danielle's annuity back to Date 5, the second calculation must discount over the remaining 5 periods. The two-step procedure is graphed in Figure 4.12.

FIGURE 4.12
Discounting Danielle Caravello's Annuity



Step one: Discount the four payments back to Date 5 by using the annuity formula.
Step two: Discount the present value at Date 5 ($1,584.93) back to present value at Date 0.

TRICK 2: ANNUITY DUE The annuity formula of Formula 4.15 assumes that the first annuity payment begins a full period hence. This type of annuity is sometimes called an *annuity in arrears* or an *ordinary annuity.* What happens if the annuity begins today, in other words, at Date 0?

EXAMPLE 4.22

Annuity Due

In a previous example, Mark Young received $50,000 a year for 20 years from the state lottery. In that example, he was to receive the first payment a year from the winning date. Let us now assume that the first payment occurs immediately. The total number of payments remains 20.

Under this new assumption, we have a 19-date annuity with the first payment occurring at Date 1—plus an extra payment at Date 0. The present value is:

$$\underset{\text{Payment at Date 0}}{\$50{,}000} + \underset{19-\text{year annuity}}{\$50{,}000 \times \text{PVIFA}_{8\%,19}}$$
$$= \$50{,}000 + (\$50{,}000 \times 9.6036)$$
$$= \$530{,}180$$

The present value in this example, $530,180, is greater than $490,907.37, the present value in the earlier lottery example. This is to be expected because the annuity of the current example begins earlier. An annuity with an immediate initial payment is called an *annuity in advance* or, more commonly, an *annuity due.* Always remember that Formula 4.15 and Table A.2 in this book refer to an *ordinary annuity.*

TRICK 3: THE INFREQUENT ANNUITY The following example treats an annuity with payments occurring less frequently than once a year.

EXAMPLE 4.23

Infrequent Annuities

Ms. Ann Chen receives an annuity of $450, payable once every two years. The annuity stretches out over 20 years. The first payment occurs at Date 2, that is, two years from today. The annual interest rate is 6 percent.

The trick is to determine the interest rate over a two-year period. The interest rate over two years is:

$$(1.06 \times 1.06) - 1 = 12.36\%$$

That is, $100 invested over two years will yield $112.36.

What we want is the present value of a $450 annuity over 10 periods, with an interest rate of 12.36 percent per period. This is:

$$\$450 \times \left[\frac{1 - \frac{1}{(1+.1236)^{10}}}{.1236}\right] = \$450 \times \text{PVIFA}_{12.36\%,10} = \$2{,}505.57$$

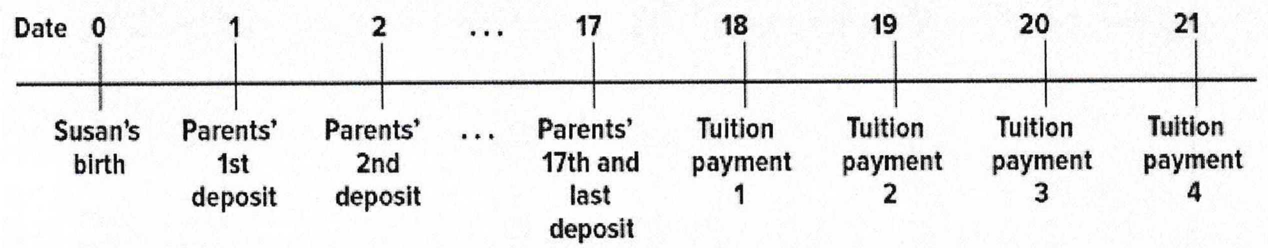
TRICK 4: EQUATING PRESENT VALUE OF TWO ANNUITIES The following example equates the present value of inflows with the present value of outflows.

EXAMPLE 4.24

Working with Annuities

Harold and Helen Nash are saving for the college education of their newborn daughter, Susan. The Nashes estimate that college expenses will run $30,000 per year when their daughter reaches college in 18 years. The annual interest rate over the next few decades will be 14 percent. How much money must they deposit in the bank each year so that their daughter will be completely supported through four years of college?

To simplify the calculations, we assume that Susan is born today. Her parents will make the first of her four annual tuition payments on her 18th birthday. They will make equal bank deposits on each of her first 17 birthdays, but no deposit at Date 0. This is illustrated as:



Mr. and Ms. Nash will be making deposits to the bank over the next 17 years. They will be withdrawing \$30,000 per year over the following four years. We can be sure they will be able to withdraw fully \$30,000 per year if the present value of the deposits is equal to the present value of the four \$30,000 withdrawals.

This calculation requires three steps. The first two determine the present value of the withdrawals. The final step determines yearly deposits that will have a present value equal to that of the withdrawals.

1. We calculate the present value of the four years at college using the annuity formula:

$$\$30{,}000 \times \left[\frac{1 - \dfrac{1}{(1.14)^4}}{.14}\right] = \$30{,}000 \times \text{PVIFA}_{14\%,4}$$

$$= \$30{,}000 \times 2.9137 = \$87{,}411.37$$

We assume that Susan enters college on her 18th birthday. Given our discussion in Trick 1, \$87,411.37 represents the present value at Date 17.

2. We calculate the present value of the college education at Date 0 as:

$$\frac{\$87{,}411.37}{(1.14)^{17}} = \$9{,}422.92$$

3. Assuming that Helen and Harold Nash make deposits to the bank at the end of each of the 17 years, we calculate the annual deposit that will yield a present value of all deposits of \$9,422.92. This is calculated as:

$$C \times \text{PVIFA}_{14\%,17} = \$9{,}422.92$$

$$\text{Because PVIFA}_{14\%,17} = 6.3729$$

$$C = \frac{\$9{,}422.92}{6.3729} = \$1{,}478.60$$

Thus, deposits of \$1,478.60 made at the end of each of the first 17 years and invested at 14 percent will provide enough money to make tuition payments of \$30,000 over the following four years. Alternatively, we could have set \$84,411.37 as the future value of an annuity and solved for the payment that way. Do this yourself and see if you don't get the same annuity payment.

An alternative method would be to (1) calculate the present value of the tuition payments at Susan's 18th birthday and (2) calculate annual deposits such that the future value of the deposits at her 18th birthday equals the present value of the tuition payments at that date. Although this technique can also provide the right answer, we have found that it is more likely to lead to errors. Therefore, we only equate present values in our presentation.

Growing Annuity

Cash flows in business are very likely to grow over time, due either to real growth or to inflation. The growing perpetuity, which assumes an infinite number of cash flows, provides one formula to handle this growth. We now consider a **growing annuity**, which is a end of the third year, the borrower would return the $1,000 along with another $100 in interest for that year. Similarly, a 50-year interest-only loan would call for the borrower to pay interest every year for the next 50 years and then repay the principal. In the extreme, the borrower pays the interest every period forever and never repays any principal. As we discussed earlier in the chapter, the result is a perpetuity.

Most corporate bonds have the general form of an interest-only loan. Because we will be considering bonds in some detail in the next chapter, we will defer further discussion of them for now.

Amortized Loans

With a pure discount or interest-only loan, the principal is repaid all at once. An alternative is an *amortized loan,* with which the lender may require the borrower to repay parts of the loan amount over time. The process of providing for a loan to be paid off by making regular principal reductions is called *amortizing* the loan.

A simple way of amortizing a loan is to have the borrower pay the interest each period plus some fixed amount. This approach is common with medium-term business loans. For example, suppose a business takes out a $5,000, five-year loan at 9 percent. The loan agreement calls for the borrower to pay the interest on the loan balance each year and to reduce the loan balance each year by $1,000. Because the loan amount declines by $1,000 each year, it is fully paid in five years.

In the case we are considering, notice that the total payment will decline each year. The reason is that the loan balance goes down, resulting in a lower interest charge each year, whereas the $1,000 principal reduction is constant. For example, the interest in the first year will be $5,000 × .09 = $450. The total payment will be $1,000 + 450 = $1,450. In the second year, the loan balance is $4,000, so the interest is $4,000 × .09 = $360, and the total payment is $1,360. We can calculate the total payment in each of the remaining years by preparing a simple *amortization schedule* as follows:

YEAR	BEGINNING BALANCE	TOTAL PAYMENT	INTEREST PAID	PRINCIPAL PAID	ENDING BALANCE
1	$5,000	$1,450	$ 450	$1,000	$4,000
2	4,000	1,360	360	1,000	3,000
3	3,000	1,270	270	1,000	2,000
4	2,000	1,180	180	1,000	1,000
5	1,000	1,090	90	1,000	0
Totals		$6,350	$1,350	$5,000	

Notice that in each year, the interest paid is given by the beginning balance multiplied by the interest rate. Also notice that the beginning balance is given by the ending balance from the previous year.

Probably the most common way of amortizing a loan is to have the borrower make a single, fixed payment every period. Almost all consumer loans (such as car loans) and mortgages work this way. For example, suppose our five-year, 9 percent, $5,000 loan was amortized this way. How would the amortization schedule look?

We first need to determine the payment. From our discussion earlier in the chapter, we know that this loan's cash flows are in the form of an ordinary annuity. In this case, we can solve for the payment as follows:

$$\$5{,}000 = C \times \{[1 - (1/1.09^5)]/.09\}$$
$$= C \times [(1 - .6499)/.09]$$

This gives us:

C = $5,000/3.8897
= $1,285.46

The borrower will therefore make five equal payments of $1,285.46. Will this pay off the loan? We will check by filling in an amortization schedule.

In our previous example, we knew the principal reduction each year. We then calculated the interest owed to get the total payment. In this example, we know the total payment. We will thus calculate the interest and then subtract it from the total payment to calculate the principal portion in each payment.

In the first year, the interest is $450, as we calculated before. Because the total payment is $1,285.46, the principal paid in the first year must be:

Principal paid = $1,285.46 − 450 = $835.46

The ending loan balance is thus:

Ending balance = $5,000 − 835.46 = $4,164.54

The interest in the second year is $4,164.54 × .09 = $374.81, and the loan balance declines by $1,285.46 − 374.81 = $910.65. We can summarize all of the relevant calculations in the following schedule:

YEAR	BEGINNING BALANCE	TOTAL PAYMENT	INTEREST PAID	PRINCIPAL PAID	ENDING BALANCE
1	$5,000.00	$1,285.46	$ 450.00	$ 835.46	$4,164.54
2	4,164.54	1,285.46	374.81	910.65	3,253.88
3	3,253.88	1,285.46	292.85	992.61	2,261.27
4	2,261.27	1,285.46	203.51	1,081.95	1,179.32
5	1,179.32	1,285.46	106.14	1,179.32	0.00
Totals		$6,427.30	$1,427.31	$5,000.00	

Because the loan balance declines to zero, the five equal payments do pay off the loan. Notice that the interest paid declines each period. This isn't surprising because the loan balance is going down. Given that the total payment is fixed, the principal paid must be rising each period. To see how to calculate this loan in Excel, see the upcoming *Spreadsheet Techniques* box.

If you compare the two loan amortizations in this section, you will see that the total interest is greater for the equal total payment case: $1,427.31 versus $1,350. The reason for this is that the loan is repaid more slowly early on, so the interest is somewhat higher. This doesn't mean that one loan is better than the other; it means that one is effectively paid off faster than the other. For example, the principal reduction in the first year is $835.46 in the equal total payment case as compared to $1,000 in the first case.

EXAMPLE 4.28

Partial Amortization, or "Bite the Bullet"

A common arrangement in real estate lending might call for a 5-year loan with, say, a 15-year amortization. What this means is that the borrower makes a payment every month of a fixed amount based on a 15-year amortization. However, after 60 months, the borrower makes a single, much larger payment called a "balloon" or "bullet" to pay off the loan. Because the monthly payments don't fully pay off the loan, the loan is said to be partially amortized.

Suppose we have a $100,000 commercial mortgage with a 12 percent APR and a 20-year (240-month) amortization. Further suppose the mortgage has a five-year balloon. What will the monthly payment be? How big will the balloon payment be?

The monthly payment can be calculated based on an ordinary annuity with a present value of $100,000. There are 240 payments, and the interest rate is 1 percent per month. The payment is:

$$\begin{aligned}\$100{,}000 &= C \times [(1 - 1/1.01^{240})/.01] \\ &= C \times 90.8194 \\ C &= \$1{,}101.09\end{aligned}$$

Now, there is an easy way and a hard way to determine the balloon payment. The hard way is to actually amortize the loan for 60 months to see what the balance is at that time. The easy way is to recognize that after 60 months, we have a 240 − 60 = 180-month loan. The payment is still $1,101.09 per month, and the interest rate is still 1 percent per month. The loan balance is thus the present value of the remaining payments:

$$\begin{aligned}\text{Loan balance} &= \$1{,}101.09 \times [(1 - 1/1.01^{180})/.01] \\ &= \$1{,}101.09 \times 83.3217 \\ &= \$91{,}744.69\end{aligned}$$

The balloon payment is a substantial $91,744. Why is it so large? To get an idea, consider the first payment on the mortgage. The interest in the first month is $100,000 × .01 = $1,000. Your payment is $1,101.09, so the loan balance declines by only $101.09. Because the loan balance declines so slowly, the cumulative "pay down" over five years is not great.

We will close this section with an example that may be of particular relevance. Federal Stafford loans are an important source of financing for many college students, helping to cover the cost of tuition, books, new cars, condominiums, and many other things. Sometimes students do not seem to fully realize that Stafford loans have a serious drawback: They must be repaid in monthly installments, usually beginning six months after the student leaves school.

Some Stafford loans are subsidized, meaning that the interest does not begin to accrue until repayment begins (this is a good thing). If you are a dependent undergraduate student under this particular option, the total debt you can run up is, at most, $23,000. For loans between July 2015 and July 2016, the interest rate is 4.29 percent, or 4.29/12 = .3575 percent per month. Under the "standard repayment plan," the loans are amortized over 10 years (subject to a minimum payment of $50).

Suppose you max out borrowing under this program and also get stuck paying the maximum interest rate. Beginning six months after you graduate (or otherwise depart the ivory tower), what will your monthly payment be? How much will you owe after making payments for four years?

Given our earlier discussions, see if you don't agree that your monthly payment assuming a $23,000 total loan is $236.05 per month. Also, as explained in Example 4.28, after making payments for four years, you still owe the present value of the remaining payments. There are 120 payments in all. After you make 48 of them (the first four years), you have 72 to go. By now, it should be easy for you to verify that the present value of $236.05 per month for 72 months at .3575 percent per month is just under $15,000, so you still have a long way to go.

Of course, it is possible to rack up much larger debts. According to the Association of American Medical Colleges, students who borrowed to attend medical school and graduated in 2014 had an average student loan balance of $176,000. Ouch! How long will it take the average student to pay off her medical school loans?

Loan Amortization Using a Spreadsheet **SPREADSHEET TECHNIQUES**

Loan amortization is a common spreadsheet application. To illustrate, we will set up the problem that we examined earlier: a five-year, $5,000, 9 percent loan with constant payments. Our spreadsheet looks like this:

	A	B	C	D	E	F	G	H
1								
2				**Using a spreadsheet to amortize a loan**				
3								
4			Loan amount:	$5,000				
5			Interest rate:	.09				
6			Loan term:	5				
7			Loan payment:	**$1,285.46**				
8				Note: Payment is calculated using PMT(rate, nper, -pv, fv).				
9			*Amortization table:*					
10								
11		Year	Beginning	Total	Interest	Principal	Ending	
12			Balance	Payment	Paid	Paid	Balance	
13		1	$5,000.00	$1,285.46	$450.00	$835.46	$4,164.54	
14		2	4,164.54	1,285.46	374.81	910.65	3,253.88	
15		3	3,253.88	1,285.46	292.85	992.61	2,261.27	
16		4	2,261.27	1,285.46	203.51	1,081.95	1,179.32	
17		5	1,179.32	1,285.46	106.14	1,179.32	0.00	
18		Totals		6,427.31	1,427.31	5,000.00		
19								
20			*Formulas in the amortization table:*					
21								
22		Year	Beginning	Total	Interest	Principal	Ending	
23			Balance	Payment	Paid	Paid	Balance	
24		1	=+D4	=$D$7	=+$D$5*C13	=+D13-E13	=+C13-F13	
25		2	=+G13	=$D$7	=+$D$5*C14	=+D14-E14	=+C14-F14	
26		3	=+G14	=$D$7	=+$D$5*C15	=+D15-E15	=+C15-F15	
27		4	=+G15	=$D$7	=+$D$5*C16	=+D16-E16	=+C16-F16	
28		5	=+G16	=$D$7	=+$D$5*C17	=+D17-E17	=+C17-F17	
29								
30		Note: Totals in the amortization table are calculated using the SUM formula.						
31								

Let's say she makes a monthly payment of $1,200, and the loan has an interest rate of 7 percent per year, or .5833 percent per month. See if you agree that it will take 333 months, or just about 28 years, to pay off the loan. Maybe MD really stands for "mucho debt!"

4.6 WHAT IS A FIRM WORTH?

Suppose you are in the business of trying to determine the value of small companies. (You are a business appraiser.) How can you determine what a firm is worth? One way to think about the question of how much a firm is worth is to calculate the present value of its future cash flows.

Let us consider the example of a firm that is expected to generate net cash flows (cash inflows minus cash outflows) of $5,000 in the first year and $2,000 for each of the next five years. The firm can be sold for $10,000 seven years from now. The owners of the firm would like to be able to make 10 percent on their investment in the firm.

The value of the firm is found by multiplying the net cash flows by the appropriate present value factor. The value of the firm is the sum of the present values of the individual net cash flows.

The present value of the net cash flows is given next.

THE PRESENT VALUE OF THE FIRM

END OF YEAR	NET CASH FLOW OF THE FIRM	PRESENT VALUE FACTOR (10%)	PRESENT VALUE OF NET CASH FLOWS
1	$ 5,000	.90909	$ 4,545.45
2	2,000	.82645	1,652.89
3	2,000	.75131	1,502.63
4	2,000	.68301	1,366.03
5	2,000	.62092	1,241.84
6	2,000	.56447	1,128.95
7	10,000	.51316	5,131.58
		Present value of firm	$16,569.38

We can also use the simplifying formula for an annuity to give us:

$$\frac{\$5{,}000}{1.1} + \frac{(2{,}000 \times PVIFA_{10\%,5})}{1.1} + \frac{10{,}000}{(1.1)_7} = \$16{,}569.38$$

Suppose you have the opportunity to acquire the firm for $12,000. Should you acquire the firm? The answer is yes because the NPV is positive.

$$NPV = PV - Cost$$

$$\$4{,}569.38 = \$16{,}569.38 - 12{,}000$$

The incremental value (NPV) of acquiring the firm is $4,569.38.

EXAMPLE 4.29

Firm Valuation

The Trojan Pizza Company is contemplating investing $1 million in four new outlets in Los Angeles. Andrew Lo, the firm's chief financial officer (CFO), has estimated that the investments will pay out cash flows of $200,000 per year for nine years and nothing thereafter. (The cash flows will occur at the end of each year and there will be no cash flow after Year 9.) Mr. Lo has determined that the relevant discount rate for this investment is 15 percent. This is the rate of return that the firm can earn at comparable projects. Should the Trojan Pizza Company make the investments in the new outlets?

The decision can be evaluated as:

$$NPV = -\$1{,}000{,}000 + \frac{\$200{,}000}{1.15} + \frac{\$200{,}000}{(1.15)^2} + \cdots + \frac{\$200{,}000}{(1.15)^9}$$

$$= -\$1{,}000{,}000 + \$200{,}000 \times PVIFA_{15\%,9}$$

$$= -\$1{,}000{,}000 + \$954{,}316.78$$

$$= -\$45{,}683.22$$

The present value of the four new outlets is only $954,316.78. The outlets are worth less than they cost. The Trojan Pizza Company should not make the investment because the NPV is −$45,683.22. If the Trojan Pizza Company requires a 15 percent rate of return, the new outlets are not a good investment.

SUMMARY AND CONCLUSIONS

1. Two basic concepts, *future value* and *present value,* were introduced in the beginning of this chapter. With a 10 percent interest rate, an investor with $1 today can generate a future value of $1.10 in a year, $1.21 [=$1 × $(1.10)^2$] in two years, and so on. Conversely, present value analysis places a current value on a later cash flow. With the same 10 percent interest rate, a dollar to be received in one year has a present value of $.909 [=$1/1.10] in Year 0. A dollar to be received in two years has a present value of $.826 [=$1/$(1.10)^2$].

2. One commonly expresses the interest rate as, say, 12 percent per year. However, one can speak of the interest rate as 3 percent per quarter. Although the annual percentage rate remains 12 percent [=3 percent × 4], the effective annual interest rate is 12.55 percent [=$(1.03)^4 - 1$]. In other words, the compounding process increases the future value of an investment. The limiting case is continuous compounding, where funds are assumed to be reinvested every infinitesimal instant.

3. A basic quantitative technique for financial decision making is net present value analysis. The net present value formula for an investment that generates cash flows (C_i) in future periods is:

$$\text{NPV} = -C_0 + \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_T}{(1+r)^T} = -C_0 + \sum_{i=1}^{T} \frac{C_i}{(1+r)^i}$$

   The formula assumes that the cash flow at Date 0 is the initial investment (a cash outflow).

4. Frequently, the actual calculation of present value is long and tedious. The computation of the present value of a long-term mortgage with monthly payments is a good example of this. We presented four simplifying formulas:

$$\textbf{Perpetuity: } \text{PV} = \frac{C}{r}$$

$$\textbf{Growing perpetuity: } \text{PV} = \frac{C}{r-g}$$

$$\textbf{Annuity: } \text{PV} = C\left[\frac{1 - \frac{1}{(1+r)^T}}{r}\right]$$

$$\textbf{Growing annuity: } \text{PV} = C\left[\frac{1 - \left(\frac{1+g}{1+r}\right)^T}{r-g}\right]$$

5. We stressed a few practical considerations in the application of these formulas:

   a. The numerator in each of the formulas, *C,* is the cash flow to be received *one full period hence.*

b. Cash flows are generally irregular in practice. To avoid unwieldy problems, assumptions to create more regular cash flows are made both in this textbook and in the real world.

c. A number of present value problems involve annuities (or perpetuities) beginning a few periods hence. Students should practice combining the annuity (or perpetuity) formula with the discounting formula to solve these problems.

d. Annuities and perpetuities may have periods of every two or every *n* years, rather than once a year. The annuity and perpetuity formulas can easily handle such circumstances.

e. One frequently encounters problems where the present value of one annuity must be equated with the present value of another annuity.

6. Many loans are annuities. The process of providing for a loan to be paid off gradually is called amortizing the loan, and we discussed how amortization schedules are prepared and interpreted.