Step 1 of 6

Consider the differential equation $y''+by'+4y=0$ with initial conditions $y(0)=1$ and $y'(0)=0$.

The objective is to study the study the effect of changing the parameter by solving the differential equation for $b=5,\ b=4,\ b=2$.

Further, the objective is to sketch the solution for each value of *b*.

Consider the initial value problem $y''+by'+4y=0, y(0)=1, y'(0)=0$ **for** $b=5$.

The differential equation becomes $y''+5y'+4y=0$.

The corresponding auxiliary equation is $r^2+5r+4=0$.

Solve the auxiliary equation as follows:

$$\begin{aligned} r^2+5r+4&=0 \\ r^2+4r+r+4&=0 \\ r(r+4)+1(r+4)&=0 \\ (r+1)(r+4)&=0 \end{aligned}$$

$$r=-4,-1$$

The auxiliary equation has two distinct real roots.

Therefore, the solution of the differential equation is as follows:

$$y(t)=c_1e^{-4t}+c_2e^{-t} \qquad ......(1)$$

Differentiate equation (1) with respect to *t*.

$$y'(t)=-4c_1e^{-4t}-c_2e^{-t} \qquad ......(2)$$

Apply initial condition $y(0)=1$ on equation (1).

$$\begin{aligned} y(t)&=c_1e^{-4t}+c_2e^{-t} \\ y(0)&=c_1e^0+c_2e^0 \\ 1&=c_1+c_2 \\ c_2&=1-c_1 \qquad ......(3) \end{aligned}$$

Apply initial condition $y'(0)=0$ on equation (2).

$$\begin{aligned} y'(t)&=-4c_1e^{-4t}-c_2e^{-t} \\ y'(0)&=-4c_1e^0-c_2e^0 \\ 0&=-4c_1-c_2 \\ 0&=-4c_1-(1-c_1) \qquad \text{By equation (3)} \end{aligned}$$

$$-3c_1=1$$

$$\boxed{c_1=-\frac{1}{3}}$$

Use in equation (3).

$$\begin{aligned} c_2&=1-c_1 \\ &=1-\left(-\frac{1}{3}\right) \end{aligned}$$

$$\boxed{c_2=\frac{4}{3}}$$

Hence, the required solution of the differential equation for parameter $b=5$ is as follows:

$$\boxed{y(t)=-\frac{1}{3}e^{-4t}+\frac{4}{3}e^{-t}}.$$

Step 3 of 6

Consider the initial value problem $y''+by'+4y=0, y(0)=1, y'(0)=0$ **for** $b=4$.

The differential equation becomes $y''+4y'+4y=0$.

The corresponding auxiliary equation is $r^2+4r+4=0$.

Solve the auxiliary equation as follows:

$$r^2+4r+4=0$$
$$r^2+2r+2r+4=0$$
$$r(r+2)+2(r+2)=0$$
$$(r+2)(r+2)=0$$

$$r=-2,-2$$

The auxiliary equation has repeated real roots.

Therefore, the solution of the differential equation is as follows:

$$y(t)=c_1e^{-2t}+c_2te^{-2t} \qquad ......(4)$$

Differentiate equation (4) with respect to *t*.

$$y'(t)=-2c_1e^{-4t}+c_2\left(e^{-2t}-2te^{-2t}\right) \qquad ......(5)$$

Apply initial condition $y(0)=1$ on equation (4).

$$y(t)=c_1e^{-2t}+c_2te^{-2t}$$
$$y(0)=c_1e^0+c_2(0)e^0$$
$$\boxed{1=c_1}$$

Step 4 of 6

Apply initial condition $y'(0)=0$ on equation (5).

$$y'(t)=-2c_1e^{-4t}+c_2\left(e^{-2t}-2te^{-2t}\right)$$
$$y'(0)=-2c_1e^0+c_2\left(e^0-0\right)$$
$$0=-2(1)+c_2 \qquad \text{As } c_1=1$$
$$\boxed{c_2=2}$$

Hence, the required solution of the differential equation for parameter $b=4$ is as follows:

$$\boxed{y(t)=e^{-2t}+2te^{-2t}}.$$

Consider the initial value problem $y''+by'+4y=0, y(0)=1, y'(0)=0$ **for** $b=2$.

The differential equation becomes $y''+2y'+4y=0$.

The corresponding auxiliary equation is $r^2+2r+4=0$.

Solve the auxiliary equation as follows:

$$r^2+2r+4=0$$

$$\begin{aligned} r &= \frac{-2\pm\sqrt{2^2-4(1)(4)}}{2} \\ &= \frac{-2\pm\sqrt{-12}}{2} \\ &= \frac{-2\pm i2\sqrt{3}}{2} \end{aligned}$$

$$=-1\pm i\sqrt{3}$$

The auxiliary equation has two complex conjugate roots.

Therefore, the solution of the differential equation is as follows:

$$y(t)=e^{-t}\left(c_1\cos\left(\sqrt{3}t\right)+c_2\sin\left(\sqrt{3}t\right)\right) \qquad ......(6)$$

Differentiate equation (6) with respect to *t*.

$$y'(t)=e^{-t}\left(-\sqrt{3}c_1\sin\left(\sqrt{3}t\right)+\sqrt{3}c_2\cos\left(\sqrt{3}t\right)\right)-e^{-t}\left(c_1\cos\left(\sqrt{3}t\right)+c_2\sin\left(\sqrt{3}t\right)\right) \qquad ......(7)$$

Apply initial condition $y(0)=1$ on equation (6).

$$y(0)=e^0\left(c_1\cos 0+c_2\sin 0\right)$$

$$\boxed{1=c_1}$$

Apply initial condition $y'(0)=0$ on equation (7).

$$y'(0)=e^0\left(-\sqrt{3}c_1\sin 0+\sqrt{3}c_2\cos 0\right)-e^0\left(c_1\cos 0+c_2\sin 0\right)$$

$$0=\sqrt{3}c_2-c_1$$

$$0=\sqrt{3}c_2-1 \qquad \{\text{As } c_1=1\}$$

$$\boxed{c_2=\frac{1}{\sqrt{3}}}$$

Hence, the required solution of the differential equation for parameter $b=2$ is as follows:

$$\boxed{y(t)=e^{-t}\left(\cos\left(\sqrt{3}t\right)+\frac{1}{\sqrt{3}}\sin\left(\sqrt{3}t\right)\right)}.$$

Step 6 of 6

Figure 1 shows the sketch of solutions of given initial value problem for different values of parameter *b*.

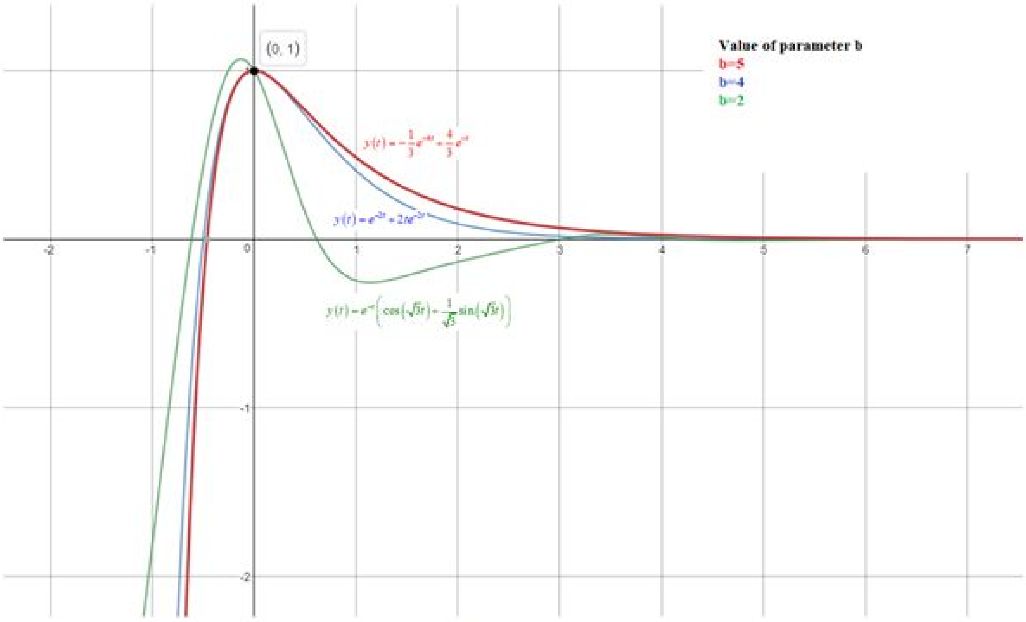

Figure 1

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