Discounted Cash Flow Valuation 4

OPENING CASE

The signing of big-name athletes is often accompanied by great fanfare, but the numbers are often misleading. For example, in late 2015, catcher Matt Weiters reached a one-year deal with the Baltimore Orioles, signing a contract with a reported value of $15.8 million. Not bad, especially for someone who makes a living using the "tools of ignorance" (jock jargon for a catcher's equipment). Another example is the contract signed by David Price of the Boston Red Sox, which had a stated value of $217 million.

It looks like Matt and David did pretty well, but the Orioles weren't done as they signed first baseman Chris Davis to a contract that has a stated value of $161 million, but this amount was actually payable over several years. The contract called for $17 million per year for the first six years, plus $42 million in future salary to be paid in the years 2023 through 2037. David Price's payments were similarly spread over time, although his payments were only for seven years. Because two of the three contracts called for payments that are made at future dates, we must consider the time value of money, which means none of these players received the quoted amounts. How much did they really get? This chapter gives you the "tools of knowledge" to answer this question.

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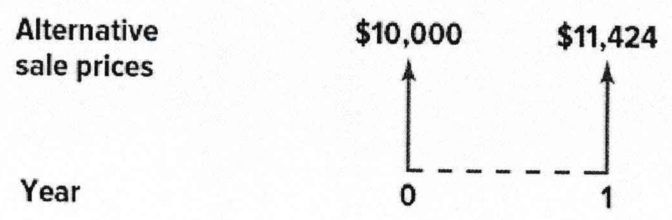
4.1 VALUATION: THE ONE-PERIOD CASE

Keith Vaughan is trying to sell a piece of raw land in Alaska. Yesterday, he was offered $10,000 for the property. He was about ready to accept the offer when another individual offered him $11,424. However, the second offer was to be paid a year from now. Keith has satisfied himself that both buyers are honest and financially solvent, so he has no fear that the offer he selects will fall through. These two offers are pictured as cash flows in Figure 4.1. Which offer should Mr. Vaughan choose?

FIGURE 4.1
Cash Flow for Mr. Vaughan's Sale



Jim Ellis, Keith's financial adviser, points out that if Keith takes the first offer, he could invest the $10,000 in a bank at an insured rate of 12 percent.[1] At the end of one year, he would have:

$$\$10{,}000 + (.12 \times \$10{,}000) = \$10{,}000 \times 1.12 = \$11{,}200$$

Return of principal — **Interest**

Because this is less than the $11,424 Keith could receive from the second offer, Mr. Ellis recommends that he take the latter. This analysis uses the concept of **future value**, or **compound value**, which is the value of a sum after investing over one or more periods. The compound, or future value, of $10,000 at 12 percent is $11,200.

An alternative method employs the concept of **present value**. One can determine present value by asking the following question: How much money must Keith put in the bank today at 12 percent so that he will have $11,424 next year? We can write this algebraically as:

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$$PV \times 1.12 = \$11{,}424$$

We want to solve for present value (PV), the amount of money that yields $11,424 if invested at 12 percent today. Solving for PV, we have:

$$PV = \frac{\$11{,}424}{1.12} = \$10{,}200$$

The formula for PV can be written as:

Present Value of Investment:

$$PV = \frac{C_1}{1 + r} \qquad [4.1]$$

where C_1 is cash flow at Date 1 and r is the rate of return that Keith Vaughan requires on his land sale. It is sometimes referred to as the *discount rate.*

Present value analysis tells us that a payment of $11,424 to be received next year has a present value of $10,200 today. In other words, at a 12 percent interest rate, Mr. Vaughan is indifferent between $10,200 today or $11,424 next year. If you gave him $10,200 today, he could put it in the bank and receive $11,424 next year.

Because the second offer has a present value of $10,200, whereas the first offer is for only $10,000, present value analysis also indicates that Mr. Vaughan should take the second offer. In other words, both future value analysis and present value analysis lead to the same decision. As it turns out, present value analysis and future value analysis must always lead to the same decision.

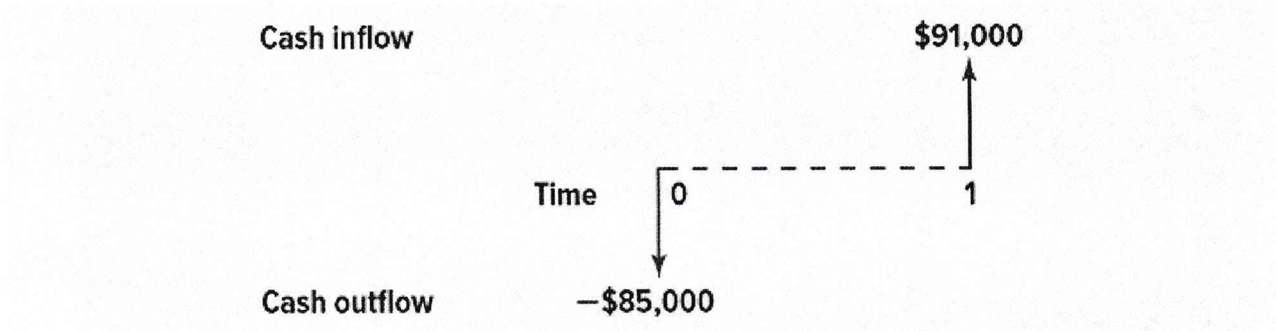
As simple as this example is, it contains the basic principles that we will be working with over the next few chapters. We now use another example to develop the concept of net present value.

EXAMPLE 4.1

Present Value

Diane Badame, a financial analyst at Kaufman & Broad, a leading real estate firm, is thinking about recommending that Kaufman & Broad invest in a piece of land that costs $85,000. She is certain that next year the land will be worth $91,000, a sure $6,000 gain. Given that the guaranteed interest rate in the bank is 10 percent, should Kaufman & Broad undertake the investment in land? Ms. Badame's choice is described in Figure 4.2 with the cash flow time chart.

FIGURE 4.2
Cash Flows for Land Investment


Cash inflow
$91,000
Time
0
1
Cash outflow
−$85,000

A moment's thought should be all it takes to convince her that this is not an attractive business deal. By investing $85,000 in the land, she will have $91,000 available next year. Suppose, instead, that Kaufman & Broad puts the same $85,000 into the bank. At the interest rate of 10 percent, this $85,000 would grow to:

$$(1 + .10) \times \$85{,}000 = \$93{,}500$$

next year.

It would be foolish to buy the land when investing the same $85,000 in the financial market would produce an extra $2,500 (that is, $93,500 from the bank minus $91,000 from the land investment). This is a future value calculation.

Alternatively, she could calculate the present value of the sale price next year as:

$$\text{Present value} = \frac{\$91{,}000}{1.10} = \$82{,}727.27$$

Because the present value of next year's sales price is less than this year's purchase price of $85,000, present value analysis also indicates that she should not recommend purchasing the property.

Frequently, business people want to determine the exact *cost* or *benefit* of a decision. The decision to buy this year and sell next year can be evaluated as

Net Present Value of Investment:

$$\underset{}{-\$2{,}273} = \underset{\text{Cost of land today}}{-\$85{,}000} + \underset{\text{Present value of next year's sales price}}{\frac{\$91{,}000}{1.10}}$$

The formula for NPV can be written as:

$$\text{NPV} = -\text{Cost} + \text{PV} \qquad [4.2]$$

Equation 4.2 says that the value of the investment is −$2,273, after stating all the benefits and all the costs as of Date 0. We say that −$2,273 is the **net present value (NPV)** of the investment. That is, NPV is the present value of future cash flows minus the present value of the cost of the investment. Because the net present value is negative, Diane Badame should not recommend purchasing the land.

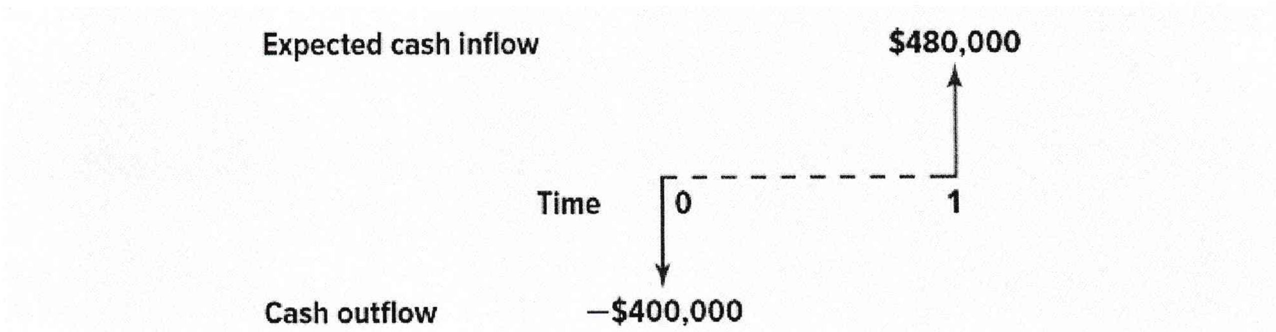
Both the Vaughan and the Badame examples deal with perfect certainty. That is, Keith Vaughan knows with perfect certainty that he could sell his land for $11,424 next year. Similarly, Diane Badame knows with perfect certainty that Kaufman & Broad could receive $91,000 for selling its land. Unfortunately, business people frequently do not know future cash flows. This uncertainty is treated in the next example.

EXAMPLE 4.2

Uncertainty and Valuation

Professional Artworks, Inc., is a firm that speculates in modern paintings. The manager is thinking of buying an original Picasso for $400,000 with the intention of selling it at the end of one year. The manager expects that the painting will be worth $480,000 in one year. The relevant cash flows are depicted in Figure 4.3.

FIGURE 4.3
Cash Flows for Investment in Painting



Of course, this is only an expectation—the painting could be worth more or less than $480,000. Suppose the guaranteed interest rate granted by banks is 10 percent. Should the firm purchase the piece of art?

Our first thought might be to discount at the interest rate, yielding:

$$\frac{\$480{,}000}{1.10} = \$436{,}364$$

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Because \$436,364 is greater than \$400,000, it looks at first glance as if the painting should be purchased. However, 10 percent is the return we have assumed one can earn on a riskless investment. Because the painting is quite risky, a higher *discount rate* is called for. The manager chooses a rate of 25 percent to reflect this risk. In other words, he argues that a 25 percent expected return is fair compensation for an investment as risky as this painting.

The present value of the painting becomes:

$$\frac{\$480{,}000}{1.25} = \$384{,}000$$

Thus, the manager believes that the painting is currently overpriced at \$400,000 and does not make the purchase.

The preceding analysis is typical of decision making in today's corporations, though real-world examples are, of course, much more complex. Unfortunately, any example with risk poses a problem not faced by a riskless example. In an example with riskless cash flows, the appropriate required return (i.e., discount rate) can be determined by checking the current returns on U.S. Treasury securities. Conceptually, the correct discount rate for a risky expected cash flow is the expected return available in the market on other investments of the same risks. This is the correct discount rate to apply because it represents the economic opportunity cost to investors. It is the expected return they will require before committing funding to an investment. However, the actual selection of the discount rate for a risky investment is quite a difficult task. We don't know at this point whether the discount rate on the painting should be 11 percent, 25 percent, 52 percent, or some other percentage.

Because the choice of a discount rate is so difficult, we merely wanted to broach the subject here. We must wait until the specific material on risk and return is covered in later chapters before a risk-adjusted analysis can be presented.

4.2 THE MULTIPERIOD CASE

The previous section presented the calculation of future value and present value for one period only. We will now perform the calculations for the multiperiod case.

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Future Value and Compounding

Suppose an individual were to make a loan of \$1. At the end of the first year, the borrower would owe the lender the principal amount of \$1 plus the interest on the loan at the interest rate of r. For the specific case where the interest rate is, say, 9 percent, the borrower owes the lender:

$$\$1 \times (1 + r) = \$1 \times 1.09 = \$1.09$$

At the end of the year, though, the lender has two choices. She can either take the $1.09— or, more generally, $(1 + r)$—out of the financial market, or she can leave it in and lend it again for a second year. The process of leaving the money in the financial market and lending it for another year is called **compounding**.

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Suppose that the lender decides to compound her loan for another year. She does this by taking the proceeds from her first one-year loan, \$1.09, and lending this amount for the next year. At the end of next year, then, the borrower will owe her:

$$\$1 \times (1 + r) \times (1 + r) = \$1 \times (1 + r)^2 = 1 + 2r + r^2$$
$$\$1 \times (1.09) \times (1.09) = \$1 \times (1.09)^2 = \$1 + \$.18 + \$.0081 = \$1.1881$$

This is the total she will receive two years from now by compounding the loan.

In other words, by providing a ready opportunity for lending, the capital market enables the investor to transform \$1 today into \$1.1881 at the end of two years. At the end of three years, the total cash will be $\$1 \times (1.09)^3 = \$1.2950$.

The most important point to notice is that the total amount that the lender receives is not just the \$1 that she lent out plus two years' worth of interest on \$1:

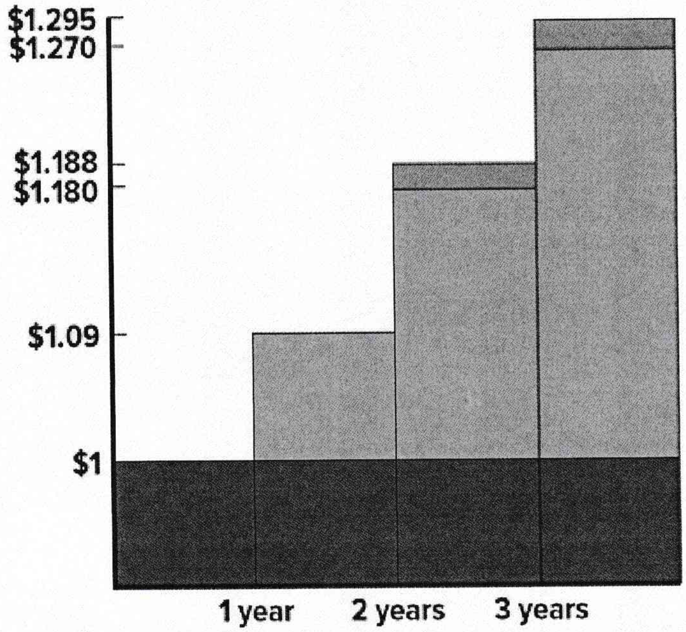
$$2 \times r = 2 \times \$.09 = \$.18$$

The lender also gets back an amount r^2, which is the interest in the second year on the interest that was earned in the first year. The term, $2 \times r$, represents **simple interest** over the two years, and the term, r^2, is referred to as the *interest on interest.* In our example this latter amount is exactly:

$$r^2 = \$.09^2 = \$.0081$$

When cash is invested at **compound interest**, each interest payment is reinvested. With simple interest, the interest is not reinvested. Benjamin Franklin's statement, "Money makes money and the money that money makes makes more money," is a colorful way of explaining compound interest. The difference between compound interest and simple interest is illustrated in Figure 4.4. In this example, the difference does not amount to much because the loan is for \$1. If the loan were for \$1 million, the lender would receive \$1,188,100 in two years' time. Of this amount, \$8,100 is interest on interest. The lesson is that those small numbers beyond the decimal point can add up to big dollar amounts when the transactions are for big amounts. In addition, the longer-lasting the loan, the more important interest on interest becomes.

FIGURE 4.4
Simple and Compound Interest



The purple-shaded area represents the initial investment.
The green-shaded area represents the simple interest.
The blue-shaded area represents interest on interest.

The general formula for an investment over many periods can be written as:

Future Value of an Investment:

$$FV = C_0 \times (1 + r)^T \qquad [4.3]$$

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where C_0 is the cash to be invested at Date 0 (i.e., today), r is the interest rate per period, and T is the number of periods over which the cash is invested.

EXAMPLE 4.3

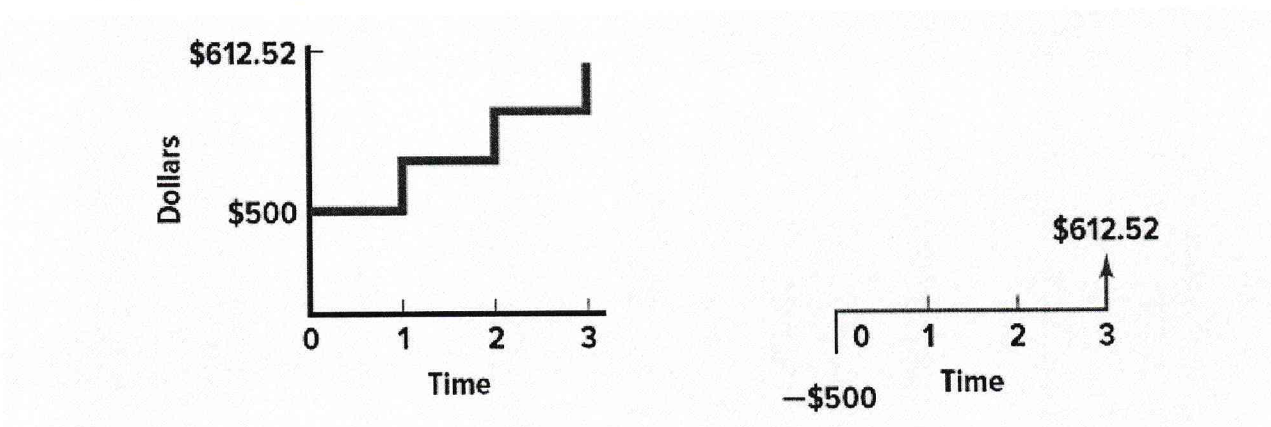
Interest on Interest

Suh-Pyng Ku has put $500 in a savings account at the First National Bank of Kent. The account earns 7 percent, compounded annually. How much will Ms. Ku have at the end of three years?

$$\$500 \times 1.07 \times 1.07 \times 1.07 = \$500 \times (1.07)^3 = \$612.52$$

Figure 4.5 illustrates the growth of Ms. Ku's account.

FIGURE 4.5
Suh-Pyng Ku's Savings Account



EXAMPLE 4.4

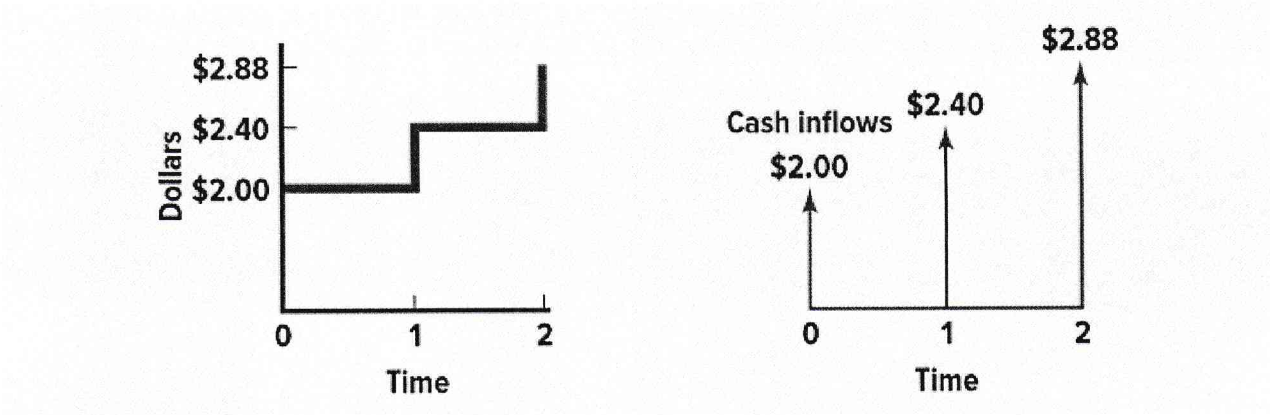
Compound Growth

Jay Ritter invested $1,000 in the stock of the SDH Company. The company pays a current dividend of $2, which is expected to grow by 20 percent per year for the next two years. What will the dividend of the SDH Company be after two years?

$$\$2 \times (1.20)^2 = \$2.88$$

Figure 4.6 illustrates the increasing value of **SDH's** dividends.

FIGURE 4.6
The Growth of the SDH Dividends



The two previous examples can be calculated in any one of four ways. The computations could be done by hand, by calculator, by spreadsheet, or with the help of a table. The appropriate table is Table A.3, which appears in the back of the text. This table presents *future value of $1 at the end of* T *periods.* The table is used by locating the appropriate interest rate on the horizontal axis and the appropriate number of periods on the vertical axis.

For example, Suh-Pyng Ku would look at the following portion of Table A.3:

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	INTEREST RATE		
PERIOD	6%	7%	8%
1	1.0600	1.0700	1.0800
2	1.1236	1.1449	1.1664
3	1.1910	1.2250	1.2597
4	1.2625	1.3108	1.3605

She could calculate the future value of her $500 as:

$500	×	1.2250	= $612.50
Initial investment		Future value of $1	

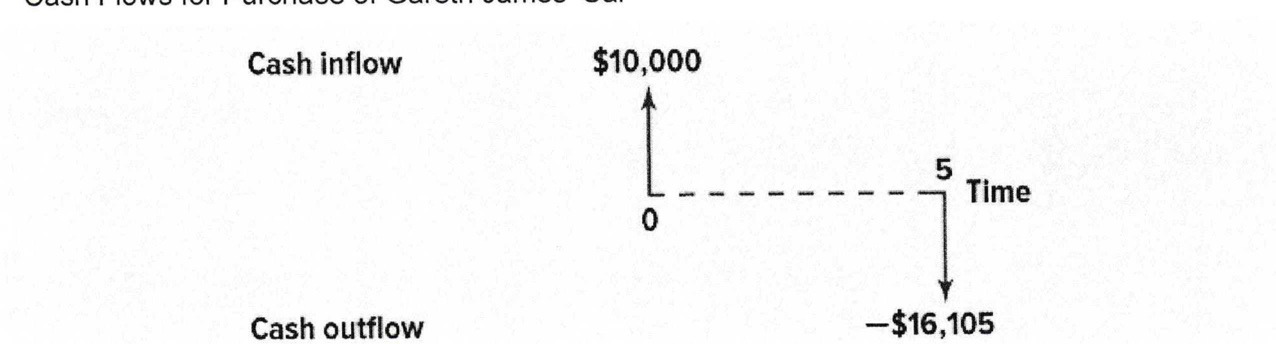
In the example concerning Suh-Pyng Ku, we gave you both the initial investment and the interest rate and then asked you to calculate the future value. Alternatively, the interest rate could have been unknown, as shown in the following example:

EXAMPLE 4.5

Finding the Rate

Gareth James, who recently won $10,000 in the lottery, wants to buy a car in five years. Gareth estimates that the car will cost $16,105 at that time. His cash flows are displayed in Figure 4.7.

FIGURE 4.7
Cash Flows for Purchase of Gareth James' Car



What interest rate must he earn to be able to afford the car?
The ratio of purchase price to initial cash is:

$$\frac{\$16,105}{\$10,000} = 1.6105$$

Thus, he must earn an interest rate that allows \$1 to become \$1.6105 in five years. Table A.3 tells us that an interest rate of 10 percent will allow him to purchase the car.

One can express the problem algebraically as:

$$\$10{,}000 \times (1 \times r)^5 = \$16{,}105$$

where r is the interest rate needed to purchase the car. Because \$16,105/\$10,000 = 1.6105, we have

$$(1 + r)^5 = 1.6105$$

Either the table or a calculator solves for r.

The Power of Compounding: A Digression

Most people who have had any experience with compounding are impressed with its power over long periods of time. In fact, compound interest has been described as the "eighth wonder of the world" and "the most powerful force in the universe."[2] Take the

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stock market, for example. Ibbotson and Sinquefield have calculated what the stock market returned as a whole from 1926 through 2015.[3] They find that one dollar placed in these stocks at the beginning of 1926 would have been worth $5,384.08 at the end of 2015. This is 10.02 percent compounded annually for 90 years, i.e., $(\$1.1002)^{90} = \$5{,}384.08$, ignoring a small rounding error.

The example illustrates the great difference between compound and simple interest. At 10.02 percent, simple interest on $1 is .1002 cents a year (i.e., $.1002). Simple interest over 90 years is $9.01 (90 × $.1002). That is, an individual withdrawing .1002 cents every year would have withdrawn $9.01 (90 × $.1002) over 90 years. This is quite a bit below the $5,384.08 that was obtained by reinvestment of all principal and interest.

The results are more impressive over even longer periods of time. A person with no experience in compounding might think that the value of $1 at the end of 180 years would be twice the value of $1 at the end of 90 years, if the yearly rate of return stayed the same. Actually the value of $1 at the end of 180 years would be the *square* of the value of $1 at the end of 90 years. That is, if the annual rate of return remained the same, a $1 investment in common stocks should be worth $28,988,317.45 [$1 × (5,384.08 × 5,384.08)].

A few years ago, an archaeologist unearthed a relic stating that Julius Caesar lent the Roman equivalent of one penny to someone. Since there was no record of the penny ever being repaid, the archaeologist wondered what the interest and principal would be if a descendant of Caesar tried to collect from a descendant of the borrower in the 20th century. The archaeologist felt that a rate of 6 percent might be appropriate. To his surprise, the principal and interest due after more than 2,000 years was vastly greater than the entire wealth on earth.

The power of compounding can explain why the parents of well-to-do families frequently bequeath wealth to their grandchildren rather than to their children. That is, they skip a generation. The parents would rather make the grandchildren very rich than make the children moderately rich. We have found that in these families the grandchildren have a more positive view of the power of compounding than do the children.

EXAMPLE 4.6

How Much for That Island?

Some people have said that it was the best real estate deal in history. Peter Minuit, director-general of New Netherlands, the Dutch West India Company's colony in North America, in 1626 allegedly bought Manhattan Island from native Americans for 60 guilders' worth of trinkets. This sounds cheap, but did the Dutch really get the better end of the deal? It is reported that 60 guilders was worth about $24 at the prevailing exchange rate. If the native Americans had sold the trinkets at a fair market value and invested the $24 at 5 percent (tax-free), it would now, about 390 years later, be worth about $4.4 billion. Today, Manhattan is undoubtedly worth more than $4.4 billion, and so, at a 5 percent rate of return, the native Americans got the worst of the deal. However, if invested at 10 percent, the amount of money they received would be worth about:

$$\$24(1+r)^T = 24 \times 1.1^{390} = \$330.7 \text{ quadrillion}$$

This is a lot of money. In fact, $330.7 quadrillion is more than all the real estate in the world is worth today. Note that no one in the history of the world has ever been able to find an investment yielding 10 percent every year for 390 years.

Present Value and Discounting

We now know that an annual interest rate of 9 percent enables the investor to transform $1 today into $1.1881 two years from now. In addition, we would like to know:

How much would an investor need to lend today so that she could receive $1 two years from today?

Algebraically, we can write this as:

$$PV \times (1.09)^2 = \$1$$

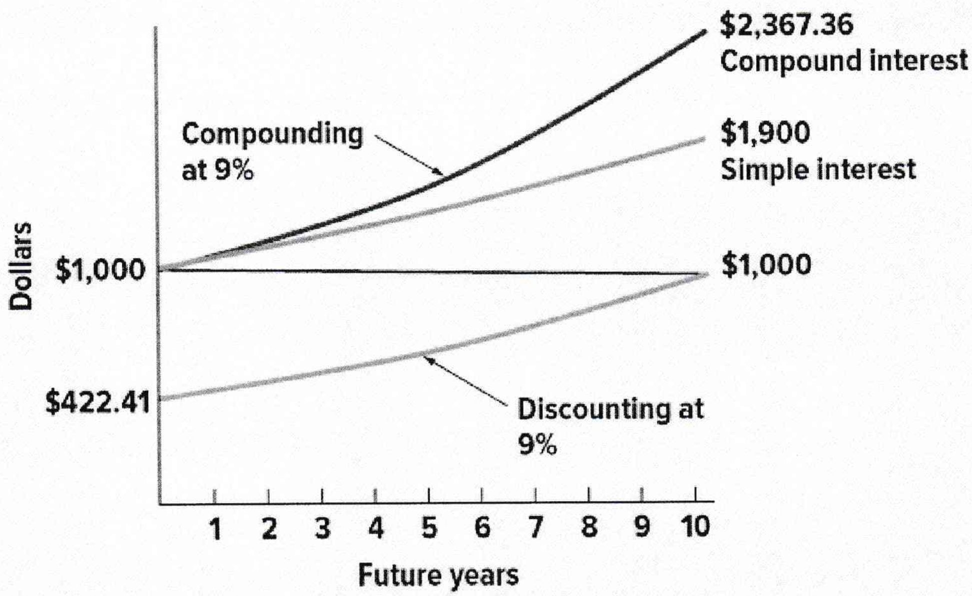
In the preceding equation, PV stands for present value, the amount of money we must lend today in order to receive $1 in two years' time.

Solving for PV in this equation, we have:

$$PV = \frac{\$1}{1.1881} = \$.84$$

This process of calculating the present value of a future cash flow is called **discounting**. It is the opposite of compounding. The difference between compounding and discounting is illustrated in Figure 4.8.

FIGURE 4.8
Compounding and Discounting



The top line shows the growth of $1,000 at compound interest with the funds invested at 9 percent: $\$1,000 \times (1.09)^{10} = \$2,367.36$. Simple interest is shown on the next line. It is $\$1,000 + [10 \times (\$1,000 \times .09)] = \$1,900$. The bottom line shows the discounted value of $1,000 if the interest rate is 9 percent.

To be certain that $.84 is in fact the present value of $1 to be received in two years, we must check whether or not, if we loaned out $.84 and rolled over the loan for two years, we would get exactly $1 back. If this were the case, the capital markets would be saying that $1 received in two years' time is equivalent to having $.84 today. Checking the exact numbers, we get:

$$\$.84168 \times 1.09 \times 1.09 = \$1$$

In other words, when we have capital markets with a sure interest rate of 9 percent, we are indifferent between receiving $.84 today or $1 in two years. We have no reason to treat these two choices differently from each other, because if we had $.84 today and loaned it out for two years, it would return $1 to us at the end of that time. The value $[1/(1.09)^2]$ is called the **present value factor**. It is the factor used to calculate the present value of a future cash flow.

In the multiperiod case, the formula for PV can be written as:

Present Value of Investment:

$$PV = \frac{C_T}{(1 + r)^T} \qquad [4.4]$$

where C_T is cash flow at Date T and r is the appropriate discount rate.

EXAMPLE 4.7

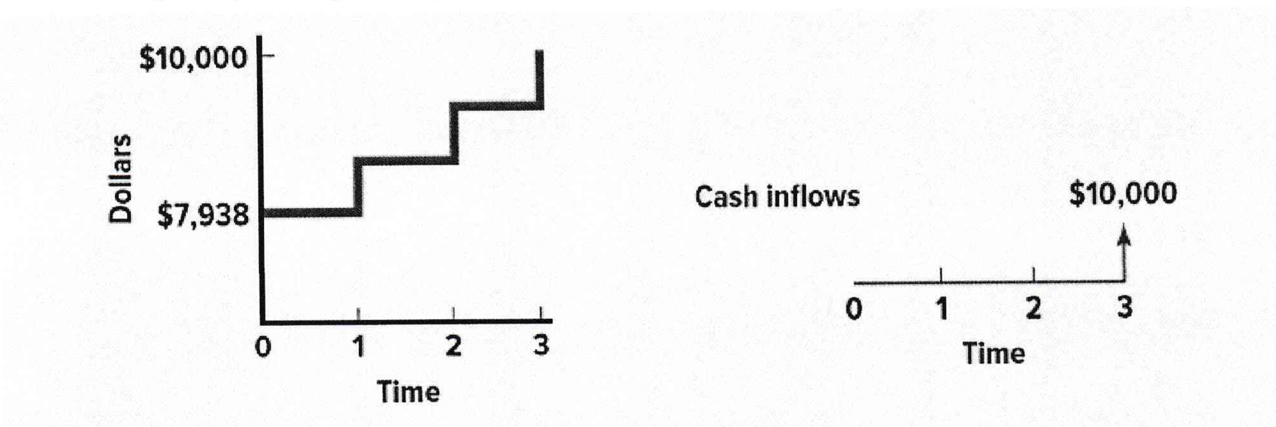
Multiperiod Discounting

Harry DeAngelo will receive $10,000 three years from now. Harry can earn 8 percent on his investments, so the appropriate discount rate is 8 percent. What is the present value of his future cash flow?

$$\begin{aligned} PV &= \$10{,}000 \times \left(\frac{1}{1.08}\right)^3 \\ &= \$10{,}000 \times .7938 \\ &= \$7{,}938 \end{aligned}$$

Figure 4.9 illustrates the application of the present value factor to Harry's investment.

FIGURE 4.9
Discounting Harry DeAngelo's Opportunity



When his investments grow at an 8 percent rate of interest, Harry DeAngelo is equally inclined toward receiving $7,938 now and receiving $10,000 in three years' time. After all, he could convert the $7,938 he receives today into $10,000 in three years by lending it at an interest rate of 8 percent.

Harry DeAngelo could have reached his present value calculation in one of three ways. The computation could have been done by hand, by calculator, or with the help of Table A.1, which appears in the back of the text. This table presents *present value of $1 to be received after* T *periods.* The table is used by locating the appropriate interest rate on the horizontal and the appropriate number of periods on the vertical. For example, Harry De Angelo would look at the following portion of Table A.1:

	INTEREST RATE		
PERIOD	7%	8%	9%
1	.9346	.9259	.9174

2	.8734	.8573	.8417
3	.8163	.7938	.7722
4	.7629	.7350	.7084

The appropriate present value factor is .7938.

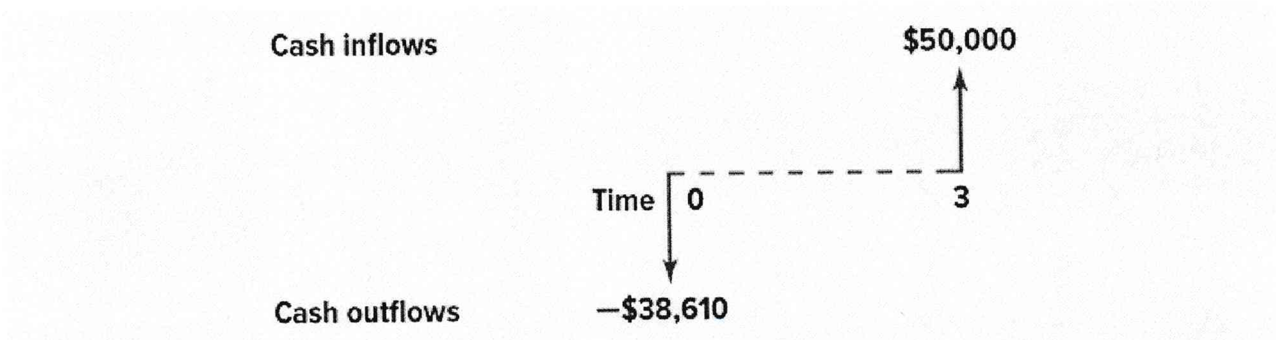
In the preceding example, we gave both the interest rate and the future cash flow. Alternatively, the interest rate could have been unknown.

EXAMPLE 4.8

Finding the Rate

A customer of the Beatty Corp. wants to buy a tugboat today. Rather than paying immediately, he will pay $50,000 in three years. It will cost the Beatty Corp. $38,610 to build the tugboat immediately. The relevant cash flows to Beatty Corp. are displayed in Figure 4.10. By charging what interest rate would the Beatty Corp. neither gain nor lose on the sale?

FIGURE 4.10
Cash Flows for Tugboat



The ratio of construction cost to sale price is:

$$\frac{\$38{,}610}{\$50{,}000} = .7722$$

We must determine the interest rate that allows $1 to be received in three years to have a present value of $.7722. Table A.1 tells us that 9 percent is that interest rate.

Frequently, an investor or a business will receive more than one cash flow. The present value of the set of cash flows is the sum of the present values of the individual cash flows. This is illustrated in the following examples:

EXAMPLE 4.9

Cash Flow Valuation

Dennis Draper has won the Kentucky state lottery and will receive the following set of cash flows over the next two years:

YEAR	CASH FLOW
1	$2,000

2	5,000

Mr. Draper can currently earn 6 percent in his money market account, so, the appropriate discount rate is 6 percent. The present value of the cash flows is:

YEAR	CASH FLOW × PRESENT VALUE FACTOR = PRESENT VALUE		
1	$\$2{,}000 \times \frac{1}{1.06} = \$2{,}000 \times .943$	=	\$1,887
2	$\$5{,}000 \times \left(\frac{1}{1.06}\right)^2 = \$5{,}000 \times .890$	=	4,450
	Total		\$6,337

In other words, Mr. Draper is equally inclined toward receiving \$6,337 today and receiving \$2,000 and \$5,000 over the next two years.

EXAMPLE 4.10

NPV

Finance.com has an opportunity to invest in a new high-speed computer that costs $50,000. The computer will generate cash flows (from cost savings) of $25,000 one year from now, $20,000 two years from now, and $15,000 three years from now. The computer will be worthless after three years, and no additional cash flows will occur. Finance.com has determined that the appropriate discount rate is 7 percent for this investment. Should Finance.com make this investment in a new high-speed computer? What is the present value of the investment?

The cash flows and present value factors of the proposed computer are as follows:

	CASH FLOWS	PRESENT VALUE FACTOR		
Year 0	−$50,000	1	=	1
1	25,000	$\frac{1}{1.07}$	=	.9346
2	20,000	$\left(\frac{1}{1.07}\right)^2$	=	.8734
3	15,000	$\left(\frac{1}{1.07}\right)^3$	=	.8163

The present values of the cash flows are:

Cash flows × Present value factor = Present value

Year 0	−$50,000×1	=	−$50,000.00
1	$25,000× .9346	=	23,364.49
2	$20,000× .8734	=	17,468.77
3	$15,000× .8163	=	12,244.47
		Total	$ 3,077.73

Finance.com should invest in a new high-speed computer because the present value of its future cash flows is greater than its cost. The NPV is $3,077.73.

The Algebraic Formula

To derive an algebraic formula for the net present value of a cash flow, recall that the PV of receiving a cash flow one year from now is:

$$\mathbf{PV} = C_1/(1+r)$$

and the PV of receiving a cash flow two years from now is:

$$\mathbf{PV} = C_2/(1+r)^2$$

We can write the NPV of a T-period project as:

$$\mathrm{NPV} = -C_0 + \frac{C_1}{1+r} + \frac{C_2}{(1+r)^2} + \cdots + \frac{C_T}{(1+r)^T} = -C_0 + \sum_{t=1}^{T} \frac{C_i}{(1+r)^T} \qquad [4.5]$$

The initial flow, $-C_0$, is assumed to be negative because it represents an investment. The Σ is shorthand for the sum of the series.

We will close this section by answering the question we posed at the beginning of the chapter concerning baseball player Chris Davis' contract. The terms of the contract called for $17 million per year for 2016 through 2022, $3.5 million per year for 2023 through

2032, and $1.4 million per year for 2033 through 2037. If 12 percent is the appropriate interest rate, what kind of deal did the Oriole's first baseman snag?

To answer, we can calculate the present value by discounting each year's salary back to the present as follows (notice we assume that all the payments are made at year-end and that 12 percent is the appropriate discount rate):

Year 1 (2016):	**$17,000,000 × 1/1.12^1**	=	**$15,178,571.43**
Year 2 (2017):	**$17,000,000 × 1/1.12^2**	=	**$13,552,295.92**
Year 3 (2018):	**$17,000,000 × 1/1.12^3**	=	**$12,100,264.21**
. . . .			
Year 8 (2023):	**$ 3,500,000 × 1/1.12^8**	=	**$ 1,413,591.30**
. . . .			
Year 22 (2037):	**$ 1,400,000 × 1/1.12^{22}**	=	**$ 115,699.51**

If you fill in the missing rows and then add (do it for practice), you will see that Chris' contract had a present value of about $87.3 million, or only about 54 percent of the stated $161 million value (but still pretty good).

As you have probably noticed, doing extensive present value calculations can get to be pretty tedious, so a nearby *Spreadsheet Techniques* box shows how we recommend doing them. As an application, we take a look at lottery payouts in a nearby *Finance Matters* box.

How to Calculate Present Values with Multiple Future Cash Flows Using a Spreadsheet

SPREADSHEET TECHNIQUES

We can set up a basic spreadsheet to calculate the present values of the individual cash flows as follows. Notice that we have calculated the present values one at a time and added them up:

	A	B	C	D	E
1					
2	**Using a spreadsheet to value multiple future cash flows**				
3					
4	What is the present value of $200 in one year, $400 the next year, $600 the next year, and				
5	$800 the last year if the discount rate is 12 percent?				
6					
7	Rate:	.12			
8					
9	Year	Cash flows	Present values	Formula used	
10	1	$200	$178.57	=PV(B7,A10,0,−B10)	
11	2	$400	$318.88	=PV(B7,A11,0,−B11)	
12	3	$600	$427.07	=PV(B7,A12,0,−B12)	
13	4	$800	$508.41	=PV(B7,A13,0,−B13)	
14					
15		Total PV:	**$1,432.93**	=SUM(C10:C13)	
16					
17	Notice the negative signs inserted in the PV formulas. These just make the present values have				
18	positive signs. Also, the discount rate in cell B7 is entered as B7 (an "absolute" reference)				
19	because it is used over and over. We could have just entered ".12" instead, but our approach is more				
20	flexible.				
21					
22					

FINANCE MATTERS

JACKPOT!

If you or someone you know is a regular lottery player, you probably already understand that you are 1,300 times more likely to get struck by lightning than you are to win a big lottery jackpot. What are your odds of winning? Below you will find a table with your chances of winning the Mega Millions Lottery compared to other events.

Odds of winning a Mega Millions jackpot	1:175,711,536*
Odds of being killed in a fireworks discharge	1:652,046
Odds of being killed by a dog	1:144,899
Odds of being killed by lightning	1:134,906
Odds of being killed in an earthquake	1:97,807
Odds of being killed by bees	1:79,842
Odds of being killed by air transport	1:7,178
Odds of being killed by or in a car	1:368

*Source: Mega Millions Lottery website. All other odds from the National Safety Council.

Sweepstakes may have different odds than lotteries, but these odds may not be much better. At one time, the largest advertised potential grand prize ever was Pepsi's "Play for a Billion," which, you guessed it, had a $1 billion (*billion!*) prize. Not bad for a day's work, but you still have to read the fine print. It turns out that the winner would be paid $5 million per year for the next 20 years, $10 million per year for years 21 through 39, and a lump sum $710 million in 40 years. From what you have learned, you know the value of the sweepstakes wasn't even close to $1 billion. In fact, at an interest rate of 10 percent, the present value is about $70.7 million.

In January 2016, three winners split the record $1.584 billion Powerball jackpot. Each winner was given the option of receiving the jackpot as $328 million immediately or $7.9 million per year for the next 30 years, with the first payment to be made immediately. In a unique twist, the payments will increase at 5 percent pear year. So, what discount rate does this imply? After you learn about growing annuities in the next section, see if you don't agree that the interest rate is about 2.79 percent.

Some lotteries make your decision a little tougher. The Ontario Lottery will pay you either $2,000 a week for the rest of your life or $1.3 million now. (That's in Canadian dollars, or "loonies," by the way.) Of course, there is the chance you might die in the near future, so the lottery guarantees that your heirs will collect the $2,000 weekly payments until the 20th anniversary of the first payment, or until you would have turned 91, whichever comes first. This payout scheme complicates your decision quite a bit. If you live for only the 20-year minimum, the break-even interest rate between the two options is about 5.13 percent per year, compounded weekly. If you expect to live longer than the 20-year minimum, you might be better off accepting $2,000 per week for life. Of course, if you manage to invest the $1.3 million lump sum at a rate of return of about 8 percent per year (compounded weekly), you can have your cake and eat it too because the investment will return $2,000 at the end of each week forever! Taxes complicate the decision in this case because the lottery payments are all on an after-tax basis. Thus, the rates of return in this example would have to be after-tax as well.

4.3 COMPOUNDING PERIODS

So far we have assumed that compounding and discounting occur yearly. Sometimes compounding may occur more frequently than just once a year. For example, imagine that a bank pays a 10 percent interest rate "compounded semiannually." This means that a $1,000 deposit in the bank would be worth $1,000 × 1.05 = $1,050 after six months, and $1,050 × 1.05 = $1,102.50 at the end of the year.

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The end-of-the-year wealth can be written as:

$$\$1{,}000\left(1+\frac{.10}{2}\right)^2 = \$1{,}000 \times (1.05)^2 = \$1{,}102.50$$

Of course, a \$1,000 deposit would be worth \$1,100 (= \$1,000 × 1.10) with yearly compounding. Note that the future value at the end of one year is greater with semiannual compounding than with yearly compounding. With yearly compounding, the original \$1,000 remains the investment base for the full year. The original \$1,000 is the investment base only for the first six months with semiannual compounding. The base over the second six months is \$1,050. Hence, one gets *interest on interest* with semiannual compounding.

Because \$1,000 × 1.1025 = \$1,102.50, 10 percent compounded semiannually is the same as 10.25 percent compounded annually. In other words, a rational investor could not care less whether she is quoted a rate of 10 percent compounded semiannually or a rate of 10.25 percent compounded annually.

Quarterly compounding at 10 percent yields wealth at the end of one year of:

$$\$1{,}000\left(1+\frac{.10}{4}\right)^4 = \$1{,}103.81$$

More generally, compounding an investment m times a year provides end-of-year wealth of:

$$C_0\left(1+\frac{r}{m}\right)^m \qquad [4.6]$$

where C_0 is one's initial investment and r is the **annual percentage rate (APR)**. The APR is the annual interest rate without consideration of compounding. Banks and other financial institutions may use other names for the APR.[4]

EXAMPLE 4.11

EARs

What is the end-of-year wealth if Fernando Zapatero receives an annual percentage rate of 24 percent compounded monthly on a \$1 investment?

Using Equation 4.6, his wealth is:

$$\$1\left(1+\frac{.24}{12}\right)^{12} = \$1 \times (1.02)^{12} = \$1.2682$$

The annual rate of return is 26.82 percent. This annual rate of return is either called the **effective annual rate (EAR)** or the **effective annual yield (EAY)**. Due to compounding, the effective annual interest rate is greater than the annual percentage rate of 24 percent. Algebraically, we can rewrite the effective annual interest rate as:

Effective Annual Rate:

$$\left(1+\frac{r}{m}\right)^{m}-1 \qquad [4.7]$$

Students are often bothered by the subtraction of 1 in Equation 4.7. Note that end-of-year wealth is composed of both the interest earned over the year and the original principal. We remove the original principal by subtracting 1 in Equation 4.7.

EXAMPLE 4.12

Compounding Frequencies

If an annual percentage rate of 8 percent is compounded quarterly, what is the effective annual rate?

Using Equation 4.7, we have:

$$\left(1+\frac{r}{m}\right)^{m}-1=\left(1+\frac{.08}{4}\right)^{4}-1=.0824=8.24\%$$

Referring back to our earlier example where C_0 = $1,000 and r = 10%, we can generate the following table:

C_0	COMPOUNDING FREQUENCY (m)	C_1	EFFECTIVE ANNUAL
$1,000	Yearly (m = 1)	$1,100.00	.10
1,000	Semiannually (m = 2)	1,102.50	.1025
1,000	Quarterly (m = 4)	1,103.81	.10381
1,000	Daily (m = 365)	1,105.16	.10516

Distinction between Annual Percentage Rate and Effective Annual Rate

The distinction between the annual percentage rate (APR) and the effective annual rate (EAR) is frequently quite troubling to students. One can reduce the confusion by noting that the APR becomes meaningful only if the compounding interval is given. For example, for an APR of 10 percent, the future value at the end of one year with semiannual compounding is $[1 + (.10/2)]^2 = 1.1025$. The future value with quarterly compounding is $[1 + (.10/4)]^4 = 1.1038$. If the APR is 10 percent but no compounding interval is given, one cannot calculate future value. In other words, one does not know whether to compound semiannually, quarterly, or over some other interval.

By contrast, the EAR is meaningful *without* a compounding interval. For example, an EAR of 10.25 percent means that a $1 investment will be worth $1.1025 in one year. One can think of this as an APR of 10 percent with semiannual compounding or an APR of 10.25 percent with annual compounding, or some other possibility.

There can be a big difference between an APR and an EAR when interest rates are high. For example, consider "payday loans." Payday loans are short-term loans made to consumers, often for less than two weeks. They are offered by companies such as Check Into Cash and AmeriCash

Platinum. The loans work like this: You write a check today that is postdated. When the check date arrives, you go to the store and either pay the cash for the check or the company cashes the check. For example, in one particular state, Check Into Cash allows you to write a check for $115 dated 14 days in the future, for which they give you $100 today. So what are the APR and EAR of this arrangement? First, we need to find the interest rate, which we can find by the FV equation as follows:

$$\text{FV} = \text{PV} \times (1 + r)^1$$
$$\$115 = \$100 \times (1 + r)^1$$
$$1.15 = (1 + r)$$
$$r = .15, \text{ or } 15\%$$

That doesn't seem too bad until you remember this is the interest rate for *14 days!* The APR of the loan is:

$$\text{APR} = .15 \times 365/14$$

$$\text{APR} = 3.9107, \text{ or } 391.07\%$$

And the EAR for this loan is:

$$\text{EAR} = (1 + \text{Quoted rate}/m)^m - 1$$

$$\text{EAR} = (1 + .15)^{365/14} - 1$$

$$\text{EAR} = 37.2366, \text{ or } 3{,}723.66\%$$

Now that's an interest rate! Just to see what a difference a small variation in fees can make, AmeriCash Platinum will make you write a check for $117.50 for the same amount today. Check for yourself that the APR of this arrangement is 456.25 percent and the EAR is 6,598.65 percent. Definitely not a loan we would like to take out!

By law, lenders are required to report the APR on all loans. In this text, we compute the APR as the interest rate per period multiplied by the number of periods in a year. According to federal law, the APR is a measure of the cost of consumer credit expressed as a yearly rate, and it includes interest and certain noninterest charges and fees. In practice, the APR can be much higher than the interest rate on the loan if the lender charges substantial fees that must be included in the federally mandated APR calculation.

Compounding over Many Years

Formula 4.6 applies for an investment over one year. For an investment over one or more (T) years, the formula becomes:

Future Value with Compounding:

$$\text{FV} = C_0\left(1 + \frac{r}{m}\right)^{mT} \qquad [4.8]$$

EXAMPLE 4.13

Multiyear Compounding

Harry DeAngelo is investing $5,000 at an annual percentage rate of 12 percent per year, compounded quarterly, for five years. What is his wealth at the end of five years?

Using Equation 4.8, his wealth is:

$$\$5{,}000 \times \left(1 + \frac{.12}{4}\right)^{4\times 5} = \$5{,}000 \times (1.03)^{20} = \$5{,}000 \times 1.8061 = \$9{,}030.50$$

Continuous Compounding

The previous discussion shows that one can compound much more frequently than once a year. One could compound semiannually, quarterly, monthly, daily, hourly, each minute, or even more often. The limiting case would be to compound every infinitesimal instant, which is commonly called **continuous compounding**. Surprisingly, banks and other financial institutions sometimes quote continuously compounded rates, which is why we study them.

Though the idea of compounding this rapidly may boggle the mind, a simple formula is involved. With continuous compounding, the value at the end of *T* years is expressed as:

$$C_0 \times e^{rT} \qquad [4.9]$$