EXAMPLE 14 How many bit strings of length n contain exactly r 1s?

Solution: The positions of r 1s in a bit string of length n form an r-combination of the set $\{1, 2, 3, \ldots, n\}$. Hence, there are $C(n, r)$ bit strings of length n that contain exactly r 1s. ◀

EXAMPLE 15 Suppose that there are 9 faculty members in the mathematics department and 11 in the computer science department. How many ways are there to select a committee to develop a discrete mathematics course at a school if the committee is to consist of three faculty members from the mathematics department and four from the computer science department?

Solution: By the product rule, the answer is the product of the number of 3-combinations of a set with nine elements and the number of 4-combinations of a set with 11 elements. By Theorem 2, the number of ways to select the committee is

$$C(9, 3) \cdot C(11, 4) = \frac{9!}{3!6!} \cdot \frac{11!}{4!7!} = 84 \cdot 330 = 27{,}720.$$

◀

Exercises

1. List all the permutations of $\{a, b, c\}$.
2. How many different permutations are there of the set $\{a, b, c, d, e, f, g\}$?
3. How many permutations of $\{a, b, c, d, e, f, g\}$ end with a?
4. Let $S = \{1, 2, 3, 4, 5\}$.
   a) List all the 3-permutations of S.
   b) List all the 3-combinations of S.
5. Find the value of each of these quantities.
   a) $P(6, 3)$ b) $P(6, 5)$
   c) $P(8, 1)$ d) $P(8, 5)$
   e) $P(8, 8)$ f) $P(10, 9)$
6. Find the value of each of these quantities.
   a) $C(5, 1)$ b) $C(5, 3)$
   c) $C(8, 4)$ d) $C(8, 8)$
   e) $C(8, 0)$ f) $C(12, 6)$
7. Find the number of 5-permutations of a set with nine elements.
8. In how many different orders can five runners finish a race if no ties are allowed?
9. How many possibilities are there for the win, place, and show (first, second, and third) positions in a horse race with 12 horses if all orders of finish are possible?
10. There are six different candidates for governor of a state. In how many different orders can the names of the candidates be printed on a ballot?
11. How many bit strings of length 10 contain
    a) exactly four 1s?
    b) at most four 1s?
    X c) at least four 1s?
    d) an equal number of 0s and 1s?
12. How many bit strings of length 12 contain
    a) exactly three 1s?
    b) at most three 1s?
    c) at least three 1s?
    d) an equal number of 0s and 1s?
13. A group contains n men and n women. How many ways are there to arrange these people in a row if the men and women alternate?
14. In how many ways can a set of two positive integers less than 100 be chosen?
15. In how many ways can a set of five letters be selected from the English alphabet?
16. How many subsets with an odd number of elements does a set with 10 elements have?
17. How many subsets with more than two elements does a set with 100 elements have?
18. A coin is flipped eight times where each flip comes up either heads or tails. How many possible outcomes
    a) are there in total?
    b) contain exactly three heads?
    c) contain at least three heads?
    d) contain the same number of heads and tails?
19. A coin is flipped 10 times where each flip comes up either heads or tails. How many possible outcomes
    a) are there in total?
    X b) contain exactly two heads?
    c) contain at most three tails?
    d) contain the same number of heads and tails?
20. How many bit strings of length 10 have
    a) exactly three 0s?
    b) more 0s than 1s?
    c) at least seven 1s?
    d) at least three 1s?

21. How many permutations of the letters $ABCDEFG$ contain

a) the string BCD?
b) the string $CFGA$?
c) the strings BA and GF?
d) the strings ABC and DE?
e) the strings ABC and CDE?
f) the strings CBA and BED?

22. How many permutations of the letters $ABCDEFGH$ contain

a) the string ED?
b) the string CDE?
c) the strings BA and FGH?
d) the strings AB, DE, and GH?
e) the strings CAB and BED?
f) the strings BCA and ABF?

23. How many ways are there for eight men and five women to stand in a line so that no two women stand next to each other? [*Hint:* First position the men and then consider possible positions for the women.]

24. How many ways are there for 10 women and six men to stand in a line so that no two men stand next to each other? [*Hint:* First position the women and then consider possible positions for the men.]

25. One hundred tickets, numbered $1, 2, 3, \ldots, 100$, are sold to 100 different people for a drawing. Four different prizes are awarded, including a grand prize (a trip to Tahiti). How many ways are there to award the prizes if

a) there are no restrictions?
b) the person holding ticket 47 wins the grand prize?
c) the person holding ticket 47 wins one of the prizes?
d) the person holding ticket 47 does not win a prize?
e) the people holding tickets 19 and 47 both win prizes?
f) the people holding tickets 19, 47, and 73 all win prizes?
g) the people holding tickets 19, 47, 73, and 97 all win prizes?
h) none of the people holding tickets 19, 47, 73, and 97 wins a prize?
i) the grand prize winner is a person holding ticket 19, 47, 73, or 97?
j) the people holding tickets 19 and 47 win prizes, but the people holding tickets 73 and 97 do not win prizes?

26. Thirteen people on a softball team show up for a game.

a) How many ways are there to choose 10 players to take the field?
b) How many ways are there to assign the 10 positions by selecting players from the 13 people who show up?
c) Of the 13 people who show up, three are women. How many ways are there to choose 10 players to take the field if at least one of these players must be a woman?

27. A club has 25 members.

a) How many ways are there to choose four members of the club to serve on an executive committee?
b) How many ways are there to choose a president, vice president, secretary, and treasurer of the club, where no person can hold more than one office?

28. A professor writes 40 discrete mathematics questions. Of the statements in these questions true. If the questions can be positioned in any many different answer keys are possible?

*29. How many 4-permutations of the positive integers ceeding 100 contain three consecutive integers $k + 2$, in the correct order

a) where these consecutive integers can perhaps arated by other integers in the permutation?
b) where they are in consecutive positions in the tation?

30. Seven women and nine men are on the faculty mathematics department at a school.

a) How many ways are there to select a committee five members of the department if at least one must be on the committee?
b) How many ways are there to select a committee five members of the department if at least one and at least one man must be on the committee?

31. The English alphabet contains 21 consonants vowels. How many strings of six lowercase letters English alphabet contain

a) exactly one vowel?
b) exactly two vowels?
c) at least one vowel?
d) at least two vowels?

32. How many strings of six lowercase letters from glish alphabet contain

a) the letter a?
b) the letters a and b?
c) the letters a and b in consecutive positions preceding b, with all the letters distinct?
d) the letters a and b, where a is somewhere of b in the string, with all the letters distinct?

33. Suppose that a department contains 10 men women. How many ways are there to form a tee with six members if it must have the same men and women?

34. Suppose that a department contains 10 men women. How many ways are there to form a tee with six members if it must have more men?

35. How many bit strings contain exactly eight 0s if every 0 must be immediately followed by a

36. How many bit strings contain exactly five 0s every 0 must be immediately followed by two

37. How many bit strings of length 10 contain at 1s and at least three 0s?

38. How many ways are there to select 12 countries United Nations to serve on a council if 3 from a block of 45, 4 are selected from a block the others are selected from the remaining 69

Question 6.5

(3) How many strings of six letters are there?

(19) Suppose that a large family has 14 children, including two sets of identical triplets, three sets of identical twins, and two individual children. How many ways are there to seat these children in a row of chairs if the identical tripplets or twins cannot be distinguished from one another?

(31) How many ways are there to distribute six distinguishable objects into four indistinguishable boxes so that each of the boxes contains at least one object?

Question 7.1

(13) What is the probability that a five-card poker hand contains at least one ace

(17) What is probability that a five-card poker hand contains a straight, that is, five cards that have consecutive kinds? (Note that an ace can be considered either the lowest card of an A-2-3-4-5 straight or the highest card of a 10-J-Q-K-A straight.)

(25) b. find the probability of winning a lottery by selecting the correct six integers, where the order in which these integers are selected does not matter, from the positive integers not exceeding 52.

Question 7.2

(5) A pair of dice is loaded. The probability that a 4 appears on the first die is 2/7, and the probability that a 3 appears on the second die is 2/7. Other outcomes for each die appear with probability 1/7. What is the probability of 7 appearing as the sum of the numbers when the two dice are rolled?

What is the probability of these events when we randomly select a permutation of the 26 lowercase letters of the English alphabet?

a) The first 13 letters of the permutation are in alphabetical order.
b) a is the first letter of the permutation and z is the last letter.
c) a and z are next to each other in the permutation.
d) a and b are not next to each other in the permutation.
e) a and z are separated by at least 23 letters in the permutation.
f) z precedes both a and b in the permutation.

Suppose that E and F are events such that $p(E) = 0.7$ and $p(F) = 0.5$. Show that $p(E \cup F) \geq 0.7$ and $p(E \cap F) \geq 0.2$.

Suppose that E and F are events such that $p(E) = 0.8$ and $p(F) = 0.6$. Show that $p(E \cup F) \geq 0.8$ and $p(E \cap F) \geq 0.4$.

Show that if E and F are events, then $p(E \cap F) \geq p(E) + p(F) - 1$. This is known as **Bonferroni's inequality**.

Use mathematical induction to prove the following generalization of Bonferroni's inequality:

$$p(E_1 \cap E_2 \cap \cdots \cap E_n) \geq p(E_1) + p(E_2) + \cdots + p(E_n) - (n-1),$$

where $E_1, E_2, \ldots, E_n$ are n events.

Show that if $E_1, E_2, \ldots, E_n$ are events from a finite sample space, then

$$p(E_1 \cup E_2 \cup \cdots \cup E_n) \leq p(E_1) + p(E_2) + \cdots + p(E_n).$$

This is known as **Boole's inequality**.

Show that if E and F are independent events, then $\overline{E}$ and $\overline{F}$ are also independent events.

If E and F are independent events, prove or disprove that $\overline{E}$ and F are necessarily independent events.

Exercises 18, 20, and 21 assume that the year has 366 days all birthdays are equally likely. In Exercise 19 assume it equally likely that a person is born in any given month of year.

a) What is the probability that two people chosen at random were born on the same day of the week?
b) What is the probability that in a group of n people chosen at random, there are at least two born on the same day of the week?
c) How many people chosen at random are needed to make the probability greater than 1/2 that there are at least two people born on the same day of the week?

19a What is the probability that two people chosen at random were born during the same month of the year?
b What is the probability that in a group of n people chosen at random, there are at least two born in the same month of the year?
19c How many people chosen at random are needed to make the probability greater than 1/2 that there are at least two people born in the same month of the year?

19c only

20. Find the smallest number of people you need to choose at random so that the probability that at least one of them has a birthday today exceeds 1/2.

21. Find the smallest number of people you need to choose at random so that the probability that at least two of them were both born on April 1 exceeds 1/2.

***22.** February 29 occurs only in leap years. Years divisible by 4, but not by 100, are always leap years. Years divisible by 100, but not by 400, are not leap years, but years divisible by 400 are leap years.

a) What probability distribution for birthdays should be used to reflect how often February 29 occurs?
b) Using the probability distribution from part (a), what is the probability that in a group of n people at least two have the same birthday?

23. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?

24. What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up tails?

25. What is the conditional probability that a randomly generated bit string of length four contains at least two consecutive 0s, given that the first bit is a 1? (Assume the probabilities of a 0 and a 1 are the same.)

26. Let E be the event that a randomly generated bit string of length three contains an odd number of 1s, and let F be the event that the string starts with 1. Are E and F independent?

27. Let E and F be the events that a family of n children has children of both sexes and has at most one boy, respectively. Are E and F independent if

a) $n = 2$? **b)** $n = 4$? **c)** $n = 5$?

28. Assume that the probability a child is a boy is 0.51 and that the sexes of children born into a family are independent. What is the probability that a family of five children has

a) exactly three boys?
b) at least one boy?
c) at least one girl?
d) all children of the same sex?

29. A group of six people play the game of "odd person out" to determine who will buy refreshments. Each person flips a fair coin. If there is a person whose outcome is not the same as that of any other member of the group, this person has to buy the refreshments. What is the probability that there is an odd person out after the coins are flipped once?

30. Find the probability that a randomly generated bit string of length 10 does not contain a 0 if bits are independent and if

a) a 0 bit and a 1 bit are equally likely.
b) the probability that a bit is a 1 is 0.6.
c) the probability that the ith bit is a 1 is $1/2^i$ for $i = 1, 2, 3, \ldots, 10$.

Question 7.3

(1) Suppose that E and F are events in a sample space and $P(E) = 1/3$, $P(F) = 1/2$, and ~~P(E)~~ $P(E|F) = 2/5$. Find $P(F|E)$.

7(b) Suppose that a test for opium use has a 2% false positive rate and a 5% false negative rate. That is 2% of people who do not use opium test positive for opium, and 5% of opium users test negative for opium. Furthermore, suppose that 1% of people actually use opium.

(b) find the probability that someone who test positive for opium use actually uses opium.

(11) An electronics company is planning to introduce a new camera phone. The company commissions a marketing report for each new product that predict either the success or the failure of the product. Of ~~which~~ new products introduced by the company, 60% have been successes. Furthermore, 70% of their successful products were predicted to be successes. Find the probability that this new camera phone will be successful if its success has been predicted.

(19) Suppose that a Bayesian spam filter is trained on a set of 1000 spam messages and and 400 messages that are not spam. The word "opportunity" appears in 175 spam messages and 20 messages ~~and~~ that are not spam. Would an incoming message be rejected as spam if ~~when~~ when it contains the word "opportunity" and the threshold for rejecting a message is 0.9?