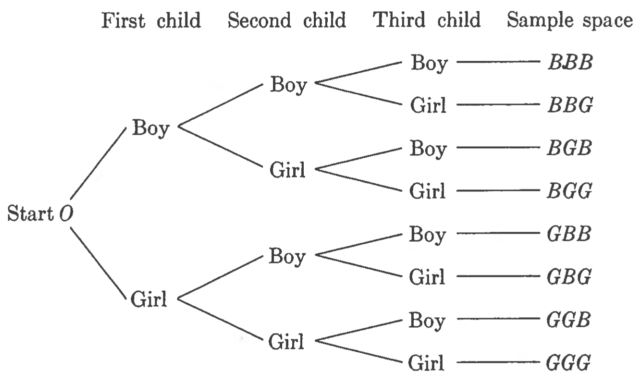

Fig. 1–1 Tree for three-child families.

draws two slips from the hat, one after the other, without replacement. Describe a sample space for the experiment.

Solution We may consider that each outcome of the experiment is represented by an ordered pair of numbers (x, y), where x is the number on the first slip and y is the number on the second. The restrictions on x and y are as follows:

$$1 \leq x \leq 4, \qquad 1 \leq y \leq 4, \qquad x \neq y.$$

Table 1–10 shows a sample space.

Table 1–10 Sample space for 2 numbered slips

		y: number on second slip			
	$x \backslash y$	1	2	3	4
x: number on first slip	1		(1, 2)	(1, 3)	(1, 4)
	2	(2, 1)		(2, 3)	(2, 4)
	3	(3, 1)	(3, 2)		(3, 4)
	4	(4, 1)	(4, 2)	(4, 3)	

EXERCISES FOR SECTION 1–7

1. A coin is tossed and then a die is thrown. List a sample space for this experiment. Illustrate with a tree graph.
2. Three coins are tossed. List two sample spaces for this experiment.
3. Two letters are randomly chosen, one after another, from the word *tack*. List a sample space.

4. A boy has in his pocket a penny, a nickel, a dime, and a quarter. He takes two coins out of his pocket, one after the other. List a sample space. Illustrate with a tree graph.
5. Suppose you plan to make a survey of families having two children. You want to record the sex of each child, in the order of their births. For example, if the first child is a boy and the second a girl, you record (boy, girl). This is one point in the sample space. List all the sample points.
6. If the survey in Exercise 5 is undertaken for families having four children, list an appropriate sample space. How many sample points does it have? How many of these points correspond to families having 3 boys and 1 girl? How many correspond to families in which the first child is a girl?
7. Two dice, one black and one red, are tossed and the numbers of dots on their upper faces are noted. List a sample space for the experiment. [*Note:* A tabular arrangement is convenient.]
8. An engineer's ruler has a cross section that is an equilateral triangle. Two such rulers, one red and one green, have their faces numbered 1, 2, and 3. The rulers are tossed onto the floor and the numbers on the bottom faces are read when they come to rest. Set up a table for the sample space of outcomes.
9. An experiment consists of selecting 3 radios from a lot of 25 and testing them. The test shows that a radio is defective (D), or nondefective (N). List a sample space for this experiment.
10. From five different books, A, B, C, D, and E, three are selected. List a suitable sample space of outcomes. In your sample space: (a) How many sample points correspond to a selection including A? (b) How many correspond to a selection without A? (c) How many correspond to a selection including both B and C? (d) How many correspond to a selection including either D or E?
11. A letter is chosen at random from the word *ground*. Which of the following sets are acceptable as sample spaces for the experiment and which are not?

    a) $\{g, r, o, u, n, d\}$; b) $\{\text{vowel}, g, r, n, d\}$;
    c) $\{r, o, u, n, d\}$; d) $\{\text{vowel}, \text{consonant}\}$;
    e) $\{\text{consonant}, u\}$.
12. A bag contains a number of marbles, identical in every way except that some are red, some white, and some blue, at least 2 of each color. Two marbles are drawn, one after the other, without replacement. What is a sample space for this experiment? How many points of the sample space correspond to drawing two marbles of the same color? How many points correspond to drawing marbles of different colors? How many correspond to drawing a red *and* a blue marble? How many correspond to drawing a red *or* a blue marble?
13. In the sample space of Exercise 7, how many sample points correspond to a total of more than 10 dots? To a total of less than 5 dots? To an even total? To the black die showing more than 5 dots? To the red die showing less than 3 *and* the black die showing more than 5? To the red die showing an even number *or* the black die showing 3?
14. There are 12 ordered pairs listed in Table 1–10. If these represent equally likely outcomes, what is the probability that

    a) $x + y = 5$? b) $x + y > 5$? c) x is even and y is odd?

3. *To obtain the probability of an event E, add the probabilities assigned to the elements of the subset of S that corresponds to E.* Since the empty set has no elements, its probability is zero.

Example 2 For a chronic disease, there are five standard ameliorative treatments: a, b, c, d, and e. A doctor has resources for conducting a comparative study of three of these treatments. If he chooses the three treatments for study at random from the five, what is the probability that (a) treatment a will be chosen, (b) treatments a and b will be chosen, (c) at least one of a and b will be chosen?

Solution In Table 1–13, we list the 10 possible selections of the 5 treatments, taken 3 at a time. For reference, the sample points are numbered.

Table 1–13 S for study of treatments

1	2	3	4	5	6	7	8	9	10
abc	*abd*	*abe*	*acd*	*ace*	*ade*	*bcd*	*bce*	*bde*	*cde*

Next, we assign probability $\frac{1}{10}$ to each sample point, since we assume that all 10 selections are equally likely. Then the probability that treatment a is chosen is $\frac{6}{10}$, because there are 6 selections corresponding to the event "treatment a is chosen".

Similarly, the probability that treatments a and b are both chosen is $\frac{3}{10}$, because there are exactly 3 selections containing both a and b. Finally, the probability that at least one of the treatments a and b is chosen is $\frac{9}{10}$; only the tenth sample point contains neither a nor b.

EXERCISES FOR SECTION 1–8

Exercises 1 through 7 refer to the two-dice experiment. Consult the sample space in Table 1–11.

1. What is the probability of not throwing a double?
2. What is the probability that the number on one die is double the number on the other?
3. What is the probability that one die gives a 5 and the other die a number less than 5?
4. What is the probability that the clear die gives a number less than 3 and the red die a number greater than 3?
5. Evaluate:

   a) $P(r + c = 6)$ b) $P(r + c = 8)$ c) $P(r + c < 5)$

   d) $P(r + c > 9)$ e) $P(r \geq c + 4)$

6. Give algebraic descriptions of the following verbally described events: (a) not throwing a double, (b) red die shows two less than clear die, (c) clear die shows number at least 2 greater than red die, (d) number on red die twice that on clear die.

7. Give verbal descriptions of the following algebraically described events:

a) $r = 3c$ b) $r - c = 1$ c) $r \neq c$
d) $r + c > 8$ e) $c = r^2$ f) $r \geq c$

8. For the sample space of Exercise 4 at the end of Section 1–7, answer the following: (a) What is the probability that he takes out 35 cents? (b) What is the probability that the value of the coins selected is less than 20 cents? Less than 15 cents? More than 15 cents? A prime number? A number divisible by 10?

9. In Exercise 6, Section 1–7, assume that all points in the sample space have the same probability. What is the probability that in a family of four children the first two are girls? What is the probability that three are boys and one is a girl? That there are two boys and two girls?

10. In Exercise 8, Section 1–7, assume that all points of the sample space have equal probabilities. Let r denote the number on the red ruler, and g the number on the green. Evaluate:

a) $P(r = g)$ b) $P(r + g > 3)$
c) $P(r > g)$ d) $P(r \neq g)$
e) $P(r = g^2)$

11. In the ancient Indian game of Tong, two players simultaneously show their right hands to each other, exhibiting either one or two or three extended fingers. If each player is equally likely to extend one, two, or three fingers, what is the probability that the total number of fingers extended is even? Odd? Greater than 4? Less than 2? Prime? [*Note:* Set up a sample space as a first step.]

12. Two rods, one black and one white, have square cross sections. Each rod has its faces numbered 1, 2, 3, and 4. The rods are rolled on the floor, and the numbers on their upper faces are read after they come to rest. Set up a table for a sample space of outcomes. If b is the number on the upper face of the black rod, and w that on the upper face of the white rod, evaluate:

a) $P(b + w = 5)$ b) $P(b = w)$ c) $P(b > w + 1)$
d) P(black 1 or 3 and white 2 or 4) e) P(sum of numbers even)
f) P(larger number shown is a 4)

13. Suppose that you have a black rod from Exercise 12 and a red engineer's ruler from Exercise 8, Section 1–7. Rod and ruler are rolled on the floor, and the number on the top face of the rod and that on the bottom face of the ruler are noted. Set up a sample space and find the probability that the number on the black rod is greater than that on the red ruler. What is the probability that both numbers are the same? That the sum of the numbers is prime?

14. In the sample space of Exercise 10, Section 1–7, assume that all points are equally likely. What is the probability that B will be included in the selection? That both A and B will be included? That either A or B will be included? That the selection will be C, D, and E?

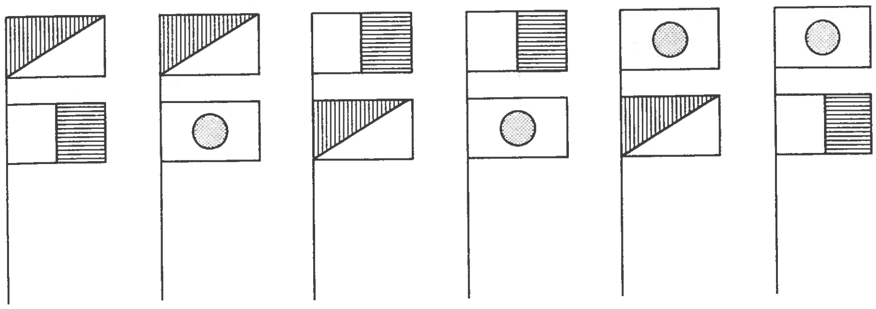

Fig. 2-3 Signals using 2 flags out of 3.

Now we have only one signal to arrange, and this signal may be a two-flag signal or a three-flag signal, but not both together. It is a question of performing the first operation *or* the second, not the first operation *and then* the second. The operations are mutually exclusive: they cannot both occur together. The total number of signals is therefore $6 + 6 = 12$.

2-3 The addition principle

If two operations are mutually exclusive, and the first can be done in m ways and the second in n ways, then one operation or the other can be done in $m + n$ ways.

This principle is readily generalized to include any finite number of operations. The statement is left as an exercise.

EXERCISES FOR SECTION 2-1

Use the multiplication principle or the addition principle to solve the following exercises.

1. In how many ways can 8 people line up at a theater box office?
2. How many 5-digit numbers can be formed from the integers 1, 2, 4, 6, 7, 8, if no integer can be used more than once? How many of these numbers will be even? How many odd?
3. If the call letters of a broadcasting station must begin with the letter W, how many different stations could be designated by using only 3 letters, with repetitions of a letter allowed? How many by using 4 letters, without repetitions?
4. In how many ways can 3 letters be mailed in 6 mailboxes, if each letter must be mailed in a different box? If the letters are not necessarily mailed in different boxes, how many ways are there of posting them? If the letters are mailed at random, and not necessarily in different boxes, what is the probability that all the letters are put in the same mailbox?
5. There are 7 seats available in a sedan. In how many ways can 7 persons be seated for a journey if only 3 are able to drive? [*Hint:* See note following Example 6(c).]

6. A passenger train has 9 coaches. In how many ways can 4 people be assigned to coaches if they must ride in different coaches?
7. In how many ways can 6 students be seated in a classroom with 30 desks?
8. Twelve boys try out for the basketball team. Two can play only at center, four only as right or left guard, and the rest can play only as right or left forward. In how many ways could the coach assign a team?
9. How many numbers, each with at least 3 digits, can be formed from the 5 digits 1, 2, 3, 4, 5, if no digit may be used more than once?
10. In how many ways can 5 boys and 5 girls be seated alternately in a row of 10 chairs, numbered from 1 to 10, if a boy always occupies chair number one?
11. In how many ways can 3 different presents, A, B, and C, be given to any 3 of 15 persons? If a specified person must receive A, and if no person is to receive more than one present, in how many ways can the presents be distributed?
12. In how many ways can a selection of at least one book be made from 8 different books? [*Hint:* See Example 8 of the text.]
13. Given 4 flags of different colors, how many different signals can be made by arranging them on a vertical mast, if at least 2 flags must be used for each signal?
14. An encyclopedia consists of 9 volumes numbered 1 to 9. In how many ways can the 9 volumes be arranged together on a shelf so that some or all of the volumes are out of order? If the 9 volumes are placed on the shelf at random, what is the probability that they are not in correct order?
15. How many 5-digit numbers can be formed? How many of these begin with 2 and end with 4? How many do not contain the digit 5? How many are divisible by 5? If a 5-digit number is chosen at random from a table listing all possible 5-digit numbers, what is the probability that the chosen number does not contain the digit 5? What is the probability that the chosen number is divisible by 5?
16. How many different parties of 2 or more can be formed from 9 people?
17. Five men compete in a race. In how many ways can the first two places be taken?
18. (a) How many subsets, including the empty and universal sets, can be formed from a set of 10 different objects? (b) From a set of n different objects?
19. How many ordered pairs of symbols (x, y) can be formed if x can be replaced by a or b or c, and y can be replaced by 1 or 2 or 3 or 4? Draw a tree diagram exhibiting the set of possible ordered pairs (x, y).
20. How many permutations are there of n different objects, taken r at a time, with repetitions allowed? (It is assumed that there are at least r copies of each of the n objects available.)
21. On stepping off a train, a man finds that he has a nickel, a dime, a quarter, and a half-dollar in his pocket. In how many ways can he give the porter a tip?
22. Six people, designated A, B, C, D, E, and F, line up at random at a theater box office. What is the probability that A is beside B in the lineup?
23. A farm is divided into 64 plots of land. An agricultural experimenter wishes to compare, on this farm, the yield of 5 strains of beets using 3 kinds of insecticide and 4 different fertilizers. Will he have enough plots to compare all possible combinations of strain, insecticide, and fertilizer?

the multiplication or addition principle, some special device, or a combination of these methods.

Example 6 Twenty-six persons $A, B, \ldots, Z$ are to be arranged in a line. If this is done randomly, what is the probability that A and B will be adjacent in the lineup?

Solution There are 26! ways of lining up the 26 persons. To compute the number of ways of having A and B together, we first pick two adjacent places (which can be done in 25 ways), then put A and B in these places (which can be done in 2 ways), and finally put the remaining 24 persons in the remaining 24 places (which can be done in 24! ways). Therefore, the desired probability is

$$P(A \text{ and } B \text{ together}) = \frac{25 \times 2 \times 24!}{26!} = \frac{2}{26} = \frac{1}{13}.$$

EXERCISES FOR SECTION 2-2

Note. A "word", as used in these exercises, means any arrangement of letters.

1. Evaluate the following: ${}_9P_3$, ${}_mP_1$, ${}_7P_7$, ${}_kP_2$.
2. Compute ${}_nP_0$ and interpret it.
3. How many words can be formed from the letters of the word *fragments* (a) taken all at a time, (b) taken 8 at a time, (c) taken 4 at a time?
4. A student has 4 examinations to write and there are 10 examination periods available. How many possible arrangements are there of his examination program?
5. (a) A musical concert is to consist of 3 songs and 2 violin selections. In how many ways can the program be arranged so that the concert begins and ends with a song, and neither violin selection follows immediately after the other? (b) If the 5 selections are arranged at random, what is the probability that the program follows the plan in part (a)?
6. Prove that the number of 3-letter words that can be formed from the letters of the word *background* is the same as the number of words that can be made by rearranging the letters of the word *ground*.
7. Use your knowledge of whole numbers to find the number of 4-digit whole numbers. (What is the first such whole number? What is the last?) Then check your answer by using the multiplication principle.
8. How many automobile license plates bearing 5-digit numbers can be made if no license number starts with 0? If letters of the alphabet are used in place of the first digit and the next digit is not 0, how many plates can be made?
9. A passenger train consists of 2 baggage cars, 4 day coaches, and 3 parlor cars. In how many ways can the train be made up if the 2 baggage cars must come in front, and the 3 parlor cars must come in the rear?
10. If the passenger train in Exercise 9 is assembled at random, what is the probability that the 2 baggage cars will be in front and the 3 parlor cars will be in the rear?

11. If there are 3 roads from town A to town B, and 4 roads from town B to town C, in how many ways can one make a trip from A to C by way of B, and return from C to A by way of B?
12. In the Hotel Superba, 7 rooms in a row are assigned at random to 7 guests, 2 of whom are from Minnesota. What is the probability that the guests from Minnesota are assigned rooms side by side? What are the odds against such an assignment?
13. In geometry, polygons are commonly labeled by placing letters at their vertices. How many ways are there of labeling a triangle with letters of the alphabet? A pentagon? A decagon? (Do not multiply out the answers.)
14. How many 5-letter words can be made from 10 different letters (a) if any letter may be repeated any number of times, (b) if repetitions of a letter are not allowed? (c) In how many of the words of (a) will repeated letters actually occur?
15. Ten different letters are marked on 10 identical discs, and the discs are then placed in a bag and mixed. A disc is drawn at random from the bag, its letter is noted, and the disc is returned to the bag. This operation is repeated until 5 letters are recorded. Find the probability that repeated letters occur among the 5 letters recorded.
16. (Refer to Example 3, page 46.) If the memory expert decides to use always the same first step in each ordered 3-step rule, how many experiments are needed to study all admissible 3-step rules? If each experiment costs \$200, how much money is needed for the complete investigation?
17. Use the fact noted after formula (2) to show that (the number of permutations of n things taken r at a time) $\times$ (the number of permutations of the remaining $n - r$ things) = (the number of permutations of n things). Explain why this should be so.

2-3 COMBINATIONS

In Section 2-2, we obtained some formulas for counting permutations. These formulas enable us to count the number of ways objects can be arranged in an *ordered* row. Sometimes a problem requires us to make a *selection* of objects without regard to order; in such cases a change in order does *not* yield a new selection. For example, in choosing a committee of 3, "Smith, Jones, and White" is the same committee as "White, Jones, and Smith", and these two triples are counted as one selection, not two. The total number of such selections is not obtained by the formulas of Section 2-2, as the following example shows.

Example 1 Four drugs, designated A, B, C, and D, are beneficial in the treatment of a certain chronic disease. A research scientist has funds sufficient for a study of the comparative efficacy of 3 of the drugs. How many different studies are possible?

Solution The scientist makes a selection of 3 drugs from A, B, C, and D without taking order into account. He has only 4 possible selections:

$$ABC, \quad ABD, \quad ACD, \quad BCD. \tag{1}$$

Table 2–3 Pascal's triangle for $\binom{n}{r}$, $0 \le r \le n \le 10$

$$\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}, \qquad 1 \le r \le n$$

n \ r	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1	1									
2	1	2	1								
3	1	3	3	1							
4	1	4	6	4	1						
5	1	5	10	10 +	5	1					
6	1	6	15	20	= 15	6	1				
7	1	7	21	35	35	21	7	1			
8	1	8	28	56	70	56	28	8	1		
9	1	9	36	84	126	126	84	36	9	1	
10	1	10	45	120	210	252	210	120	45	10	1

to the number in which the specified object does not occur, since no other cases are possible. Therefore, the total number is

$$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}. \quad \square$$

Note The foregoing proof is an application of the additional principle, not of the multiplication principle. The problem presents us with two operations, either of which is admissible separately, but not both simultaneously. It is a question of *either* this operation *or* that operation. Such operations are *mutually exclusive;* they cannot both occur together. Each operation forms the basis of a separate problem, and the final result is obtained by addition, not by multiplication.

Pascal's Rule gives a simple way of building a table of values of $\binom{n}{r}$, known as *Pascal's triangle.* Table 2–3 shows the part of Pascal's triangle for values of n from 0 through 10. The rows of the table correspond to values of n; the columns, to values of r. The first and last entries in each row are 1 because $\binom{n}{0} = \binom{n}{n} = 1$. The entry other than the first or last in each row is the sum of the entry immediately above it and the entry to the left of that one, by Pascal's Rule. Thus, for example, the little box illustrates

$$\binom{6}{4} = \binom{5}{3} + \binom{5}{4} \qquad \text{or} \qquad 15 = 10 + 5.$$

EXERCISES FOR SECTION 2–3

1. Evaluate the following: $\binom{9}{3}$, $\binom{9}{6}$, $\binom{m}{3}$, $\binom{k}{1}$, $\binom{5}{5}$.
2. Show that $\binom{n}{0} = 1$, and interpret it in terms of selections.

3. Solve the following equations for n:

a) $\binom{n}{2} = 45$; b) $\dfrac{{}_nP_4}{\binom{n-1}{3}} = 60$; c) $\binom{n}{8} = \binom{n}{12}$.

4. In how many ways can a committee of 5 be chosen from 8 people?

5. A contractor needs 4 carpenters and 10 apply for the jobs. In how many ways can he pick out 4?

6. In how many ways can a selection of fruit be made from 7 plums, 4 lemons, and 9 oranges? (Assume that the 7 plums are indistinguishable; likewise for the lemons and for the oranges.)

7. How many selections of 1 or more letters can be made from 2 A's, 5 B's, and 9 C's.

8. Ten points are taken on the circumference of a circle. How many chords can be drawn by joining them in all possible ways? With these 10 points as vertices, how many triangles can be drawn? How many hexagons?

9. In how many ways can a selection of 4 musical records be made from 9? If a certain record must be chosen, in how many ways can the selection be made? In how many ways can it be made if a certain record must be left?

10. A company of 20 men is to be divided into 3 sections so that there are 3 men in the first, 5 in the second, and 12 in the third. In how many ways can this be done? (Don't multiply out.)

11. A sports car club plans to purchase 3 different cars selected from 8 different makes in the same price–performance group. Each club member is to vote for one 3-car selection. To facilitate matters, the ballot committee prepares a list of sixty 3-car selections. The committee then wonders whether the list has omissions or duplications. Give the committee some good advice.

12. Write a symbol for the number of combinations of 20 objects taken 4 at a time, and for the number of combinations of 100 objects taken 98 at a time. Compute the numerical value of each symbol, and find which is the greater.

13. A pack of playing cards contains 52 different cards. If a hand is made up of 5 cards, use the factorial notation to express the number of possible hands. (Disregard order in the hands.)

14. In how many ways can 2 booksellers divide between them 300 copies of one book, 200 copies of another, and 100 copies of a third, if neither bookseller is to get all the books? (Don't multiply out.)

15. A bridge deck of cards is made up of 13 spades, 13 hearts, 13 diamonds, and 13 clubs. How many different hands can be formed if each hand contains 5 spades, 4 hearts, 2 diamonds, and 2 clubs? (Don't multiply out.)

16. Six candidates contest an election for two similar offices. If a voter may mark his ballot either for one or for two candidates, in how many ways can he cast his vote?

17. How many 5-letter words, each consisting of 3 consonants and 2 vowels, can be formed from the letters of the word *equations*?

18. Twenty persons are to travel in a double-decker bus that can carry 12 passengers inside and 8 outside. If 4 of the persons will not travel inside, and 5 will not

travel outside, in how many ways can the passengers be seated (a) if the arrangement of the passengers inside, or outside, is not considered, and (b) if the arrangement inside and outside is considered?

19. In how many ways can 4 persons be selected from 5 married couples (a) if the selection must consist of 2 women and 2 men, and (b) if a husband and wife cannot both be selected?

20. (a) Find the number of ways in which at least one musical record can be selected from 4 identical stereo records and 8 identical monophonic records. (b) What would be the answer to part (a) if the 12 records were all different?

21. Verify that the entries for $n = 4$ in Table 2–3 satisfy the conditions $\binom{4}{0} = \binom{4}{4} = 1$ and $\binom{4}{r} = \binom{3}{r} + \binom{3}{r-1}$ for $1 \leq r \leq 3$.

22. Write out the entries that would be in the rows for $n = 11$ and $n = 12$ in Pascal's triangle, Table 2–3.

EXERCISES ON PERMUTATIONS AND COMBINATIONS

PART A

1. In how many ways can a man choose 3 gifts from 10 different articles?

2. A railway has 50 stations. If the names of the point of departure and the destination are printed on each ticket, how many different kinds of single tickets must be printed? How many kinds are needed if each ticket may be used in either direction between two towns?

3. In how many ways can 15 different objects be divided among A, B, and C, if A must receive 2 objects, B must receive 3 objects, and C must receive 10 objects?

4. Given 20 points, no three of which are in a straight line, find the number of straight lines that can be drawn by joining pairs of these points.

5. Given 4 noncoplanar points in space, how many planes can be determined by selecting triples of these points?

6. A ring of 8 boys is to be enlarged by the addition of 5 girls. In how many ways can this be done if no two girls are to stand beside each other? (Note that order counts here because people are distinguishable.)

7. A town council is made up of a mayor and 6 aldermen. How many different committees of 4 can be formed (a) if the mayor is on each committee, and (b) if the mayor is on no committee?

8. How many 4-letter words can be made from the letters of the word *zephyr*? How many of these words will not contain the letter r? How many will contain r? How many will begin with z and end with r?

9. In how many ways can a coach choose a team of 5 from 10 boys (a) if 2 specified boys must be included, and (b) if there are no restrictions?

10. A man has 8 different pairs of gloves. In how many ways can he select a right-hand glove and a left-hand glove that do not match?

11. A 35-mm colored slide is mounted in a 2″ × 2″ square holder. How many wrong ways are there of inserting the mounted slide into a projector?