Problem 1 (20 pts)

Let $W = \{f(x) \in P_2 \mid f(1) = f(2)\}$ where P_2 is the vector space of polynomials with real coefficients of degree at most 2.

(a) Show that W is a subspace of P_2.

(b) Determine the dimension of W and find a basis for the subspace.

Problem 2 (10 pts)

Determine whether or not $\left\{ \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix}, \begin{pmatrix} 2 & -7 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 5 \\ 1 & 2 \end{pmatrix} \right\}$ is a set of linearly independent vectors in $M_{2\times 2}(\mathbb{R})$.

Problem 3 (10 pts)

Let

$$A = \begin{pmatrix} 1 & 1 & 7 & -3 \\ -3 & -2 & -3 & 5 \\ 4 & -1 & 2 & -1 \end{pmatrix}.$$

(a) Find the rank and nullity of A.

(b) Give a basis for the column space of A.

Problem 4 (20 pts)

Let $B = \{5 + 3x, 3 + 2x\}$ and let $C = \{2 + 3x, 1 + x\}$ be bases of P_1.

(a) Find the coordinate vector of $5 - x$ with respect to B

(b) Find the change-of-basis matrix from B to C.

(c) If $\Phi = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$ is the matrix of the linear transformation $T(a_0 + a_1 x) = 2a_1 - a_0 x$ with respect to the standard basis $\{1, x\}$, what is the matrix of T with respect to C?

Problem 5 (20 pts)

(a) Find an invertible matrix B that diagonalizes the following matrix:

$$A = \begin{pmatrix} 0 & -3 \\ 1 & 4 \end{pmatrix}.$$

(b) Use B to compute A^6. Use exponents to express the solution

Problem 6 (20 pts)

Let A be the following 3×3 matrix:

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 0 & 1 \\ 0 & 2 & 3 \end{pmatrix}.$$

Test A for diagonalizability. If A is diagonalizable, find a 3×3 invertible matrix B and a 3×3 diagonal matrix D such that $B^{-1}AB = D$.

Bonus Problem (10 pts)

Let U and W be subspaces of a vector space V. Prove that the union $U \cup W = \{v | \, v \in U \, \text{or} \, v \in W\}$ of U and W is a subspace of V if and only if $U \subseteq W$ or $W \subseteq U$.