37. Two runners start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed. [*Hint:* Consider $f(t) = g(t) - h(t)$, where g and h are the position functions of the two runners.]

Runners run in the interval (a, b)

$f(t) = g(t) - h(t)$ $\quad f'(t) = g'(t) - h'(t)$

$f(t)$ is continuous in the interval $[a, b]$

differentiate in the interval (a, b)

From the information given,

$f(a) = f(b)$

By Rolle's Theorem, there exist a number c in the interval (a, b), and $f'(c) = 0$

$\therefore f'(c) = g'(c) - h'(c) = 0.$

$\therefore g'(c) = h'(c)$

Hence the speed is same at some time during the race.

12. $f(x) = x^3 - 3x + 2$, $[-2, 2]$

$a = -2$, $b = 2$

$f'(x) = 3x^2 - 3$

$f'(c) = 3c^2 - 3$

By the Mean Value Theorem, there exist a number $c \in (-2, 2)$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{8 - 6 + 2 - [(-8) + 6 + 2]}{4}$$

$$= 1$$

$\therefore 3c^2 - 3 = 1$

$$c^2 = \frac{4}{3}$$

$$\therefore c = \pm \frac{2\sqrt{3}}{3}$$

$\pm \frac{2\sqrt{3}}{3} \in (-2, 2)$

$$\therefore c = \pm \frac{2\sqrt{3}}{3}$$

14. $f(x) = \frac{1}{x}$, $[1, 3]$

$a = 1$, $b = 3$

$f'(x) = -\frac{1}{x^2}$

$f'(c) = -\frac{1}{c^2}$

By the Mean Value Theorem, there exist a number $c \in (1, 3)$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$= \frac{f(3) - f(1)}{3 - 1}$$

$$= \frac{\frac{1}{3} - 1}{2} = -\frac{1}{3}$$

$\because -\frac{1}{c^2} = -\frac{1}{3}$

$\therefore c = \pm\sqrt{3}$

$\because c \in (1, 3)$

$\therefore c = \sqrt{3}$.

4.2 Exercises

9. Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but there is no number c in $(-1, 1)$ such that $f'(c) = 0$. Why does this not contradict Rolle's Theorem?

$f(-1) = 1-(-1)^{\frac{2}{3}} = 1-[(-1)^2]^{\frac{1}{3}} = 0$

$f(1) = 1-(1)^{\frac{2}{3}} = 0$

$\therefore f(-1) = f(1) = 0$

$f'(x) = -\frac{2}{3}x^{-\frac{1}{3}} = -\frac{2}{3\sqrt[3]{x}}$

$\therefore f'(x)$ is not defined at $x=0$.

$f'(0)$ deos not exist.

$\therefore f(x)$ is not differentiable on the open interval $(-1,1)$, which is one of conditions for the Rolle's Theorem to be established.

Hence, This not contradict Rolle's Theorem.

11–14 Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

11. $f(x) = 2x^2 - 3x + 1$, $[0, 2]$

$a=0, b=2$

$f'(x) = 4x-3$

$\therefore f'(c) = 4c-3$

By the Mean Value Theorem, there exist a number $c \in (0,2)$

$$f'(c) = \frac{f(2)-f(0)}{2-0}$$

$$= \frac{8-6+1-1}{2}$$

$$= 1$$

$\therefore 4c-3 = 1$

$c = 1$

$1 \in (0,2)$

$\therefore c = 1$

47–62 Find the absolute maximum and absolute minimum values of f on the given interval.

~~49~~. $f(x) = 2x^3 - 3x^2 - 12x + 1, \quad [-2, 3]$

$f'(x) = 6x^2 - 6x - 12$

$\quad = 6(x+1)(x-2)$

Let $f'(x) = 0$.

$\therefore x = -1$ or $x = 2$

$-1, 2$ is in the interval $[-2, 3]$

Substitute $x = -1$ in $f(x)$

$\therefore f(-1) = -2 - 3 + 12 + 1 = 8$

Substitute $x = 2$ in $f(x)$

$\therefore f(2) = 16 - 12 - 24 + 1 = -19$.

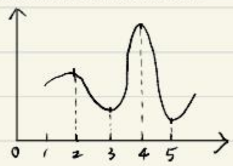
Hence, the absolute minimum value is $f(2) = -19$

the absolute maximum value is $f(-1) = 8$

4.1 Exercises

7–10 Sketch the graph of a function f that is continuous on $[1, 5]$ and has the given properties.

8. Absolute maximum at 4, absolute minimum at 5, local maximum at 2, local minimum at 3



29–44 Find the critical numbers of the function.

30. $f(x) = x^3 + 6x^2 - 15x$

$f'(x) = 3x^2 + 12x - 15$

Let $f'(x) = 0$.

$\therefore 3x^2 + 12x - 15 = 0$

$x^2 + 4x - 5 = 0$

$(x-1)(x+5) = 0$

$\therefore x = 1$ or $x = -5$

Hence, the critical numbers are 1 and -5.

35. $g(y) = \dfrac{y-1}{y^2 - y + 1}$

$g'(y) = \dfrac{y^2 - y + 1 - (y-1)(2y-1)}{(y^2 - y + 1)^2} = \dfrac{-y^2 + 2y}{(y^2 - y + 1)^2}$

Let $g'(y) = 0$.

$\therefore -y^2 + 2y = 0$

$y(-y + 2) = 0$

$\therefore y = 0$ or $y = 2$

Hence the critical numbers are 0 and 2.

40. $g(\theta) = 4\theta - \tan\theta$

$g'(\theta) = 4 - \sec^2\theta$

Let $g'(\theta) = 0$

$\therefore 4 - \sec^2\theta = 0$

$4 = \sec^2\theta$

$\therefore \cos\theta = \pm\frac{1}{2}$

$\therefore \theta = \pm\frac{\pi}{3} + 2k\pi,\ k \in \mathbb{Z}$

or

$\theta = \pm\frac{2}{3}\pi + 2k\pi,\ k \in \mathbb{Z}$

Hence, the critical numbers are $\pm\frac{\pi}{3} + 2k\pi\ (k \in \mathbb{Z})$ and $\pm\frac{2\pi}{3} + 2k\pi\ (k \in \mathbb{Z})$

43. $f(x) = x^2 e^{-3x}$

$f'(x) = 2x \cdot e^{-3x} + x^2 \cdot e^{-3x} \cdot (-3)$

$= e^{-3x}(-3x^2 + 2x)$

Let $f'(x) = 0$.

$\therefore -3x^2 + 2x = 0$

$x(-3x + 2) = 0$

$\therefore x = 0$ or $x = \frac{2}{3}$.

Hence the critical numbers are 0 and $\frac{2}{3}$.

40. When blood flows along a blood vessel, the flux F (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius R of the blood vessel:

$$F = kR^4$$

(This is known as Poiseuille's Law; we will show why it is true in Section 8.4.) A partially clogged artery can be expanded by an operation called angioplasty, in which a balloon-tipped catheter is inflated inside the artery in order to widen it and restore the normal blood flow.

Show that the relative change in F is about four times the relative change in R. How will a 5% increase in the radius affect the flow of blood?

From the information given,

$F = kR^4$

$\therefore dF = 4kR^3 dR$

$\therefore$ The relative change in F is

$$\frac{dF}{F} = \frac{4kR^3 dR}{kR^4} = \frac{4dR}{R}$$

Hence the relative change in F is about four times the relative change in R.

From the information given,

$$\frac{dR}{R} = 5\% \Rightarrow \frac{dF}{F} = \frac{4dR}{R} = 20\%.$$

Hence a 5% increase in the radius will cause a 20% increase in the flow of blood.

15–18 (a) Find the differential dy and (b) evaluate dy for the given values of x and dx.

15. $y = e^{x/10}, \quad x = 0, \quad dx = 0.1$

(a) $\frac{dy}{dx} = e^{\frac{x}{10}} \cdot \frac{1}{10}$

$\therefore dy = \frac{1}{10} e^{\frac{x}{10}} dx$

(b) Substitute $x=0, dx=0.1$

$\therefore dy = \frac{1}{10} \cdot e^0 \times 0.1 = 0.01$

18. $y = \frac{x+1}{x-1}, \quad x = 2, \quad dx = 0.05$

$\frac{dy}{dx} = \frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2}$

$\therefore dy = \frac{-2}{(x-1)^2} dx$

(b) Substitute $x=2, dx=0.05$

$\therefore dy = \frac{-2}{1^2} \times 0.05 = -0.1$

19–22 Compute Δy and dy for the given values of x and $dx = \Delta x$. Then sketch a diagram like Figure 5 showing the line segments with lengths dx, dy, and Δy.

22. $y = e^x, \quad x = 0, \quad \Delta x = 0.5$

$f(0) = e^0 = 1$

$\Delta y = f(x+\Delta x) - f(x)$

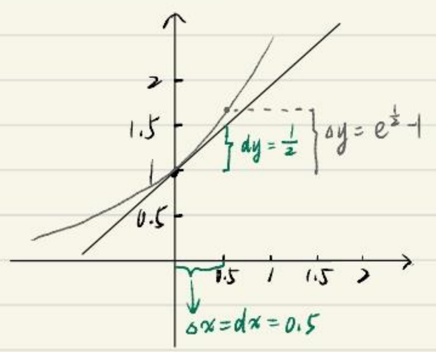
$= f(0.5) - f(1)$

$= e^{\frac{1}{2}} - 1 \approx 0.65$

$dy = f'(x)dx$

$= e^x dx$

$= e^0 \times 0.5 = \frac{1}{2}$



23–28 Use a linear approximation (or differentials) to estimate the given number.

23. $(1.999)^4$

Let the function be $f(x) = x^4$

First find the linearization $L(x)$ at $a=2$.

$\therefore f(a) = f(2) = 2^4 = 16$

$f'(x) = 4x^3$

$f'(a) = f'(2) = 4 \times 2^3 = 32$

$L(x) = f(a) + f'(a)(x-a)$

$\therefore L(x) = 16 + 32(x-2)$

$L(x) = 32x - 48$

Substitute $x = 1.999$

$\therefore L(1.999) = 32 \times 1.999 - 48$

$= 15.968$

Hence the estimation of 1.999^4 is 15.968.

3.10 Exercises

1–4 Find the linearization $L(x)$ of the function at a.

2. $f(x) = \sin x$, $a = \frac{\pi}{6}$
$f'(x) = \cos x$.
$\therefore f(a) = f(\frac{\pi}{6}) = \frac{1}{2}$, $f'(a) = f'(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$
$L(x) = f(a) + f'(a)(x-a)$
$\therefore L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6})$
$\therefore L(x) = \frac{\sqrt{3}}{2}x - \frac{\sqrt{3}\pi}{12} + \frac{1}{2}$

3. $f(x) = \sqrt{x}$, $a = 4$
$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$
$\therefore f(a) = f(4) = 2$, $f'(a) = f'(4) = \frac{1}{4}$
$L(x) = f(a) + f'(a)(x-a)$
$L(x) = 2 + \frac{1}{4}(x-4)$
$\therefore L(x) = \frac{1}{4}x + 1$

7–10 Verify the given linear approximation at $a = 0$. Then determine the values of x for which the linear approximation is accurate to within 0.1.

10. $e^x \cos x \approx 1 + x$

10. Let $f(x) = e^x \cos x$.
$f'(x) = e^x(\cos x - \sin x)$
$\therefore f(0) = e^0 \cos 0 = 1$
$f'(0) = e^0(\cos 0 - \sin 0) = 1$
$L(x) = f(a) + f'(a)(x-a)$
$\therefore L(x) = 1 + 1(x-0) = 1+x$
Hence, the linear approximation is verified.

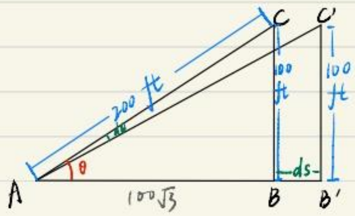
$|e^x \cos x - (1+x)| < 0.1$
$-0.1 < e^x \cos x - (1+x) < 0.1$
$0.9 < e^x \cos x - x < 1.1$
Using CAS, the inequation is satisfied for $-0.763 < x < 0.607$.
Hence, the linear approximation is accurate to within 0.1 in the interval $(-0.763, 0.607)$.

11–14 Find the differential of each function.

13. (a) $y = \tan\sqrt{t}$
$\frac{dy}{dt} = \frac{d\tan\sqrt{t}}{dt}$
$\frac{dy}{dt} = \sec^2\sqrt{t} \cdot \frac{1}{2\sqrt{t}}$
$\therefore dy = \frac{\sec^2\sqrt{t}}{2\sqrt{t}}\,dt$

(b) $y = \frac{1-v^2}{1+v^2}$
$\frac{dy}{dv} = \frac{-2v\cdot(1+v^2) - 2v\cdot(1-v^2)}{(1+v^2)^2}$
$\frac{dy}{dv} = \frac{-4v}{(1+v^2)^2}$
$\therefore dy = \frac{-4v}{(1+v^2)^2}\,dv$

30. A kite 100 ft above the ground moves horizontally at a speed of 8 ft/s. At what rate is the angle between the string and the horizontal decreasing when 200 ft of string has been let out?



From the information given, draw a picture.

$\sin\theta = \frac{BC}{AC} = \frac{100}{200} = \frac{1}{2}$

$\therefore \theta = \frac{\pi}{6}$

$AB = AC\cos\theta = 200 \times \cos\frac{\pi}{6} = 200 \times \frac{\sqrt{3}}{2} = 100\sqrt{3}$ ft.

let s = horizontal distance.

$\therefore \frac{ds}{dt} = 8 ft/s$.

$\therefore \tan\theta = \frac{100}{s}$

Differentiate both side,

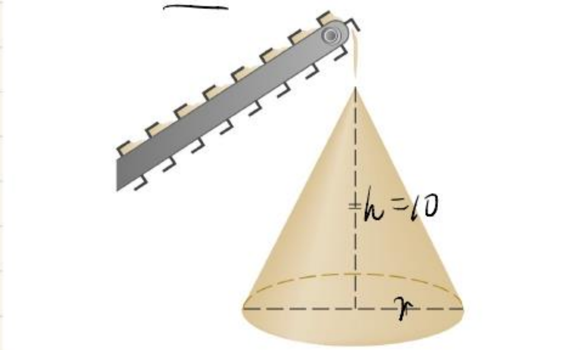
$\sec^2\theta \frac{d\theta}{dt} = -\frac{100}{s^2} \cdot \frac{ds}{dt}$

$\therefore \left(\frac{2}{\sqrt{3}}\right)^2 \frac{d\theta}{dt} = -\frac{100}{(100\sqrt{3})^2} \times 8$

$\therefore \frac{d\theta}{dt} = -\frac{1}{50}$ rad/s

Hence, the angle is decreasing at a rate of $\frac{1}{50}$ rad/s.

29. Gravel is being dumped from a conveyor belt at a rate of $30 \text{ ft}^3/\text{min}$, and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?



Let V = volume, h = height, r = radius

given that $r = \frac{h}{2}$, $\frac{dV}{dt} = 30$ ft/min

$\therefore V = \frac{1}{3} \cdot \pi \cdot \left(\frac{h}{2}\right)^2 \cdot h = \frac{\pi}{12} h^3$

$$V = \frac{\pi}{12} h^3$$

Differentiate both side with respect to 't'

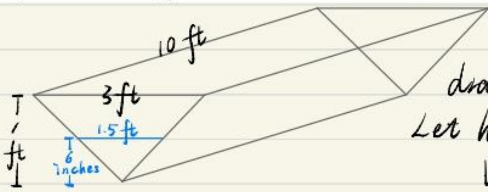
$$\therefore \frac{dV}{dt} = \frac{\pi}{4} \cdot h^2 \cdot \frac{dh}{dt}$$

$$30 = \frac{\pi}{4} \times 10^2 \times \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{6}{5}\pi \text{ ft/min}$$

Hence the height increases with the rate of $\frac{6\pi}{5}$ ft/min.

26. A trough (水槽) is 10 ft long and its ends have the shape of isosceles triangles (等腰) that are 3 ft across at the top and have a height of 1 ft. If the trough is being filled with water at a rate of 12 ft^3/min, how fast is the water level rising when the water is 6 inches deep?



From the information given, draw a picture.

Let h = height of water

V = volume of water

$\therefore V = \frac{1}{2} \cdot h \cdot 3h \cdot 10$

$= 15h^2$

6 inches = 0.5 ft.

$\therefore V = 15h^2$

Differentiate both side with respect to 't'

$$\frac{dV}{dt} = 15 \cdot 2h \cdot \frac{dh}{dt}$$

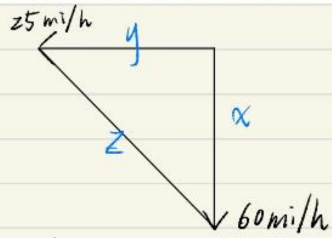
$$\therefore 12 = 15 \times 2 \times 0.5 \times \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = \frac{4}{5} \text{ ft/min}$$

Hence, the water level is rising at the rate of $\frac{4}{5}$ ft/min when the water is 6 inches deep.

17. Two cars start moving from the same point. One travels south at 60 mi/h and the other travels west at 25 mi/h. At what rate is the distance between the cars increasing two hours later?

17.



From the information given, draw a graph.

$$\therefore \frac{dx}{dt} = 60\,mi/h \qquad \frac{dy}{dt} = 25\,mi/h$$

Two hours later,

$$x = 60 \times 2 = 120\ mi$$

$$y = 25 \times 2 = 50\ mi$$

$$\therefore z = \sqrt{x^2 + y^2} = 130\ mi$$

$$z^2 = x^2 + y^2$$

Differentiate both side with respect to 't'.

$$2z \cdot \frac{dz}{dt} = 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt}$$

$$z \cdot \frac{dz}{dt} = x \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt}$$

$$\therefore \frac{dz}{dt} = \frac{120 \times 60 + 50 \times 25}{130} = 65\,mi/h$$

Hence, The car is increasing 2 hours later at the rate of 65 mi/h.

14. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

let the time be t

Let the surface area be S, and the diameter be D

From the information given, $\frac{ds}{dt} = 1\ cm^2/min$

$S = 4\pi\left(\frac{D}{2}\right)^2 = 4\pi \cdot \frac{D^2}{4} = \pi D^2$. $S = \pi D^2$.

Differentiate both side with respect to 't'.

$$\frac{dS}{dt} = \pi \cdot \frac{dS}{dD} \cdot \frac{dD}{dt}$$

$$\frac{dS}{dt} = 2\pi D \cdot \frac{dD}{dt}$$

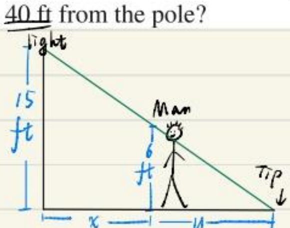
$$\therefore \quad 1 = 2\pi \times 10 \times \frac{dD}{dt}$$

$$\therefore \frac{dD}{dt} = \frac{1}{20\pi} \quad cm/min.$$

Hence, the diameter decreases at a rate of $\frac{1}{20\pi}$ cm/min.

15. A street light is mounted at the top of a 15-ft-tall pole. A man 6 ft tall walks away from the pole with a speed of 5 ft/s along a straight path. How fast is the tip of his shadow moving when he is 40 ft from the pole?

15.



Draw a picture through the given information

$\therefore \frac{dx}{dt} = 5\ ft/s$

From the picture, $\frac{15}{6} = \frac{x+y}{y}$.

$\therefore y = \frac{2}{3}x$

The rate of change of the tip of shadow is $\frac{d(x+y)}{dt}$.

$$\frac{d(x+y)}{dt} = \frac{d\left(x+\frac{2}{3}x\right)}{dt} = \frac{5}{3}\frac{dx}{dt} = \frac{25}{3}\ ft/s$$

$\therefore$ The rate of change of the tip of shadow is $\frac{25}{3}$ ft/s when he is 40 ft from the pole.

3.9 Exercises (16.11.21)

2. (a) If A is the area of a circle with radius r and the circle expands as time passes, find dA/dt in terms of dr/dt.

(b) Suppose oil spills from a ruptured tanker and spreads in a circular pattern. If the radius of the oil spill increases at a constant rate of 1 m/s, how fast is the area of the spill increasing when the radius is 30 m?

2. (a) From the information given,

$A = \pi r^2$

$\therefore \frac{dA}{dt} = \pi \cdot \frac{dA}{dr} \cdot \frac{dr}{dt}$

$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$

(b) From the information given,

$\frac{dr}{dt} = 1 m/s. \quad r = 30m.$

$\therefore$ The rate of change in Area:

$\frac{dA}{dt} = 2\pi \times 30 \times 1 = 60\pi \ m^2/s$

Hence the area of the circle is increasing with the rate $60\pi \ m^2/s$.

11. If $x^2 + y^2 + z^2 = 9$, $dx/dt = 5$, and $dy/dt = 4$, find dz/dt when $(x, y, z) = (2, 2, 1)$.

11. Differentiate the function with respect to 't'.

$\frac{dx}{dt} \cdot 2x + \frac{dy}{dt} \cdot 2y + \frac{dz}{dt} \cdot 2z = 0.$

$\frac{dx}{dt} \cdot x + \frac{dy}{dt} \cdot y + \frac{dz}{dt} \cdot z = 0$

$\therefore \ 5 \times 2 + 4 \times 2 + \frac{dz}{dt} \times 1 = 0$

$\therefore \frac{dz}{dt} = -18.$