...to identify the best method for a given integral.

Related Exercises 44, 49 ◄

8.4

EXAMPLE 6 **A secant substitution** Evaluate $\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx$.

SOLUTION This example illustrates a useful preliminary step first encountered in Section 8.1. The integrand does not contain any of the patterns in Table 8.4 that suggest a trigonometric substitution. Completing the square does, however, lead to one of those patterns. Noting that $x^2 + 4x - 5 = (x+2)^2 - 9$, we change variables with $u = x + 2$ and write the integral as

$$\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx = \int_1^4 \frac{\sqrt{(x+2)^2-9}}{x+2}\,dx \quad \text{Complete the square.}$$

$$= \int_3^6 \frac{\sqrt{u^2-9}}{u}\,du. \quad u = x+2,\ du = dx \ \text{Change limits of integration.}$$

This new integral calls for the secant substitution $u = 3\sec\theta$ (where $0 \le \theta < \pi/2$), which implies that $du = 3\sec\theta\tan\theta\,d\theta$ and $\sqrt{u^2-9} = 3\tan\theta$. We also change the limits of integration: When $u = 3$, $\theta = 0$, and when $u = 6$, $\theta = \pi/3$. The complete integration can now be done:

$$\int_1^4 \frac{\sqrt{x^2+4x-5}}{x+2}\,dx = \int_3^6 \frac{\sqrt{u^2-9}}{u}\,du \quad u = x+2,\ du = dx$$

$$= \int_0^{\pi/3} \frac{3\tan\theta}{3\sec\theta}\,3\sec\theta\tan\theta\,d\theta \quad u = 3\sec\theta,\ du = 3\sec\theta\tan\theta\,d\theta$$

$$= 3\int_0^{\pi/3} \tan^2\theta\,d\theta \quad \text{Simplify.}$$

$$= 3\int_0^{\pi/3} (\sec^2\theta - 1)\,d\theta \quad \tan^2\theta = \sec^2\theta - 1$$

$$= 3\,(\tan\theta - \theta)\Big|_0^{\pi/3} \quad \text{Evaluate the integral.}$$

$$= 3\sqrt{3} - \pi. \quad \text{Simplify.}$$

Related Exercises 40, 60 ◄

n integral containing

n integral containing

n integral containing

sin θ, for

6. Using the trigonometric substitution $x = 8\sec\theta$, $x \ge 8$ and $0 < \theta \le \frac{\pi}{2}$, express $\tan\theta$ in terms of x.

Practice Exercises

7–56. Trigonometric substitutions *Evaluate the following integrals using trigonometric substitution.*

7. $\int_0^{5/2} \frac{dx}{\sqrt{25-x^2}}$ (*Hint:* Check your answer without using trigonometric substitution.)

8.4

$\int_0^{3/2} \frac{dx}{(9 - x^2)^{3/2}}$

$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4 - x^2}}\,dx$

$\int_{1/2}^{1} \frac{\sqrt{1 - x^2}}{x^2}\,dx$

$\int \sqrt{36 - t^2}\,dt$

$\int \frac{x^2}{(25 + x^2)^2}\,dx$

$\int \frac{dx}{(1 + x^2)^{3/2}}$

$\int \frac{dx}{\sqrt{x^2 - 49}}, x > 7$

$\int \frac{dt}{t^2\sqrt{9 - t^2}}$

$\int \frac{\sqrt{9 - x^2}}{x^2}\,dx$

$\int_{\sqrt{2}}^{2} \frac{\sqrt{x^2 - 1}}{x}\,dx$

$\int_0^6 \frac{z^2}{(z^2 + 36)^2}\,dz$

$\int x^3\sqrt{1 - x^2}\,dx$

$\int \frac{dx}{(x^2 - 36)^{3/2}}, x > 6$

$\int \frac{dx}{x^3\sqrt{x^2 - 1}}, x > 1$

$\int_{8\sqrt{2}}^{16} \frac{dx}{\sqrt{x^2 - 64}}$

$\int^{\sqrt{2}} \frac{dx}{x^2\sqrt{4 - x^2}}$

$\int_{10/\sqrt{3}}^{10} \frac{dy}{\sqrt{y^2 - 25}}$

$\int \frac{dx}{\ldots}, x > \frac{1}{3}$

9. $\int_5^{5\sqrt{3}} \sqrt{100 - x^2}\,dx$

11. $\int_{1/2}^{\sqrt{3}/2} \frac{x^2}{\sqrt{1 - x^2}}\,dx$

13. $\int \frac{dx}{\sqrt{16 - x^2}}$

15. $\int \frac{dx}{x^2\sqrt{x^2 + 9}}$

17. $\int_0^2 \frac{x^2}{x^2 + 4}\,dx$

19. $\int \frac{dx}{\sqrt{x^2 - 81}}, x > 9$

21. $\int \sqrt{64 - x^2}\,dx$

23. $\int \frac{dx}{(25 - x^2)^{3/2}}$

25. $\int \frac{\sqrt{9 - x^2}}{x}\,dx$

27. $\int_0^{1/3} \frac{dx}{(9x^2 + 1)^{3/2}}$

29. $\int \frac{dx}{(4 + x^2)^2}$

31. $\int \frac{x^2}{\sqrt{16 - x^2}}\,dx$

33. $\int \frac{\sqrt{x^2 - 9}}{x}\,dx, x > 3$

35. $\int \frac{dx}{x(x^2 - 1)^{3/2}}, x > 1$

37. $\int_{1/\sqrt{3}}^{1} \frac{dx}{x^2\sqrt{1 + x^2}}$

39. $\int \frac{x^2}{(100 - x^2)^{3/2}}\,dx$

41. $\int \frac{dx}{(1 + 4x^2)^{3/2}}$

43. $\int_0^{4/\sqrt{3}} \frac{dx}{\sqrt{x^2 + 16}}$

52. $\int \frac{\sqrt{4x^2 - 1}}{x^2}\,dx, x > \frac{1}{2}$

53. $\int \frac{\sqrt{9x^2 - 25}}{x^3}\,dx, x > \ldots$

54. $\int \frac{y^4}{1 + y^2}\,dy$

55. $\int \frac{dx}{x^3\sqrt{x^2 - 100}}, x > \ldots$

56. $\int \frac{x^3}{(x^2 - 16)^{3/2}}\,dx, x < -4$

57. **Explain why or why not** Determine whether the following statements are true and give an explanation or counterexample.

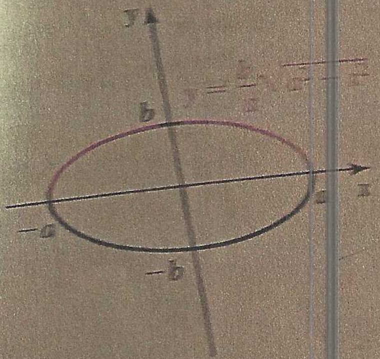
a. If $x = 4\tan\theta$, then $\csc\theta = 4/x$.

b. The integral $\int_1^2 \sqrt{1 - x^2}\,dx$ does not have a finite real value.

c. The integral $\int_1^2 \sqrt{x^2 - 1}\,dx$ does not have a finite real value.

d. The integral $\int \frac{dx}{x^2 + 4x + 9}$ cannot be evaluated using a trigonometric substitution.

58. **Area of an ellipse** The upper half of the ellipse centered at the origin with axes of length $2a$ and $2b$ is described by $y = \frac{b}{a}\sqrt{a^2 - x^2}$ (see figure). Find the area of the ellipse in terms of a and b.

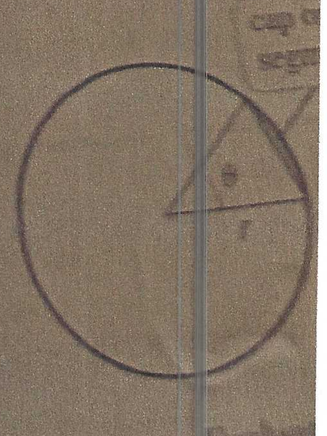


59. **Area of a segment of a circle** Use two approaches to show that the area of a cap (or segment) of a circle of radius r subtended by an angle θ (see figure) is given by

$$A_{seg} = \frac{1}{2}r^2(\theta - \sin\theta)$$

a. Find the area using geometry (no calculus).

b. Find the area using calculus.



... completing the square Evaluate ...