

3. Let  $\mathcal{M}$  be an infinite  $\sigma$ -algebra.
- $\mathcal{M}$  contains an infinite sequence of disjoint sets.
  - $\text{card}(\mathcal{M}) \geq \mathfrak{c}$ .
4. An algebra  $\mathcal{A}$  is a  $\sigma$ -algebra iff  $\mathcal{A}$  is closed under countable increasing unions (i.e., if  $\{E_j\}_1^\infty \subset \mathcal{A}$  and  $E_1 \subset E_2 \subset \dots$ , then  $\bigcup_1^\infty E_j \in \mathcal{A}$ ).
6. Complete the proof of Theorem 1.9.

**1.9 Theorem.** Suppose that  $(X, \mathcal{M}, \mu)$  is a measure space. Let  $\mathcal{N} = \{N \in \mathcal{M} : \mu(N) = 0\}$  and  $\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subset N \text{ for some } N \in \mathcal{N}\}$ . Then  $\overline{\mathcal{M}}$  is a  $\sigma$ -algebra, and there is a unique extension  $\overline{\mu}$  of  $\mu$  to a complete measure on  $\overline{\mathcal{M}}$ .

*Proof.* Since  $\mathcal{M}$  and  $\mathcal{N}$  are closed under countable unions, so is  $\overline{\mathcal{M}}$ . If  $E \cup F \in \overline{\mathcal{M}}$  where  $E \in \mathcal{M}$  and  $F \subset N \in \mathcal{N}$ , we can assume that  $E \cap N = \emptyset$  (otherwise, replace  $F$  and  $N$  by  $F \setminus E$  and  $N \setminus E$ ). Then  $E \cup F = (E \cup N) \cap (N^c \cup F)$ , so  $(E \cup F)^c = (E \cup N)^c \cup (N \setminus F)$ . But  $(E \cup N)^c \in \mathcal{M}$  and  $N \setminus F \subset N$ , so that  $(E \cup F)^c \in \overline{\mathcal{M}}$ . Thus  $\overline{\mathcal{M}}$  is a  $\sigma$ -algebra.

If  $E \cup F \in \overline{\mathcal{M}}$  as above, we set  $\overline{\mu}(E \cup F) = \mu(E)$ . This is well defined, since if  $E_1 \cup F_1 = E_2 \cup F_2$  where  $F_j \subset N_j \in \mathcal{N}$ , then  $E_1 \subset E_2 \cup N_2$  and so  $\mu(E_1) \leq \mu(E_2) + \mu(N_2) = \mu(E_2)$ , and likewise  $\mu(E_2) \leq \mu(E_1)$ . It is easily verified that  $\overline{\mu}$  is a complete measure on  $\overline{\mathcal{M}}$ , and that  $\overline{\mu}$  is the only measure on  $\overline{\mathcal{M}}$  that extends  $\mu$ ; details are left to the reader (Exercise 6). ■

The measure  $\overline{\mu}$  in Theorem 1.9 is called the **completion** of  $\mu$ , and  $\overline{\mathcal{M}}$  is called the **completion** of  $\mathcal{M}$  with respect to  $\mu$ .

8. If  $(X, \mathcal{M}, \mu)$  is a measure space and  $\{E_j\}_1^\infty \subset \mathcal{M}$ , then  $\mu(\liminf E_j) \leq \liminf \mu(E_j)$ . Also,  $\mu(\limsup E_j) \geq \limsup \mu(E_j)$  provided that  $\mu(\bigcup_1^\infty E_j) < \infty$ .
12. Let  $(X, \mathcal{M}, \mu)$  be a finite measure space.
- If  $E, F \in \mathcal{M}$  and  $\mu(E \Delta F) = 0$ , then  $\mu(E) = \mu(F)$ .
  - Say that  $E \sim F$  if  $\mu(E \Delta F) = 0$ ; then  $\sim$  is an equivalence relation on  $\mathcal{M}$ .
  - For  $E, F \in \mathcal{M}$ , define  $\rho(E, F) = \mu(E \Delta F)$ . Then  $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$ , and hence  $\rho$  defines a metric on the space  $\mathcal{M}/\sim$  of equivalence classes.

(I) Suppose  $\{\mu_n\}$  is a sequence of finite measures on  $(X, \mathcal{M})$  and  $\mu_n \nearrow \mu$  on  $\mathcal{M}$ , for some set function  $\mu$ . Show that  $\mu$  is a measure. (This is actually true if we just assume  $\mu_n \rightarrow \mu$ , but it's much harder to prove.)

(II) Let  $\mathcal{B}_{\mathbb{R}}$  be the Borel sets in  $\mathbb{R}$ , so by definition  $\mathcal{B}_{\mathbb{R}}$  is generated by the collection  $\mathcal{E}$  of all open intervals. Define the measure

$$\mu(E) = \begin{cases} 0 & \text{if } E = \emptyset, \\ \infty & \text{if } E \neq \emptyset. \end{cases}$$

Find another measure  $\nu$  such that  $\mu = \nu$  on the generators  $\mathcal{E}$ , but  $\mu$  and  $\nu$  do not agree on all of  $\mathcal{B}_{\mathbb{R}}$ .

(III) Let  $\mathcal{A}$  be a collection of pairwise disjoint subsets of a  $\sigma$ -finite measure space, and suppose each set in  $\mathcal{A}$  has strictly positive measure. Show that  $\mathcal{A}$  is at most countable.

(IV) Suppose  $\mathcal{E}, \mathcal{F}$  are subsets of  $\mathcal{P}(X)$ , with  $\mathcal{E} \subset \mathcal{F} \subset \sigma(\mathcal{E})$ . Show that  $\sigma(\mathcal{F}) = \sigma(\mathcal{E})$ .

(V) Let  $(X, \mathcal{M}, \mu)$  be a measure space and let  $\{E_j\}$  be a sequence of sets in  $\mathcal{M}$  such that each  $E_j$  intersects at most one other set in the sequence. Show that

$$\sum_{j=1}^{\infty} \mu(E_j) \leq 2\mu\left(\bigcup_{j=1}^{\infty} E_j\right).$$

**HINTS:**

(3) For (b), use (a).

(8) Consider properties of the sequences  $\{F_k\}$  and  $\{G_k\}$ , where  $F_k = \bigcap_{n=k}^{\infty} E_n$  and  $G_k = \bigcup_{n=k}^{\infty} E_n$ .

(12)(c) For general  $E, F, G$ , how are the 3 sets  $E\Delta G$ ,  $E\Delta F$  and  $F\Delta G$  related to each other? For example, is one always contained in the union, intersection, or complement of the other two?

(I) To show that  $\mu(\bigcup_{n \geq 1} E_n) = \sum_{n \geq 1} \mu(E_n)$ , show " $\leq$ " and " $\geq$ ", comparing  $\mu(\bigcup_{i=1}^{\infty} E_i)$  to  $\sum_{i=1}^k \mu(E_i)$  for  $k \geq 1$ .

(III) Why is it true in a *finite* measure space?

(V) Split every set  $E_j$  into two sets: its intersection with another  $E_i$ , and the rest of  $E_j$ .