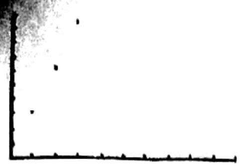
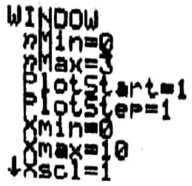
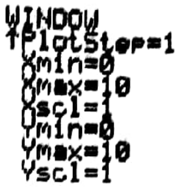
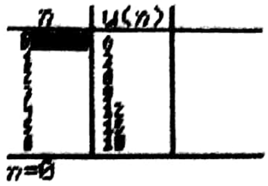




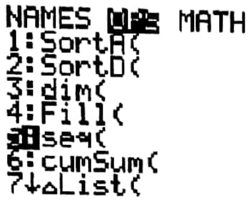
```
WINDOW
 nMin=0
 nMax=3
 PlotStart=1
 PlotStep=1
 Xmin=0
 Xmax=10
↓Xscl=1
```



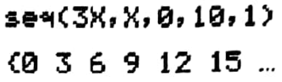
```
WINDOW
↑PlotStep=1
 Xmin=0
 Xmax=10
 Xscl=1
 Ymin=0
 Ymax=10
 Yscl=1
```



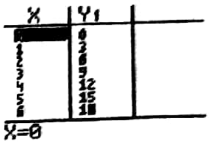
3. Using Lists
- Press 2nd STAT (LIST) and OPS keys
- Choose option 5 seq(
- Using seq(type the formula, variable, starting value for n, last value of **n needed, increment). For** ease, X may be used as the variable or type N using the alpha key.
- Press ENTER
- Press 2nd and GRAPH keys (TABLE)



```
NAMES OPS MATH
1:SortA(
2:SortD(
3:dim(
4:Fill(
5:seq(
6:cumSum(
7↓ΔList(
```



```
seq(3X,X,0,10,1)
{0 3 6 9 12 15 ...
```



Geometric Sequences (refer to task 2 above)

- It is left to the reader to prepare a detailed presentation of the concept of geometric sequence using an analogous path, so making use of the graphing calculator could enhance student conceptual understanding.

Summary/ What a mathematics teacher should know…

Arithmetic sequences

Recursive definition:

$$a_0 = a$$
$$a_n = a_{n-1} + d,\ n = 1, 2, 3, \ldots \qquad (1)$$

Explicit formula

$$a_n = dn + a,\ n = 0, 1, 2, \ldots \qquad (2$$

Geometric sequences

Recursive definition:

$$a_0 = a$$
$$a_n = ka_{n-1},\ n = 1, 2, 3, \ldots \qquad (3)$$

Explicit formula

$$a_n = ak^n,\ n = 0, 1, 2, \ldots \qquad (4)$$

Simple Interest

In an investment account at simple interest each year the interest produced by the original amount A invested $A = P$ (P is called the principal) will be added to the account balance, so the difference between the balances of two consecutive years is constant (arithmetic sequence). What is that difference and what is the amount of money in such account after t years?

P = **Principal**
I = **Interest** = Pr, where r is the APR in decimal form.
t = years

The difference between any two consecutive years is the interest. Substituting $d = Pr$ and $n = t$ in equation (2) we have a formula for the amount of money that will be in the account after t years.

$$a_t = Prt + P = P(1+rt),\ t = 0, 1, 2, \ldots \qquad (5)$$

Under compound interest, not only the principal generates interest, but the principal plus the interest generate more interest periodically or even at each instant of time (**continuously compound interest**). Let's consider the simplest case first, the **annually compound interest** (same APR).

From the simple interest we know that after a year ($t = 1$) the account will have $a_1 = Pr + P = P(1+r)$, but now this account balance will generates interest to be paid in the following year. What is the amount of money in such account after *2* years?

$$
\begin{aligned}
a_0 &= P \\
a_1 &= Pr + P = P(1+r) \\
a_2 &= P(1+r) + rP(1+r) \\
&= P(1+r)(1+r) \\
&= P(1+r)^2 \\
a_3 &= P(1+r)^2 + rP(1+r)^2 \\
&= P(1+r)^2(1+r) \\
&= P(1+r)^3 \\
&\ \ \vdots \\
a_n &= P(1+r)^{n-1} + rP(1+r)^{n-1} \\
&= P(1+r)^{n-1}(1+r) \\
&= P(1+r)^n
\end{aligned}
$$

Note that in each step an amount A is taken ($A = P$ and $A = P(1+r)$ in the first and second steps respectively) and added to itself times r, then $A + rA = A(1+r)$ by the distributive law; thus the quotient $\frac{A(1+r)}{A} = (1+r)$ is constant for any two consecutive terms of the sequence at any step and therefore this is a geometric sequence with ratio $k = (1+r)$ (could have been noticed in the first step). Substituting $k = (1+r)$, $a = P$, and $n = t$ (t-years) in equation (4) we have a formula for the amount of money that will be in the account after t years.

$$a_t = P(1+r)^t,\ t = 0,1,2,\ldots \qquad (6)$$

Idea for another Activity:

- Design an activity (sequence of tasks) to introduce the n-times annually compound interest. Extend the procedure form the beginning creating a similar problem for the introduction of the n-times annually compound interest. **Suggestion:** Organize in a chart different cases of n-times compounded interest starting with the annually compound interest. The purpose of this is to facilitate student recognition of the need for a new and more general formula as the interest paid is interpreted as the corresponding part of interest defined by the APR for each period according to the principal invested.

For a more formal deduction of the new formula read the following exposition:

Now, assume the investment account pays compounded interest n-times (n **periods**) within a year; so the APR must be divided by such number n so the interest accrued during each period could be calculated. The **periodic interest rate** is $\frac{r}{n}$. Repetitive reasoning lead to consider the geometric sequence that is generated. Substituting $k = (1+\frac{r}{n})$, $a = P$, and $n = s$ in equation (4) we have a formula for the amount of money that will be in the account after s periods.

$$a_s = P(1+\frac{r}{n})^s,\ s = 0,1,2,\ldots, \qquad (7)$$

where s indicate the periods when the interest is added to the principal; for example, months (compounded monthly), days (compounded daily), quarters (compounded quarterly), etc. Note that the index (variable n) in the original formula has been substituted by a new variable s (index), whether the letter n in the new formula is a parameter that is kept constant for each context, defining the number of periods. This formula, is very useful for example if one wants to know the amount in an investment account **compounded monthly** after 16 months (a given number of periods), however it is more common in the literature to use the formula (8) because somehow it is standard to calculate the amount of money in account periodically compounded on the annual basis (in a given number of years).

$$c_t = P(1+\frac{r}{n})^{nt},\ t = 0,1,2,\ldots, \qquad (8)$$

where t indicates the number of years. To go from formula (7) to formula (8) it is enough to notice that the number of compounded periods in t years for an n-times compounded interest is equal to nt. A more formal justification is to note that $c_t, t = 0,1,2,\ldots$ defined by formula (8) is a subsequence of $a_s, s = 0,1,2,\ldots$ defined by the formula (7) as illustrated in the diagram below:

$$\begin{tikzcd}
N \arrow[r, "a_s"] & R \\
N \arrow[u, "b_t"] \arrow[ur, "c_t"'] &
\end{tikzcd}$$

Where $b_t = s = nt$, n-fixed and $t = 0, 1, 2, 3, \ldots$, and a_s $s = 0, 1, 2, 3,\ldots$ is defined by equation (7). That is, $c_t = a_s \circ b_t$

FURTHER KNOWLEDGE TO introduce continuously compound interests to students

The number e as should be appropriately refreshed according to the students level, but a mathematics teacher should know that the number e can be defined by:

$$\lim_{n\to\infty}(1+\frac{1}{n})^n = \lim_{n\to\infty}(1+\frac{1}{n+1})^{n+1} = e \text{ and } \lim_{n\to\infty}(1+\frac{x}{n})^n = \lim_{n\to\infty}(1+\frac{x}{n+1})^{n+1} = e^x$$

To obtain a formula for a **continuously compound interest** it is required to take a step to the limit when n tends to infinite in formula (8). It requires considering the parameter n as a variable and the variable t as a parameter in this formula as shown below:

$$\lim_{n\to\infty} c_t = \lim_{n\to\infty} P(1+\frac{r}{n})^{nt} = P\lim_{n\to\infty}[(1+\frac{r}{n})^n]^t = P[\lim_{n\to\infty}(1+\frac{r}{n})^n]^t = P(e^r)^t = Pe^{rt}$$

The explicit formula for the amount of money that will be in a continuously compounded interest account after t years is:

$$a_t = Pe^{rt}, \quad t = 0,1,2,3,\ldots \qquad (9)$$

Extention: What is a series?

Series are considered infinite sums or an indicated sum (finite or infinite) of the terms of a sequence.

Some definitions of Series:

Definition 3: A series is a sequence of partial sums S_n of the terms of a sequence a_n.
Notation: A series is denote by S_n where

$$S_n = \sum_{n=0}^{\infty} a_n$$

$$S_n = \sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \ldots + a_n + \ldots$$

Partial sums: $S_0 = a_0$

$S_1 = a_0 + a_1$

$S_2 = a_0 + a_1 + a_2$

•

•

•

$S_n = a_0 + a_1 + a_2 + \ldots + a_n + \ldots = \sum_{n=0}^{\infty} a_n$

•

•

•

While a series is a sequence S_n each term of this sequence is called a ***partial sum*** instead. The language in the discipline reserves the word ***term*** of a series to name the elements of the sum a_n. This may be justified by the fact that since the antiquity series were considered infinite sums. So, the general term for a series is the *n-th* partial sum S_n.

Definition 3a: A series is a pair of sequences (a_n, S_n) where the first sequence a_n defines the terms of the series (each addend in the sum) and the second sequence is the sequence of partial sums.

In this context we also can make the distinction between finite and infinite series as well when working with sequences in general. In this sense the *n-th* partial sum of an infinite series is considered a finite series. It is an important problem to find an explicit formula for such sum, which in general could be a difficult if not an impossible task.

$S_1 = a_1$
$S_2 = S_1 + a_2 = a_1 + a_1 + d$
$S_3 = S_2 + a_3 = a_1 + a_1 + d + a_1 + 2d$

•

•

•

$S_n = a_1 + a_1 + d + a_1 + 2d + a_1 + 3d + \ldots + a_1 + (n-1)d$
$S_n = na_1 + [1 + 2 + 3 + \ldots + (n-1)]d$

$$S_n = na_1 + \frac{n(n-1)}{2}d$$

$$S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

$$S_n = \frac{n}{2}[a_1 + a_n]$$

We have use inductive reasoning to find a formula for the general term of a sequence or a finite sum. Suppose that we have a statement S_n for each positive integer n, and we wish to prove that all these statements S_n are true. A deductively proof is done using the following mathematical principle.

Principle of Mathematical Induction (PMI)

Consider the set of all integers T that make S_n true; T = $\{n\text{: } S_n \text{ is true}\}$.
In order to prove that S_n is true for all integers, we must prove that T contains all positive integers.
1 ∈ T means that S_1 is true
2 ∈ T means that S_2 is true
n ∈ T means that S_n is true and so on…
To prove that all positive integers belong to T, we use the following axiom:

A set T of integers such that
(1) 1 ∈ T and
(2) $(n + 1)$ ∈ T whenever n ∈ T
contains all positive integers.

Domino effects:
Since 1∈ T, it follows that 2∈ T
Since 2∈ T, it follows that 3∈ T
Since 3∈ T, it follows that 4∈ T and so on…

In other terms, to show that S_n is true for all positive integers, we must follow the following steps:

Step 1: Check the formula holds for $n = 1$ (S_1 is true)
Step 2: Prove that if the formula holds for any positive integer $n = k$, then it also holds for the next integer, $n = (k + 1)$

Once these steps are completed, the axiom says, we know that the formula holds for all positive integers n.

Homework

1. Using the Activity 1 as a model, think of how to use the graphing calculator for a similar activity for Geometric Sequences and Annually Compound Interest.

2. Find the sum S_n of the following series.

$$S_n = 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + \ldots + n^3$$

Hint: Use the formula $1+2+3+...+n=\frac{n(n+1)}{2}$ and inductive reasoning.

3. Find the sum S_n of the following series.

$$S_n = 1(1!) + 2(2!) + 3(3!) + ... + n(n!)$$

4. Show that

a. $\frac{1}{2^1}+\frac{1}{2^2}+\frac{1}{2^3}+...+\frac{1}{2^n}=1-\frac{1}{2^n}, n=1, 2, 3, ...$ (Show **folded paper model**)

b. $1+2+3+...+n=\frac{n(n+1)}{2}, n=1, 2, 3, ...$ (Show **square paper model**)

c. $n^3 + 2n$ is a multiple of 3, $n = 1, 2, 3, ...$

Other Starting Integer

Sometimes, instead of starting at 1, some induction arguments start at another integer.

d. Show that $n! > 3^n$ if n is large enough.

Further reflections for the classroom. What a mathematics teacher should know (cont ...)

Secondary level mathematics books often informally define a **sequence** as ***"a list of numbers,"*** while the list continues indefinitely. An example of an arithmetic sequence could look like the following:

$$0, 28, 56, 84, 112, 140, ...$$

Another way to talk about sequences is to identify them with a set of numbers arranged in a particular order. In any case, it is common to call such numbers the ***"terms"*** of a sequence. Additionally a sequence is denoted using the symbol a_n; which identifies what is called ***"the general term of a sequence."*** For the example above, one could write:

$$a_n = 28(n-1), n=1, 2, 3, ...$$

However, it is often an issue whether the teacher prefers to work with an array with sub-indices starting at zero instead of starting at 1. Thus, for the example under consideration the sequence could be written as follows:

$$a_n = 28n, n=0, 1, 2, 3, ...$$

In many different countries the set of ***natural numbers*** (counting numbers) is defined as {0, 1, 2, 3,...}. In U.S. this is called the set of ***whole numbers*** (starting from zero instead of from one) while the set of ***natural numbers*** is defined as {1, 2, 3,...} (starting form one). The set of ***natural numbers*** will play an important role in the formalization of the concept of sequence, as it will be seen soon.

An interesting problem that commonly arises in the secondary mathematics classroom is to find an explicit formula or rule for ***the general term of a sequence*** that can be used to generate all the numbers in the given list; however how can one know what will be the next term of the sequence? The list, presumably infinite, shows only a finite number of "***terms***" and we may be asking the students to tell how the list will continue to form. If it is not given enough initial information to guarantee that such list could be determined in a unique way, then we are asking for an impossible. For example, if we ask the students to