

## CHAPTER

# 2

# Exploring What It Means to Know and Do Mathematics

## LEARNER OUTCOMES

After reading this chapter and engaging in the embedded activities and reflections, you should be able to:

- 2.1 Investigate what it means to do mathematics.
- 2.2 Describe essential components of mathematical proficiency, including the importance of a relational understanding.
- 2.3 Connect learning theories to effective teaching practices.

This chapter explains how to help students learn mathematics. To get at how to help students learn, however, we must first consider what is important *to* learn. Let's look at a poorly understood topic, division of fractions, as an opening example. If a student has learned this topic well, what will they know and what should they be able to do? The answer is more than being able to successfully implement a procedure (e.g., commonly called the "invert and multiply" procedure). There is much more to know and understand about division of fractions:

- What does  $3 \div \frac{1}{4}$  mean conceptually?
- What situation might this expression represent?
- Will the result be greater than or less than 3 and why?
- What strategies can we employ to solve this problem?
- What illustration or manipulative might illustrate this problem?
- How does this expression relate to subtraction? To multiplication?

All of these questions can be answered by a student who fully understands a topic such as division of fractions. We must help students reach this level of procedural fluency and conceptual understanding.

This chapter focuses on the "what" and "how" of teaching mathematics. First, *what* does doing mathematics look like (be ready to experience this yourself through four great tasks!) and what is important to know about mathematics? Second, *how* do we help students develop a strong understanding of mathematics?



## What Does It Mean to Do Mathematics?

Doing mathematics means generating strategies for solving a problem, applying that strategy, and checking to see whether an answer makes sense. Doing mathematics means demonstrating mathematical process (see Table 1.1), which in the CCSS-M are effectively described in

eight Mathematical Practices (see Table 1.2 and Appendix A). Doing mathematics begins with establishing goals for students that reflect these practices and then posing worthwhile tasks that open up the opportunity for such processes to develop (NCTM, 2014).

## Goals for Students

One way to determine if your goals for students focus on doing mathematics is to consider the verbs in lesson and unit plans. Objectives or instructions that ask students to listen, copy, memorize, drill, and compute are lower-level thinking tasks and do not adequately prepare students for the real act of doing mathematics. In contrast, the following verbs engage students in doing higher-level mathematics:

analyze	design	justify
apply	develop	model
compare	explain	predict
connect	explore	represent
construct	formulate	solve
critique	generalize	use
describe	investigate	verify

These verbs may look familiar to you, as they are on the higher level of Bloom's (revised) Taxonomy (Anderson & Krathwohl, 2001) (see Figure 2.1).

In observing a third-grade classroom where the teacher focused on doing mathematics (i.e., focusing on the process standards), researchers found that students began to (1) connect to previous material, (2) respond with information beyond the required response, and (3) conjecture or predict (Fillingim & Barlow, 2010). When students engage in mathematical processes and practices a daily basis, they receive an empowering message: "You are capable of making sense of this—you are capable of doing mathematics!"

## An Invitation to Do Mathematics

As noted above, after selecting goals that focus on mathematical processes and practices, the next step is to pose worthwhile tasks. The purpose of this section is to provide *you* with opportunities to engage with such tasks—four different problems across the mathematical strands and across K–8. None requires mathematics beyond elementary school mathematics, but they do require higher-level thinking. For each, stop and solve first. Then read the "Explore" section. Continue to engage with the task. Then, you will be doing mathematics and seeing how others may think about the problem differently (or the same). Have fun!

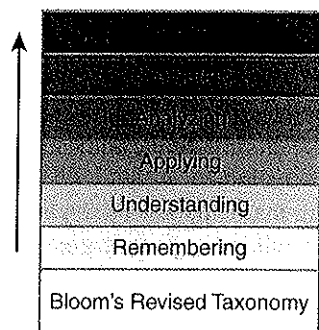
### Task 1. Pattern Search: Start and Jump Numbers.

Begin with a number (start) and add (jump) a fixed amount. For example, start with 3 and jump by 5s. Use the Start and Jump Numbers Activity Page or write the list on a piece of paper. Examine the list and record as many patterns as you see.

#### MyLab Education Activity Page: Start and Jump Numbers

**Explore.** Here are some questions to guide your pattern search:

- Do you see at least one alternating pattern?
- Have you noticed an odd/even pattern? Why is this pattern true?
- What do you notice about the numbers in the tens place?
- Do the patterns change when the numbers are greater than 100?



**FIGURE 2.1** Bloom's (Revised) Taxonomy (Anderson & Krathwohl, 2001).

Source: Anderson, L.W., & Krathwohl, D. R. (Eds.). (2001). *A taxonomy for learning, teaching, and assessing: A revision of Bloom's Taxonomy of educational objectives: Complete Edition*. New York, NY: Addison Wesley, Longman.

Have you thought about what happens to your patterns after the numbers are more than 100, for example 113? One way is as 1 hundred, 1 ten, and 3 ones. But, of course, it could also be “eleventy-three,” where the tens digit has gone from 9 to 10 to 11. How do these different perspectives affect your patterns? What would happen after 999?

**Extend.** Sometimes when you have discovered some patterns in mathematics, it is a good idea to make some changes and see how the changes affect the patterns. What changes might you make in this problem? Your changes may be even more interesting than the following suggestions. But here are some ideas:

- Change the start number but keep the jump number equal to 5. What is the same and what is different?
- Keep the same start number and explore with different jump numbers.
- What patterns do different jump numbers make? For example, when the jump number was 5, the ones-digit pattern repeated every two numbers—it had a “pattern length” of 2. But when the jump number is 3, the length of the ones-digit pattern is 10! Do other jump numbers create different pattern lengths?
  - For a jump number of 3, how does the ones-digit pattern relate to the circle of numbers in Figure 2.2? Are there similar circles of numbers for other jump numbers?
  - Using the circle of numbers for 3, find the pattern for jumps of multiples of 3, that is, jumps of 6, 9, or 12.

Calculators facilitate exploration of this problem. Using calculators make the list generation accessible for young children who can’t skip count yet, and they make it possible to readily explore bigger jump numbers like 25 or 36. Most simple calculators have an automatic constant feature that will add the same number successively. For example, if you press  $3 + 5 =$  and then keep pressing  $=$ , the calculator will keep counting by fives from the previous answer. Consider demonstrating this with an online calculator (<http://calculator-1.com>) or app for the white board so the class can observe and discuss the counting.

## Task 2. Analyzing a Situation: Two Machines, One Job.

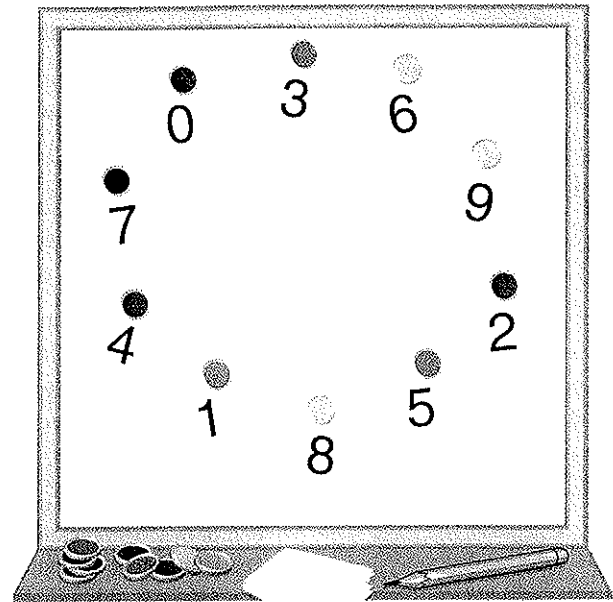
Ron’s recycle shop started when Ron bought a used paper-shredding machine. Business was good, so Ron bought a new shredding machine. The old machine could shred a truckload of paper in 4 hours. The new machine could shred the same truckload in only 2 hours. How long will it take to shred a truckload of paper if Ron runs both shredders at the same time?

**MyLab Education** Application Exercise 2.1: Observing and Responding to Student Thinking Click the link to access this exercise, then watch the video and answer the accompanying questions.



Use the Two Machines, One Job Activity Page to record your solution to this problem. Do not read on until you have an answer or are stuck. Can you check that you are correct? Can you approach the problem using a picture?

**MyLab Education** Activity Page: Two Machines, One Job



**FIGURE 2.2** For jumps of 3, this cycle of digits will occur in the ones place. The start number determines where the cycle begins.

**Explore.** This task is more challenging than the last one, though you might be surprised at how it can be solved logically. Here are some things to consider:

- Have you tried to predict approximately how much time you think it should take the two machines? For example, will it be closer to 1 hour or closer to 4 hours? Estimating can sometimes lead to a new insight.
- What type of illustration might help solve this problem? For example, could you draw a rectangle or a line segment to stand for the truckload of paper?
- Is there a manipulative (chips, plastic cubes, pennies) you might use to make a collection that represents the truckload?

**Strategies.** Hopefully you have solved this problem in some way that makes sense to you—there are lots of ways to solve this particular task! Understanding other people’s strategies can develop our own understanding. And, teachers are always in a position where they must try to figure out how their students are thinking about a problem. The following is one explanation for solving the problem, using strips (based on Schifter & Fosnot, 1993):

“This rectangle [see Figure 2.3] stands for the whole truckload. In 1 hour, the new machine will do half of this.” The rectangle is divided in half. “In 1 hour, the old machine could shred  $\frac{1}{4}$  of the paper.” The rectangle is divided accordingly. “So in 1 hour, the two machines have done  $\frac{3}{4}$  of the truck, and there is  $\frac{1}{4}$  left. What is left is one-third as much as what they already did, so it should take the two machines one-third as long to do that part as it took to do the first part. One-third of an hour is 20 minutes. That means it takes 1 hour and 20 minutes to do it all.

As with the teachers in these examples, it is important to decide whether your solution is correct through justifying why you did what you did; this reflects real problem solving rather than checking with an answer key. After you have justified that you have solved the problem in a correct manner, try to find other ways that students might solve the problem—in considering multiple ways, you are making mathematical connections.

**Task 3. Generalizing Relationships: One Up, One Down.** This problem has two parts, addition and multiplication. For your use, or to use with students, download One Up, One Down: Addition Activity Page or One Up, One Down: Multiplication Activity Page.

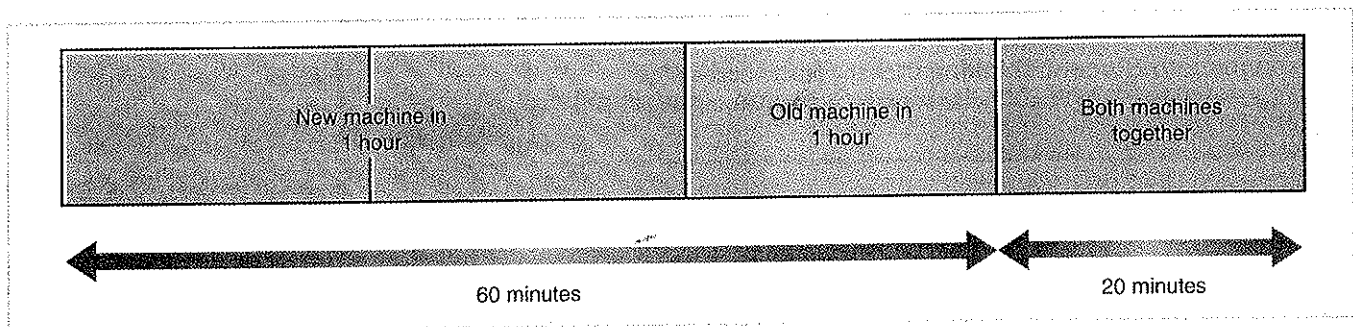
**MyLab Education Activity Page: One Up, One Down: Addition**

**MyLab Education Activity Page: One Up, One Down: Multiplication**

*Addition.* When you add  $7 + 7$ , you get 14. When you make the first number 1 more and the second number 1 less, you get the same answer:

$$\begin{array}{rcccl} & \uparrow & & \downarrow & \\ 7 & + & 7 & = & 14 \\ 8 & + & 6 & = & 14 \end{array}$$

Does it work for  $5 + 5$ ? For what other problems is this pattern true? What else do you notice about this pattern? Why is this pattern true?



**FIGURE 2.3** Cora’s solution to the paper-shredding problem.

**Multiplication.** When you multiply  $7 \times 7$  you get 14. When you make the first number 1 more and the second number 1 less, you get one less:

$$\begin{array}{r} \uparrow \\ 7 \\ \times \\ 8 \end{array} \times \begin{array}{r} \downarrow \\ 7 \\ \times \\ 6 \end{array} = 49$$

$$= 48$$

Does this work for  $5 \times 5$ ? For what other problems is this pattern true? What else do you notice about this pattern? Why is this pattern true?

**Explore.** If you explored both of these, you may have noticed that there are many more patterns or generalizations in the addition situation than in the multiplication situation. Consider:

- What manipulative or picture might illustrate the patterns?
- How is the pattern altered if the sums/products begin as two consecutive numbers (e.g.,  $8 \times 7$ )? If they differed by 2 or by 3?
- What if you instead go "Two up, two down"? (e.g.,  $7 + 7$  to  $9 + 5$  OR  $7 \times 7$  to  $9 \times 5$ )?
- What if both factors increase (i.e., one up, one up)?
- What manipulative or picture might illustrate why the patterns work?
- In what ways is the addition situation similar to and different than the multiplication situation?

**Strategies.** Let's look at the multiplication pattern using illustrations. To compare the before and after products, draw rectangles (or arrays) with a length and height of each of the factors (see Figure 2.4[a]), then draw the new rectangle (e.g., 8-by-6-unit rectangle). You may prefer to think of multiplication as equal sets. For example, using stacks of chips, 7  $\times$  7 is represented by seven stacks with seven chips in each stack (set) (see Figure 2.4[b]). The expression  $8 \times 6$  is represented by eight stacks of six (though six stacks of eight is a possible interpretation). See how the stacks for each expression compare.

Have you made some mathematical connections and conjectures in exploring this problem? In doing so you have hopefully felt a sense of accomplishment and excitement—one of the benefits of *doing* mathematics.

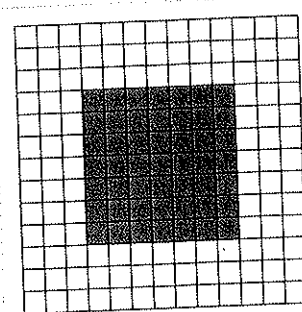
#### Task 4. Experimenting and Justifying: The Best Chance of Purple.

Samuel, Susan, and Sandu are playing a game. The first one to spin "purple" wins! Purple means that one spin lands on red and one spins lands on blue (see Figure 2.5). Each person chooses to spin each spinner once or one of the spinners twice. Samuel chooses to spin spinner A twice; Susan chooses to spin spinner B twice; and Sandu chooses to spin each spinner once. Who has the best chance of purple? (based on Lappan & Even, 1989).

Think about the problem and what you know. Experiment. Use the Best Chance of Purple Activity Page to explore this problem.

**MyLab Education** Activity Page: Best Chance of Purple

(a)



This is  $7 \times 7$  shown as an array of 7 rows of 7.

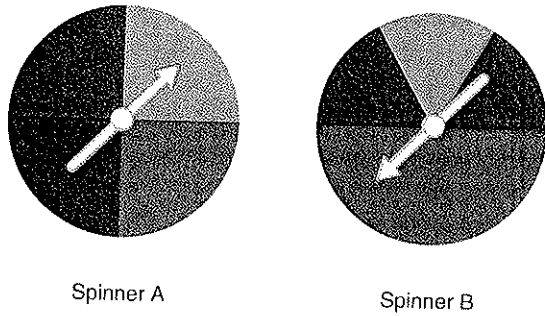
(b)



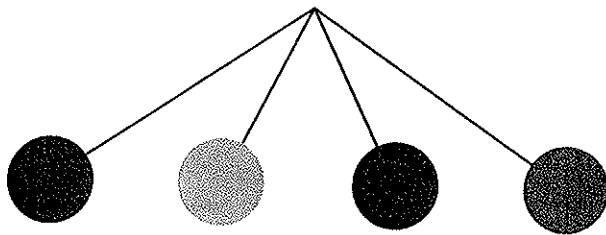
This is  $7 \times 7$  as 7 sets of 7.

What happens when you change one of these to show  $6 \times 8$ ?

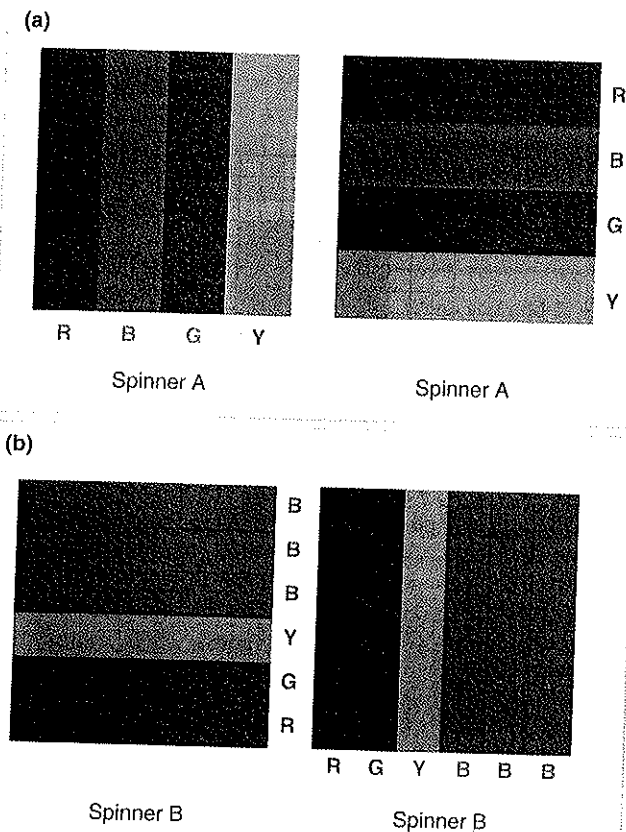
**FIGURE 2.4** Two physical ways to think about multiplication that might help in the exploration.



**FIGURE 2.5** You may spin A twice, B twice, or A then B. Which choice gives you the best chance of spinning a red and a blue (purple)?



**FIGURE 2.6** A tree diagram for spinner A in Figure 2.5.



**FIGURE 2.7** Grids can illustrate the chance of spinning purple with two spins.

**Explore.** A good strategy for learning is to first explore a problem concretely, then analyze it abstractly. This is particularly helpful in situations involving chance or probability. Use a paper clip with the spinners on your Activity Page, or use a virtual spinner. Consider the following as you explore:

- Who is most likely to win and why?
- For Sandu's turn (spinner A, then spinner B), would it matter if he spun B first and then A? Why or why not?
- How might you change one spinner so that Susan has the best chance at purple?

**Strategies.** Just like the earlier tasks, there are multiple strategies for approaching this task.

*Strategy 1: Tree Diagrams.* On spinner A, the four colors each have the same chance of coming up. You could make a tree diagram for A with four branches and all the branches would have the same chance (see Figure 2.6). Spinner B has different-sized sections, leading to the following questions:

- What is the relationship between the blue region and each of the others?
- How could you make a tree diagram for B with each branch having the same chance?
- How can you add to the diagram for spinner A so that it represents spinning A twice in succession?
- Which branches on your diagram represent getting purple?
- How could you make tree diagrams for each player's choices?
- How do the tree diagrams relate to the spinners?

Tree diagrams are only one way to approach this. If the strategy makes sense to you, stop reading and solve the problem. If tree diagrams do not seem like a strategy you want to use, read on.

*Strategy 2: Grids.* Partition squares to represent all the possible outcomes for spinner A and spinner B. Although there are many ways to divide a square into four equal parts, if you use lines going all in the same direction, you can make comparisons of all the outcomes of one event (one whole square) with the outcomes of another event (drawn on a different square). For two independent events, you can then create lines going the other direction for the second event. Samuel's two spins are represented in Figure 2.7(a). If these two squares are overlapped, you can visually see that two parts (two-sixteenths) are "blue on red" or "red on blue." Susan's probability can be determined by layering the squares in Figure 2.7(b); and Sandu's from layering one square from Figure 2.7(a) with one from Figure 2.7(b).

Why are there four parts for spinner A and 6 parts for spinner B? How is the grid strategy alike and different from the tree diagram? One strategy may make more sense to you, and one may make more sense to another. Hearing other students' explanations and reasoning for both strategies are important in building a strong understanding of mathematics.

Interesting mathematics problems such as the four presented here are plentiful. The Math Forum, for example, has

a large collection of classic problems along with discussion, solutions, and extensions. NCTM teacher journals include monthly problems, and readers submit student solutions for these tasks, which appear in an issue a few months later.

## Where Are the Answers?

Did you notice that no answers were shared for these four rich tasks? You may be wondering if your answer is correct, or if there are other answers. But, one aspect of being mathematically proficient is to rely on one's own justification and reasoning to determine if an answer is correct. Consider the message students receive when the textbook or the teacher is the source of whether an answer is correct: "Your job is to find the answers that the teacher already has." In the real world of problem solving and doing mathematics, there are no answer books. A person must be able to make sure they have used an appropriate strategy and reached a reasonable conclusion—we hope you feel this is the case for the tasks you solved in this section.

### MyLab Education Self-Check 2.1



## What Does It Mean to Know Mathematics?

In setting learning objectives for students, we often ask, "What will students know and be able to do?" The previous section focused on what students need to be able to do; this section focused on what students need to know. *Procedural knowledge* refers to *how* to complete an algorithm or procedure. *Conceptual knowledge* refers to *connected* knowledge: "mental connections among mathematical facts, procedures, and ideas" (Hiebert and Grouws 2007, p. 380). Both procedural and conceptual knowledge are foundational to procedural fluency and conceptual understanding, discussed later.

As an example, consider what is important for a student to know about a fraction such as  $\frac{6}{8}$ ? At what point do they know enough that they can claim they "understand" fractions? It is more complicated than it might first appear. Here is a partial list of what they might know or be able to do:

- Read the fraction.
- Identify the 6 and 8 as the numerator and denominator, respectively.
- Recognize it is equivalent to  $\frac{3}{4}$ .
- Know it is more than  $\frac{1}{2}$  (recognize relative size).
- Draw a region that is shaded in a way to show  $\frac{6}{8}$ .
- Locate  $\frac{6}{8}$  on a number line.
- Illustrate  $\frac{6}{8}$  of a set of 48 pennies or counters.
- Know that there are infinitely many equivalencies to  $\frac{6}{8}$ .
- Recognize  $\frac{6}{8}$  as a rational number.
- Realize  $\frac{6}{8}$  might also be describing a ratio (girls to boys, for example).
- Be able to represent  $\frac{6}{8}$  as a decimal fraction.

A number of items on this list refer to procedural knowledge (e.g., being able to find an equivalent fraction) and others refer to conceptual knowledge (e.g., recognizing  $\frac{6}{8}$  is greater than  $\frac{1}{2}$  by analyzing the relationship between the numerator and denominator). A student may know that  $\frac{6}{8}$  can be simplified to  $\frac{3}{4}$  but not recognize that  $\frac{3}{4}$  and  $\frac{6}{8}$  are equivalent (having procedural knowledge without conceptual knowledge). A student may be able to find one fraction between  $\frac{1}{2}$  and  $\frac{6}{8}$ , but not be able to find others, meaning that while they have some procedural knowledge to find a common denominator, they do not have enough conceptual knowledge to recognize they could also change to denominators of sixteenths to find more in-between fractions (and recognize there are infinitely many options).

*Understanding* is hard to define, but it can be explained as a measure of the quality and quantity of connections that an idea has with existing ideas. The extent that a student understands why an algorithm works or understands relationships is their depth of understanding.

## Relational Understanding

Understanding exists along a continuum from an *instrumental understanding*—doing something without understanding (see Figure 2.8) to a *relational understanding*—knowing what to do and why. These two terms were introduced by Richard Skemp in 1978 and continue to be a useful way to think about the depth of a student’s mathematical understanding.

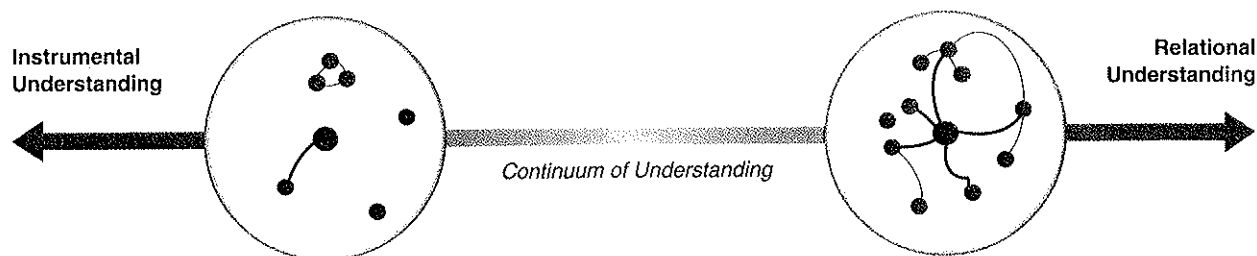
In the  $\frac{6}{8}$  example, a student who only knows a procedure for simplifying  $\frac{6}{8}$  to  $\frac{3}{4}$  has an understanding near the instrumental end of the continuum, while a student who can draw diagrams, give examples, and find numerous equivalencies, has an understanding toward the relational end of the continuum. Here we briefly share two (interrelated) important ways to nurture a relational understanding.

**Explore with Tools.** A *tool* is any object, picture, or drawing that can be used to explore a concept. *Manipulatives* are physical objects that students and teachers can use to illustrate and discover mathematical concepts, whether made specifically for mathematics (e.g., connecting cubes) or for other purposes (e.g., buttons). Choices for manipulatives (including virtual manipulatives) abound—from common objects such as lima beans to commercially produced materials such as Pattern Blocks. Figure 2.9 shows six examples, each representing a different concept, just to give a glimpse (Part II of this book is full of more options). More and more of these interactives and others (e.g., algebra tiles, geometric solids, number lines, adjustable spinners) are available in a digital format, for example, on NCTM’s Illuminations website (<http://illuminations.nctm.org>) and on the National Library of Virtual Manipulatives (NLVM) website (<http://nlvm.usu.edu>).

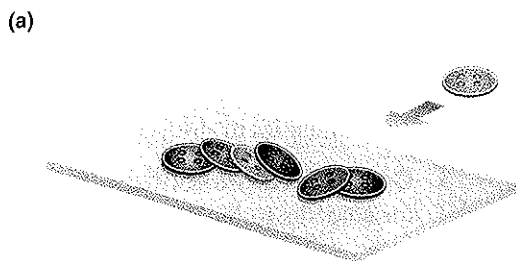
A tool does not “illustrate” a concept. The tool is used to visualize a mathematical concept and only your mind can impose the mathematical relationship on the object (Suh, 2007b; Thompson, 1994). As noted in *Task 4: Experimenting and Justifying: The Best Chance of Purple*, manipulatives can be a testing ground for emerging ideas. They are more concrete and provide insights into new and abstract relationships. Consider each of the concepts and the corresponding model in Figure 2.9. Try to separate the physical tool from the relationship that you must impose on the tool in order to *see* the concept.

The examples in Figure 2.9 are models that can show the following concepts:

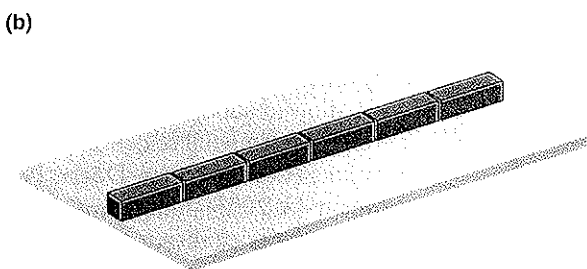
- The concept of “6” is a relationship between sets that can be matched to the words *one, two, three, four, five, or six*. Changing a set of counters by adding one changes the relationship. The difference between the set of 6 and the set of 7 is the relationship “one more than.”
- The concept of “measure of length” is a comparison. The length measure of an object is a comparison relationship of the length of the object to the length of the unit.
- The concept of “rectangle” includes both spatial and length relationships. The opposite sides are of equal length and parallel and the adjacent sides meet at right angles.
- The concept of “hundred” is not in the larger square but in the relationship of that square to the strip (“ten”) and to the little square (“one”).
- “Chance” is a relationship between the frequency of an event happening compared with all possible outcomes. The spinner can be used to create relative frequencies. These can be predicted by observing relationships of sections of the spinner.



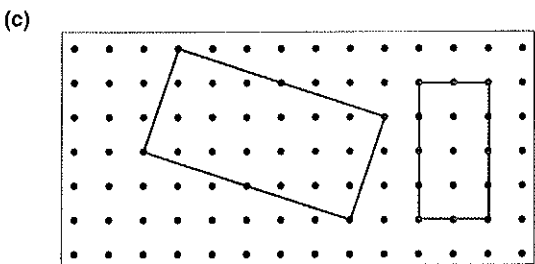
**FIGURE 2.8** Understanding is a measure of the quality and quantity of connections that a new idea has with existing ideas. The greater the number of connections to a network of ideas, the better the understanding.



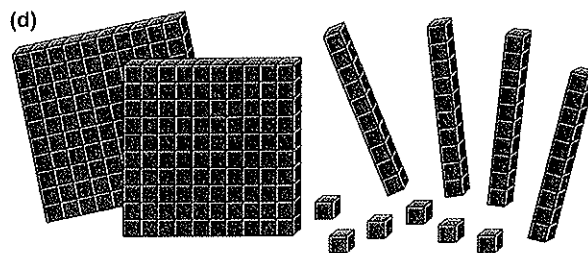
(a) Countable objects can be used to model "number" and related ideas such as "one more than."



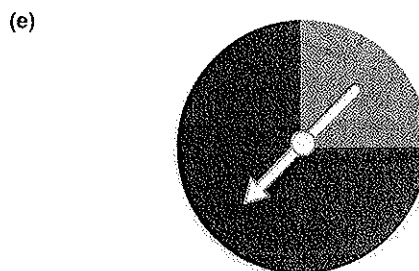
(b) "Length" involves a comparison of the length attribute of different objects. Rods can be used to measure length.



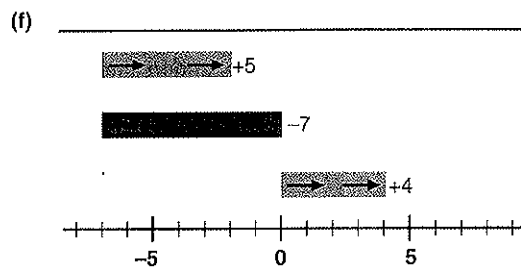
(c) "Rectangles" can be modeled on a dot grid. They involve length and spatial relationships.



(d) Base-ten concepts (ones, tens, hundreds) are frequently modeled with base 10 blocks. Sticks and bundles of sticks are also commonly used.



(e) "Chance" can be modeled by comparing outcomes of a spinner.



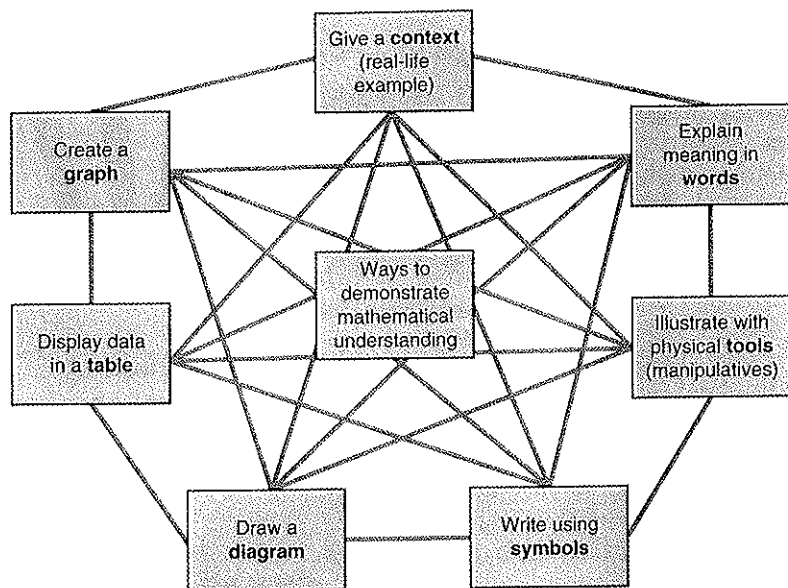
(f) "Positive" and "negative" integers can be modeled with arrows with different lengths and directions.

**FIGURE 2.9** Examples of tools to illustrate mathematics concepts.

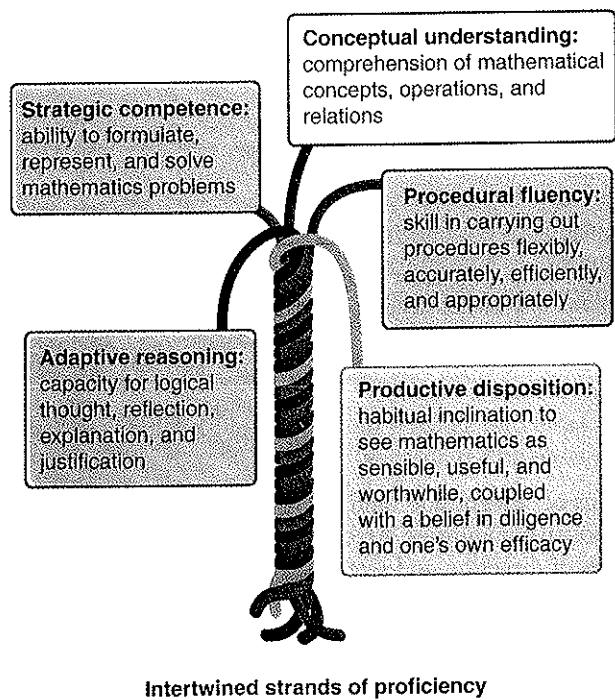
f. The concept of a "negative integer" is based on the relationships of "magnitude" and "is the opposite of." Negative quantities exist only in relation to positive quantities. Arrows on the number line model the opposite of relationship in terms of direction and size or magnitude relationship in terms of length.

A variety of tools (including calculators) should be accessible for students to select and use appropriately as they engage in doing mathematics.

While tools can be used to support relational understanding, they can be used ineffectively and not accomplish this goal. One of the most widespread misuses of tools occurs when the teacher tells students, "Do as I do." There is a natural temptation to get out the materials and show students how to use them to "show" the concept. It is just as possible to move blocks around mindlessly as it is to "invert and multiply" mindlessly. Neither promotes thinking or aids in the development of concepts (Ball, 1992; Clements & Battista, 1990; Stein & Bovalino, 2001). On the other extreme, it is ineffective to provide no focus or purpose for using the tools. This will result in nonproductive



**FIGURE 2.10** Web of Representations. Translations between and within each representation of a mathematical idea can help students build a relation understanding of a mathematical concept.



**FIGURE 2.11** *Adding It Up* describes five strands of mathematical proficiency.

Source: National Research Council. (2001). *Adding It Up: Helping Children Learn Mathematics*, p. 5. Reprinted with permission from the National Academy of Sciences, courtesy of the National Academies Press, Washington, DC.

and unsystematic investigation (Stein & Bovalino, 2001). The goal is to set up tasks with the tools so that students notice important mathematical relationships that can be discussed, connecting the concrete representations to abstract concepts.

**Connect Representations.** In order for students to build connections among ideas, different representations must be included in instruction, and opportunities must be provided for students to make connections among the representations. Figure 2.10 illustrates a Web of Representations showing ways to demonstrate understanding. Students who have difficulty translating a concept from one representation to another also have difficulty solving problems and understanding computations (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). Therefore, strengthening students' ability to move between and among these representations improves their understanding. For example, give students the Translation Task Activity Page to complete for a topic they are learning. You can fill out one box and ask them to insert the other representations, or you can invite a group to work on all four representations for a given topic (e.g., multiplication of whole numbers).

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**MyLab Education Activity Page: Translation Task**

**Mathematical Proficiency**

This chapter started with an invitation to do mathematics, engaging you in the mathematical processes or practices (See Tables 1.1, 1.2, and Appendix A). Students who are able to demonstrate these practices are mathematically proficient. In other words, proficiency isn't only knowing the list of content from your grade, it is being able to demonstrate the practices *as it applies to that content*. The mathematical practices are based on research on how students learn as described in the National Research Council (NRC) report *Adding It Up* (NRC, 2001), which can be read for free at <https://www.nap.edu/read/9822>. Figure 2.11 illustrates these interrelated and interwoven strands.

**Conceptual Understanding.** Conceptual understanding is a flexible web of connections and relationships within and between ideas, interpretations and images of mathematical concepts—a relational understanding. Consider the web of associations for ratio as shown in Figure 2.12. Students with a conceptual understanding will connect what they know about division and numbers to make sense of scaling, unit prices, and so on. Note how much is involved in having a relational understanding of ratio.

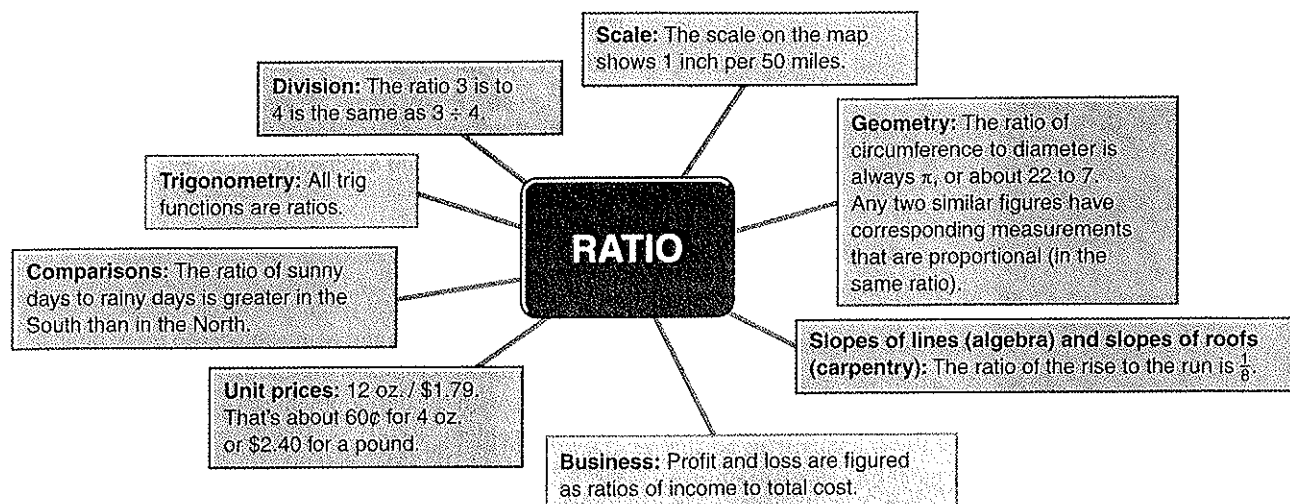


FIGURE 2.12 Potential web of ideas that could contribute to the understanding of "ratio."

Conceptual understanding includes the network of representations and interpretations of a concepts through the use of pictures, manipulatives, tables, graphs, words, and so on (see Figure 2.10). An illustration for ratios across these representations is provided in Figure 2.13.

**Procedural Fluency.** Procedural fluency is sometimes confused with being able to do standard algorithms correctly and quickly, but it is much bigger than that. Look at the four descriptors of procedural fluency in Figure 2.11. Recall that procedural knowledge is a foundation (knowing how to do an algorithm). Also important is procedural understanding (knowing why an algorithm works). But, procedural fluency is more than procedural understanding. Fluency includes four components: efficiency, accuracy, flexibility and appropriate strategy selection. Let's look at the problem  $37 + 28$ . Figure 2.14 illustrates four approaches. Which ones represent a student who has procedural fluency with two-digit addition? A fluent student does not automatically stack the numbers and apply the standard algorithm (though they know how to do this); the fluent student looks at the problem and considers which strategy will be *efficient* given the numbers in the problem and *selects an appropriate strategy*. In this case, a student might move two from the 37 to the 28 to adapt the equation to  $35 + 30$  (If this reminds you of One Up, One Down, good for you!). Or a student might add 2 onto 28, add  $37 + 30$ , and then take 2 away from the answer (see Figure 2.14[c]). Given a different problem, like  $54 + 37$ , the student might opt for a different strategy, such as adding the tens (80) and the ones (11) and combing them (91), showing *flexibility* in which strategies they use.

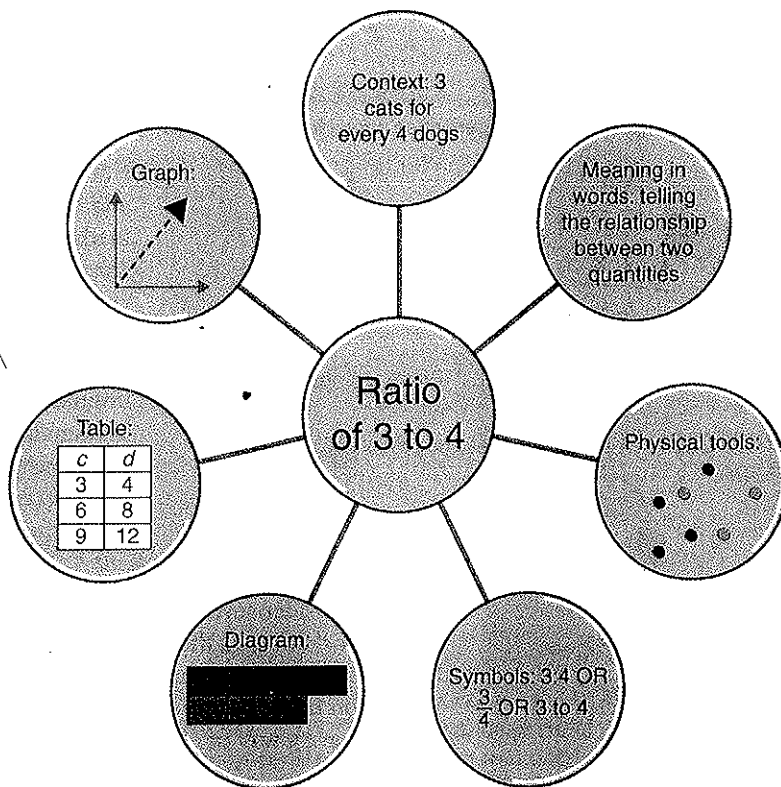
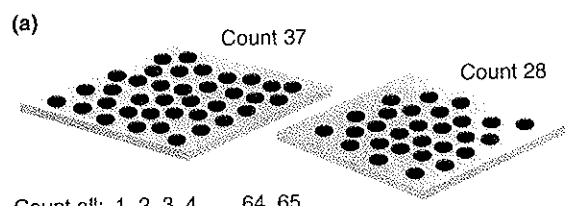
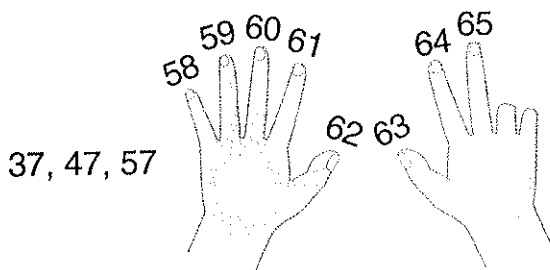


FIGURE 2.13 Multiple representations for ratio of 3 to 4.



Count all: 1, 2, 3, 4, ..., 64, 65

(b) 37 and 20 more—47, 57, 58, 59, 60, 61, 62, 63, 64, 65  
(counting on fingers)



Base 10 approach (counting 10s then 1s)

(c) Take 2 from the 37 and put it with the 28 to make 30. 30 and 35 is 65.  
 $37 + 28 = 65$   
 $35 + 30 = 65$

37 and 30 is 67, but you have to take 2 away—65.  
 $37 + 30 = 67$   
 $67 - 2 = 65$

Efficient, student generated strategies

$$\begin{array}{r} 37 \\ + 28 \\ \hline 65 \end{array}$$

Traditional algorithm

$$\begin{array}{r} 37 \\ + 28 \\ \hline 515 \end{array}$$

Errors are often made

FIGURE 2.14 A range of levels of procedural fluency for  $37 + 28$ .

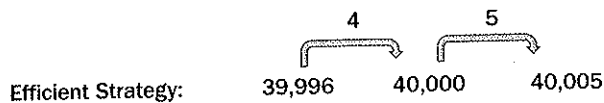
MyLab Education

Video Example 2.1

Watch Jennifer Bay-Williams discuss procedural fluency as it relates to multi-digit addition and subtraction.



Ironically, procedural fluency is often mistaken for learning standard algorithms and being able to do them quickly and accurately. An overemphasis on standard algorithms can actually interfere with the development of fluency. Think about the following problem:  $40,005 - 39,996 = \underline{\hspace{2cm}}$ . Applying the standard algorithm involves regrouping across zeros, a tedious, inefficient, and prone-to-error method. Noticing that the numbers are close together, and therefore lend to a counting up strategy, efficiently leads to a result of 9. Number lines are an important representation in building fluency:



Efficient Strategy:

Procedures support concepts (and vice versa) and both conceptual and procedural knowledge contributed to student development of procedural flexibility (Schneider, Rittle-Johnson, & Star, 2011). Both conceptual understanding and procedural fluency are crucial to becoming mathematically proficient (Baroody, Feil, & Johnson, 2007; Bransford, Brown, & Cocking, 2000).

**Productive Disposition.** As the Mathematical Practices and the Strands of Proficiency describe, being proficient at mathematics is not just what you know, but how you go about solving problems. Mathematical practices need to be interwoven with conceptual development (Kobiela & Lehrer, 2015). When teachers are intentional and explicit about mathematical practices, students' participation and confidence improves—they develop a productive disposition (Wilburne, Wildmann, Morret, & Stipanovic, 2014). For which of the questions below might a student with a productive disposition typically say “yes”?

- When you read a problem you don't know how to solve, do you think, “Great, something challenging. I can solve this.”?
- Do you consider several possible approaches before diving in to solve?
- As you work, do you draw a picture or use a manipulative?
- Do you recognize a wrong path and try something else?
- When you finish a problem, do you wonder whether it is right? If there are other answers?
- Do you have a way of convincing yourself or a peer that your answer is correct?

And, if students say yes to these questions, which mathematical practices are they accessing? Procedural fluency and conceptual understanding are supported by and support a productive disposition.



## How Do Students Learn Mathematics?

Now that you have had the chance to experience doing mathematics, you may have a series of questions: Why take the time to solve these problems— isn't it better to do a lot of shorter practice problems? Can students solve such challenging tasks? In other words, how does "doing mathematics" relate to learning mathematics? The answer lies in learning theory and research on how people learn.

Learning theories such as constructivism and sociocultural theory have influenced the way in which mathematics is taught. Your teaching practices will be influenced by how you believe people learn, which may be informed by one of these learning theories and from your own pragmatic experiences. It is important for you to attend to your own beliefs and how they relate to your teaching practice (Davis & Sumara, 2012). Here we briefly describe two theories that are important to understanding how students learn mathematics. These theories are not competing and are compatible (Norton & D'Ambrosio, 2008).

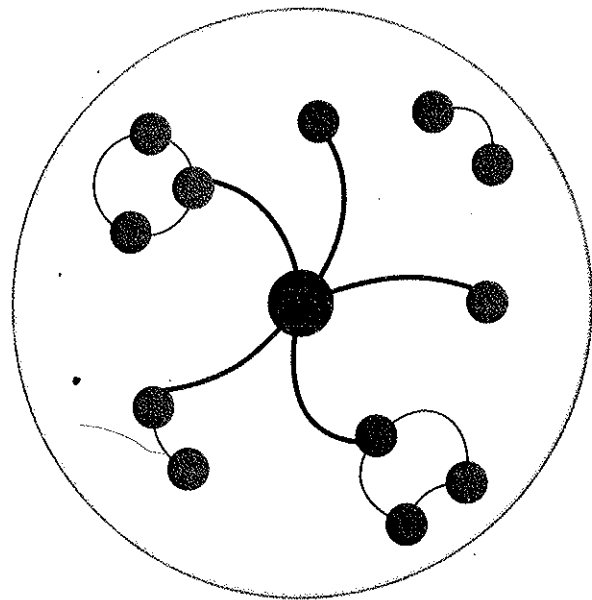
### Constructivism

*Constructivism* is rooted in Jean Piaget's work, developed in the 1930s and translated to English in the 1950s. At the heart of constructivism is the notion that learners are not blank slates but rather creators or constructors of their own learning (Cobb, 1988; Fosnot, 1996; von Glasersfeld, 1996). Integrated *networks*, or *cognitive schemas*, are formed by constructing knowledge and they are used to build new knowledge. Through *reflective thought*—the effort to connect existing ideas to new information—people modify their existing schemas to incorporate new ideas (Fosnot, 1996). All people construct or give meaning to things they perceive or think about. As you read these words you are giving meaning to them. Whether listening passively to a lecture or actively engaging in synthesizing findings in a project, your brain is applying prior knowledge in your existing schemas to make sense of the new information.

To connect to the metaphor of building construction, the *tools* we use to build understanding are our existing ideas and knowledge. The *materials* we use are things we see, hear, or touch, as well as our own thoughts and ideas. In Figure 2.15, blue and red dots are used as symbols for ideas. Consider the picture to be a small section of our cognitive makeup. The blue dots represent existing ideas. The lines joining the ideas represent our logical connections or relationships that have developed between and among ideas. The red dot is an emerging idea, one that is being constructed. Whatever existing ideas, blue dots, are used in the construction will be connected to the new idea, red dot, because those were the ideas that gave meaning to it. If a potentially relevant idea, blue dot, is not accessed by the learner when learning a new concept, red dot, then that potential connection will not be made.

### Sociocultural Theory

In the 1920s and 1930s, Lev Vygotsky, a Russian psychologist, began developing what is now called sociocultural



**FIGURE 2.15** We use the ideas we already have (blue dots) to construct a new idea (red dot), in the process developing a network of connections between ideas. The more ideas used and the more connections made, the better we understand.

theory. Like constructivism, this theory assumes active meaning-seeking on the part of the learner. An important aspect of sociocultural theory is that the way in which information is internalized, or learned, depends on whether it was within a learner's zone of proximal development (ZPD) (Vygotsky, 1978). Simply put, the ZPD refers to a range of knowledge that may be out of reach for a person to learn on his or her own, but is accessible if the learner has support from peers or more knowledgeable others. In a true mathematical community of learners there is something of a common ZPD that emerges across learners as well as the individual ZPDs of each person (Cobb, 1994; Goos, 2004). Social interaction is essential for learning to occur. And, a community of learners is affected not only by culture the teacher creates, but by the broader social and historical culture of the members of the classroom (Forman, 2003).

### Implications for Teaching Mathematics

Learning theories are not teaching strategies—theory *informs* teaching. This section outlines teaching strategies that are informed by constructivist and sociocultural perspectives. You will see these strategies revisited in more detail in Chapters 3 and 4, where a problem-based model for instruction is discussed, and in Part II of this book, where you learn how to apply these ideas to specific areas of mathematics.

Importantly, if these strategies are grounded in how people learn, it means *all* people learn this way—students with special needs, English language learners, students who struggle, and students who are gifted. Too often, when teachers make adaptations and modifications for particular learners, they trade in strategies that align with learning theories and research for methods that seem “easier” for students. These strategies, however, provide fewer opportunities for students to connect ideas and build knowledge—thereby impeding, not supporting, learning.

**Build New Knowledge from Prior Knowledge.** If you are teaching a new concept, like division, it must be developed using what students already know about equal subtraction and sharing. Consider the following task and how you might introduce it to third graders and if you are grounding your work in constructivist or sociocultural learning theories.

---

Goodies Toy Store is creating bags with 3 squishy balls in each. If they have 24 squishy balls, how many bags will they be able to make?

---

You will plan for students to access their prior knowledge, use cultural tools, and build new knowledge. You might ask students to use manipulatives or to draw pictures to solve this problem. As they work, they might have different ways of thinking about the problem (e.g., skip counting up by 3s, or skip counting down by 3s). These ideas become part of a classroom discussion, connecting what they know about equal subtraction and addition, and connecting that to multiplication and division.

Interestingly, this practice of connecting ideas is not only grounded in learning theory, but has been established through research studies. Making mathematical relationships explicit is connected with improving student conceptual understanding (Hiebert & Grouws, 2007). The teacher's role in making mathematical relationships explicit is to be sure that students are making the connections that are implied in a task. For example, asking students to relate today's topic to one they investigated last week, or by asking “How is Laila's strategy like Marco's strategy?” are ways to be “explicit” about mathematical relationships.

**Provide Opportunities to Communicate about Mathematics.** Learning is enhanced when the learner is engaged with others working on the same ideas. The interaction in such a classroom allows students to engage in reflective thinking and to internalize concepts that may be out of reach without the interaction and input from peers and their teacher. In discussions with peers, students will be adapting and expanding on their existing networks of concepts.

**MyLab Education Video Example 2.2**

Watch this video of Cathy's classroom problem solving task and how the students are communicating their process and solutions.



**Create Opportunities for Reflective Thought.** Classrooms need to provide structures and supports to help students make sense of mathematics in light of what they know. For a new idea to be interconnected in a rich web of interrelated ideas, children must be mentally engaged. They must find the relevant ideas they possess and bring them to bear on the development of the new idea. In terms of the dots in Figure 2.15 we want to activate every blue dot students have that is related to the new red dot we want them to learn. It is through student thinking, talking, and writing, that we can help them reflect on how mathematical ideas are connected to each other.

**Encourage Multiple Strategies.** Encourage students to use strategies that make sense to them. The student whose work is presented in Figure 2.16 may not understand the algorithm she used. If instead she were asked to use her own approach to find the difference, she might be able to get to a correct solution and build on her understanding of place value and subtraction.

Even learning a basic fact, like  $7 \times 8$ , can have better results if a teacher promotes multiple strategies. Imagine a class where students discuss and share ways to figure out the product. One student might think of 5 eights (40) and then 2 more eights (16) to equal 56. Another may have learned  $7 \times 7$  (49) and add on 7 more to get 56. Still another might think “8 sevens” and take half of the sevens ( $4 \times 7$ ) to get 28 and double 28 to get 56. A class discussion sharing these ideas brings to the fore a wide range of useful mathematical “dots” relating addition and multiplication concepts.

**Engage Students in Productive Struggle.** Have you ever just wanted to think through something yourself without being interrupted or told how to do it? Yet, how often in mathematics class does this happen? As soon as a student pauses in solving a problem the teacher steps in to show or explain. While this may initially get the student to an answer faster, it does not help the student learn mathematics—engaging in productive struggle is what helps students learn mathematics. As Piaget describes, learners are going to experience disequilibrium in developing new ideas. Let students know this disequilibrium is part of the process.

Productive struggle is critical to developing conceptual understanding (Hiebert & Grouws, 2007). Notice the importance of both words in “productive struggle.” Students must have the tools and prior knowledge to solve a problem, and not be given a problem that is out of reach, or they will struggle without being productive; yet students should not be given tasks that are straightforward and easy or they will not be struggling with mathematical ideas. When students, even very young students, know that struggle is expected as part of the process of doing mathematics, they embrace the struggle and feel success when they reach a solution (Carter, 2008). We must redefine what it means to “help” students. Rather than showing students how to do something, we must employ strategies, like asking probing questions, that keep students engaged in productive struggle.

**Treat Errors as Opportunities for Learning.** When students make errors, it can mean a misapplication of their prior knowledge in a new situation. Remember that from a constructivist perspective, the mind is sifting through what it knows in order to find useful approaches for the new situation. Students rarely give random responses, so their errors are insight into limited or misconceptions they might have. For example, students comparing decimals may incorrectly

$$\begin{array}{r} 5 \ 13 \\ \cancel{0} \cancel{0} \\ - 257 \\ \hline 6 \end{array}$$

There is nothing in this next column, so I'll borrow from the 6.

**FIGURE 2.16** This student's work indicates that she has a misconception about place value and regrouping.

apply “rules” of whole numbers, such as “the more digits, the bigger the number” (Martinie, 2014). Often one student’s misconception is shared by others in the class and discussing the problem publicly can help other students understand (Hoffman, Breyfogle, & Dressler, 2009). You can introduce errors and ask students to imagine what might have led to that answer (Rathouz, 2011). This public negotiation of meaning allows students to construct deeper meaning for the mathematics they are learning.

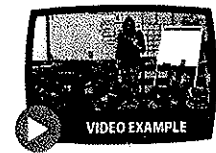
**Scaffold New Content.** The practice of *scaffolding*, often associated with sociocultural theory, is based on the idea that a task otherwise outside of a student’s ZPD can become accessible if it is carefully structured. For concepts completely new to students, the learning requires more structure or assistance, including the use of tools (e.g., manipulatives) or more assistance from peers. As students become more comfortable with the content, the scaffolds are removed and the student becomes more independent. Scaffolding can provide support for those students who may not have a robust collection of “blue dots.”

**Honor Diversity.** Finally, and importantly, these theories emphasize that each learner is unique, with a different collection of prior knowledge and cultural experiences. Since new knowledge is built on existing knowledge and experience, effective teaching incorporates and builds on what the students bring to the classroom, honoring those experiences. Thus, lesson contexts are selected based on students’ interests, knowledge and experiences. Classroom culture influences the individual learning of your students. Support a range of approaches and strategies for doing mathematics. Each students’ ideas should be valued and included in classroom discussions of the mathematics. (See also the discussion of Culturally Responsive Mathematics Instruction in Chapter 6.)

#### MyLab Education Self-Check 2.3

#### MyLab Education Application Exercise 2.2:

**How Do Students Learn Mathematics?** Click the link to access this exercise, then watch the video and answer the accompanying questions.



## Connecting the Dots

It seems appropriate to close this chapter by connecting some dots, especially because the ideas represented here are the foundation for the approach to each topic in the content chapters. This chapter began with discussing what *doing* mathematics is and challenging you to do some mathematics. Each of these tasks offered opportunities to make connections between mathematics concepts—connecting the blue dots.

Second, you read about what is important to know about mathematics—that having relational knowledge (knowledge in which blue dots are well connected) requires conceptual and procedural understanding as well as other proficiencies. The problems that you solved in the first section emphasized concepts and procedures while placing you in a position to use strategic competence, adaptive reasoning, and a productive disposition.

Finally, you read how learning theory—the importance of having opportunities to connect the dots—connects to mathematics learning. The best learning opportunities, according to constructivism and sociocultural theories, are those that engage learners in using their own knowledge and experience to solve problems through social interactions and reflection. This is what you were asked to do in the four tasks. Did you learn something new about mathematics? Did you connect an idea that you had not previously connected?

This chapter focused on connecting the dots between theory and practice—building a case that your teaching must focus on opportunities for students to develop their own networks of blue dots. As you plan and design instruction, you should constantly reflect on how to elicit prior knowledge by designing tasks that reflect the social and cultural backgrounds of students, to challenge students to think critically and creatively, and to include a comprehensive treatment of mathematics.



## RESOURCES FOR CHAPTER 2

### RECOMMENDED READINGS

#### Articles

Carter, S. (2008). Disequilibrium & questioning in the primary classroom: Establishing routines that help students learn. *Teaching Children Mathematics*, 15(3), 134–137.

*This is a wonderful teacher's story of how she infused the constructivist notion of disequilibrium and the related idea of productive struggle to support learning in her first-grade class.*

Whitacre, I., Schoen, R. C., Champagne, Z., & Goddard, A. (2016). Relational thinking: What's the difference? *Teaching Children Mathematics*, 23(5), 303–309.

*These authors illustrate the importance of relational thinking sharing interesting results from students related to subtraction, and offering strategies for developing procedural flexibility.*

Wilburne, J. M., Wildmann, T., Morret, M., & Stipanovic, J. (2014). Classroom strategies to make sense and persevere. *Mathematics Teaching in the Middle School*, 20(3), 144–151.

*Four strategies are shared that not only help build perseverance, but also develop the other mathematical practices and ultimately develop students' productive dispositions.*

#### Books

Boaler, J. (2016). *Mathematical mindsets: Unleashing student potential through creative math, inspiring messaging, and innovative teaching*. San Francisco, CA: Jossey Bass.

*Full of excellent tasks and strategies for engaging students, this very popular book is a great resource for K–12 teachers. Research and examples build a strong case for engaging students in doing mathematics to help them become mathematically proficient.*

Mason, J., Burton, L., & Stacey, K. (2010). *Thinking mathematically* (2nd ed.). Harlow, England: Pearson Education.

*This classic book is about doing mathematics. There are excellent problems to explore along the way, with strategy suggestions. It is an engaging book that will help you learn more about your own problem solving and become a better teacher of mathematics.*