5.1 Exercises

CONCEPT PREVIEW *Fill in the blank(s) to correctly complete each sentence.*

1. The solution set of the following system is $\{(1, \_\_)\}$.

$$\begin{aligned} -2x + 5y &= 18 \\ x + y &= 5 \end{aligned}$$

2. The solution set of the following system is $\{(\_\_, 0)\}$.

$$\begin{aligned} 6x + y &= -18 \\ 13x + y &= -39 \end{aligned}$$

3. One way of solving the following system by elimination is to multiply equation (2) by the integer _____ to eliminate the y-terms by direct addition.

$$\begin{aligned} 14x + 11y &= 80 \quad (1) \\ 2x + y &= 19 \quad (2) \end{aligned}$$

4. To solve the system

$$\begin{aligned} 3x + y &= 4 \quad (1) \\ 7x + 8y &= -2 \quad (2) \end{aligned}$$

by substitution, it is easiest to begin by solving equation (1) for the variable _____ and then substituting into equation (2), because no fractions will appear in the algebraic work.

5. If a system of linear equations in two variables has two graphs that coincide, there is/are ________________ (one/no/infinitely many) solutions to the system.

6. If a system of linear equations in two variables has two graphs that are parallel lines, there is/are ________________ (one/no/infinitely many) solutions to the system.

Solve each system by substitution. ***See Example 1.***

7. $4x + 3y = -13$
   $-x + y = 5$

8. $3x + 4y = 4$
   $x - y = 13$

9. $x - 5y = 8$
   $x = 6y$

10. $6x - y = 5$
    $y = 11x$

11. $8x - 10y = -22$
    $3x + y = 6$

12. $4x - 5y = -11$
    $2x + y = 5$

13. $7x - y = -10$
    $3y - x = 10$

14. $4x + 5y = 7$
    $9y = 31 + 2x$

15. $-2x = 6y + 18$
    $-29 = 5y - 3x$

16. $3x - 7y = 15$
    $3x + 7y = 15$

17. $3y = 5x + 6$
    $x + y = 2$

18. $4y = 2x - 4$
    $x - y = 4$

Solve each system by elimination. In systems with fractions, first clear denominators. ***See Example 2.***

19. $3x - y = -4$
    $x + 3y = 12$

20. $4x + y = -23$
    $x - 2y = -17$

21. $2x - 3y = -7$
    $5x + 4y = 17$

22. $4x + 3y = -1$
    $2x + 5y = 3$

23. $5x + 7y = 6$
    $10x - 3y = 46$

24. $12x - 5y = 9$
    $3x - 8y = -18$

25. $6x + 7y + 2 = 0$
$7x - 6y - 26 = 0$

26. $5x + 4y + 2 = 0$
$4x - 5y - 23 = 0$

27. $\dfrac{x}{2} + \dfrac{y}{3} = 4$
$\dfrac{3x}{2} + \dfrac{3y}{2} = 15$

28. $\dfrac{3x}{2} + \dfrac{y}{2} = -2$
$\dfrac{x}{2} + \dfrac{y}{2} = 0$

29. $\dfrac{2x-1}{3} + \dfrac{y+2}{4} = 4$
$\dfrac{x+3}{2} - \dfrac{x-y}{3} = 3$

30. $\dfrac{x+6}{5} + \dfrac{2y-x}{10} = 1$
$\dfrac{x+2}{4} + \dfrac{3y+2}{5} = -3$

Solve each system of equations. State whether it is an inconsistent system *or has* infinitely many solutions. *If the system has infinitely many solutions, write the solution set with y arbitrary.* ***See Examples 3 and 4.***

31. $9x - 5y = 1$
$-18x + 10y = 1$

32. $3x + 2y = 5$
$6x + 4y = 8$

33. $4x - y = 9$
$-8x + 2y = -18$

34. $3x + 5y = -2$
$9x + 15y = -6$

35. $5x - 5y - 3 = 0$
$x - y - 12 = 0$

36. $2x - 3y - 7 = 0$
$-4x + 6y - 14 = 0$

37. $7x + 2y = 6$
$14x + 4y = 12$

38. $2x - 8y = 4$
$x - 4y = 2$

39. $2x - 6y = 0$
$-7x + 21y = 10$

40. *Concept Check* Which screen gives the correct graphical solution of the system? (*Hint:* Solve for y first in each equation and use the slope-intercept forms to answer the question.)

$$4x - 5y = -11$$
$$2x + y = 5$$

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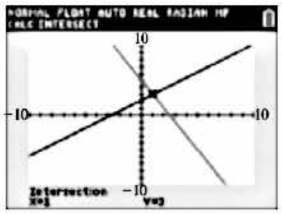

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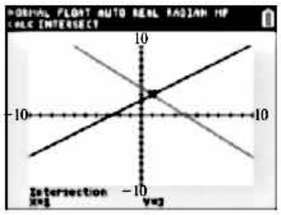

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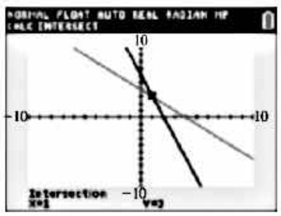

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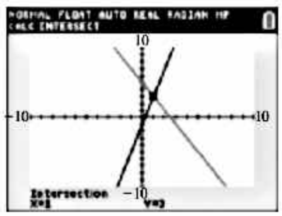

Connecting Graphs with Equations Determine the system of equations illustrated in each graph. Write equations in standard form.

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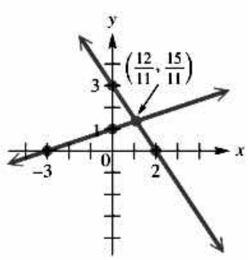

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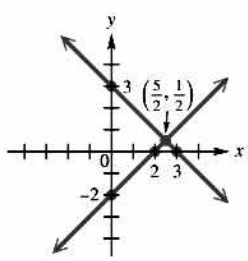


Use a graphing calculator to solve each system. Express solutions with approximations to the nearest thousandth.

43. $\frac{11}{3}x + y = 0.5$
$0.6x - y = 3$

44. $\sqrt{3}x - y = 5$
$100x + y = 9$

45. $\sqrt{7}x + \sqrt{2}y = 3$
$\sqrt{6}x - y = \sqrt{3}$

46. $0.2x + \sqrt{2}y = 1$
$\sqrt{5}x + 0.7y = 1$

Solve each system. ***See Example 6.***

47. $x + y + z = 2$
$2x + y - z = 5$
$x - y + z = -2$

48. $2x + y + z = 9$
$-x - y + z = 1$
$3x - y + z = 9$

49. $x + 3y + 4z = 14$
$2x - 3y + 2z = 10$
$3x - y + z = 9$

50. $4x - y + 3z = -2$
$3x + 5y - z = 15$
$-2x + y + 4z = 14$

51. $x + 4y - z = 6$
$2x - y + z = 3$
$3x + 2y + 3z = 16$

52. $4x - 3y + z = 9$
$3x + 2y - 2z = 4$
$x - y + 3z = 5$

53. $x - 3y - 2z = -3$
$3x + 2y - z = 12$
$-x - y + 4z = 3$

54. $x + y + z = 3$
$3x - 3y - 4z = -1$
$x + y + 3z = 11$

55. $2x + 6y - z = 6$
$4x - 3y + 5z = -5$
$6x + 9y - 2z = 11$

56. $8x - 3y + 6z = -2$
$4x + 9y + 4z = 18$
$12x - 3y + 8z = -2$

57. $2x - 3y + 2z - 3 = 0$
$4x + 8y + z - 2 = 0$
$-x - 7y + 3z - 14 = 0$

58. $-x + 2y - z - 1 = 0$
$-x - y - z + 2 = 0$
$x - y + 2z - 2 = 0$

Solve each system in terms of the arbitrary variable z. ***See Example 7.***

59. $x - 2y + 3z = 6$
$2x - y + 2z = 5$

60. $3x - 2y + z = 15$
$x + 4y - z = 11$

61. $5x - 4y + z = 9$
$y + z = 15$

62. $3x - 5y - 4z = -7$
$y - z = -13$

63. $3x + 4y - z = 13$
$x + y + 2z = 15$

64. $x - y + z = -6$
$4x + y + z = 7$

Solve each system. State whether it is an inconsistent system *or has* infinitely many solutions. *If the system has infinitely many solutions, write the solution set with z arbitrary.* ***See Examples 3, 4, 6, and 7.***

65. $3x + 5y - z = -2$
$4x - y + 2z = 1$
$-6x - 10y + 2z = 0$

66. $3x + y + 3z = 1$
$x + 2y - z = 2$
$2x - y + 4z = 4$

67. $5x - 4y + z = 0$
$x + y = 0$
$-10x + 8y - 2z = 0$

68. $2x + y - 3z = 0$
$4x + 2y - 6z = 0$
$x - y + z = 0$

Solve each system. (Hint: In Exercises 69–72, let $\frac{1}{x} = t$ and $\frac{1}{y} = u$.)

69. $\frac{2}{x} + \frac{1}{y} = \frac{3}{2}$
$\frac{3}{x} - \frac{1}{y} = 1$

70. $\frac{1}{x} + \frac{3}{y} = \frac{16}{5}$
$\frac{5}{x} + \frac{4}{y} = 5$

71. $\frac{2}{x} + \frac{1}{y} = 11$
$\frac{3}{x} - \frac{5}{y} = 10$

5.1 Systems of Linear Equations

- Linear Systems
- Substitution Method
- Elimination Method
- Special Systems
- Application of Systems of Equations
- Linear Systems with Three Unknowns (Variables)
- Application of Systems to Model Data

Linear Systems

The definition of a linear equation can be extended to more than one variable. Any equation of the form

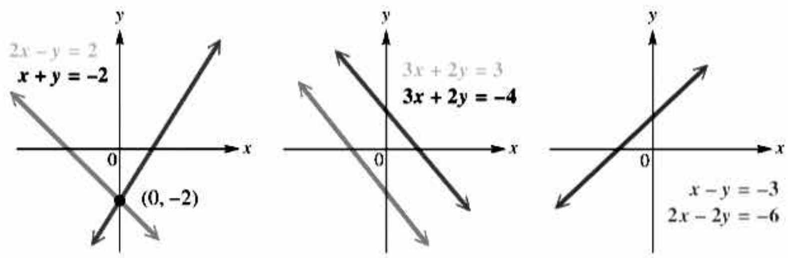
$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,$$

for real numbers $a_1, a_2, \ldots, a_n$ (all nonzero) and b, is a **linear equation,** or a **first-degree equation, in n unknowns.**

A set of equations considered simultaneously is a **system of equations.** The **solutions** of a system of equations must satisfy every equation in the system. If all the equations in a system are linear, the system is a **system of linear equations,** or a **linear system.**

The solution set of a linear equation in two unknowns (or variables) is an infinite set of ordered pairs. The graph of such an equation is a straight line, so there are three possibilities for the number of elements in the solution set of a system of two linear equations in two unknowns. See **Figure 1.** The possible graphs of a linear system in two unknowns are as follows.

1. **The graphs intersect at exactly one point,** which gives the (single) ordered-pair solution of the system. The **system is consistent** and the **equations are independent.** See **Figure 1(a).**
2. **The graphs are parallel lines,** so there is no solution and the solution set is $\emptyset$. The **system is inconsistent** and the **equations are independent.** See **Figure 1(b).**
3. **The graphs are the same line,** and there are an infinite number of solutions. The **system is consistent** and the **equations are dependent.** See **Figure 1(c).**



One solution (a) | No solution (b) | Infinitely many solutions (c)

Figure 1

Using graphs to find the solution set of a linear system in two unknowns provides a good visual perspective, but may be inefficient when the solution set contains non-integer values. Thus, we introduce two algebraic methods for solving systems with two unknowns: *substitution* and *elimination.*

Substitution Method

In a system of two equations with two variables, the **substitution method** involves using one equation to find an expression for one variable in terms of the other, and then substituting this expression into the other equation of the system.

EXAMPLE 1 Solving a System (Substitution Method)

Solve the system.

$$3x + 2y = 11 \quad (1)$$
$$-x + y = 3 \quad (2)$$

SOLUTION Begin by solving one of the equations for one of the variables. We solve equation (2) for y.

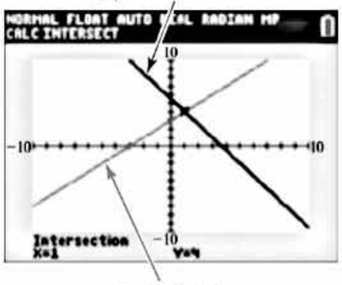
$$-x + y = 3 \quad (2)$$
$$y = x + 3 \quad \text{Add } x. \quad (3)$$

Now replace y with $x + 3$ in equation (1), and solve for x.

$$3x + 2y = 11 \quad (1)$$
$$3x + 2(x + 3) = 11 \quad \text{Let } y = x + 3 \text{ in (1).}$$
$$3x + 2x + 6 = 11 \quad \text{Distributive property}$$
$$5x + 6 = 11 \quad \text{Combine like terms.}$$
$$5x = 5 \quad \text{Subtract 6.}$$
$$x = 1 \quad \text{Divide by 5.}$$

Note the careful use of parentheses.

$y_1 = -1.5x + 5.5$



$y_2 = x + 3$

To solve the system in **Example 1** graphically, solve both equations for y.

$3x + 2y = 11$ leads to

$$y_1 = -1.5x + 5.5.$$

$-x + y = 3$ leads to

$$y_2 = x + 3.$$

Graph both y_1 and y_2 in the standard window to find that their point of intersection is $(1, 4)$.

Replace x with 1 in equation (3) to obtain

$$y = x + 3 = 1 + 3 = 4.$$

The solution of the system is the ordered pair $(1, 4)$. ***Check this solution in both equations (1) and (2).***

CHECK		
$3x + 2y = 11$ (1)		$-x + y = 3$ (2)
$3(1) + 2(4) \stackrel{?}{=} 11$		$-1 + 4 \stackrel{?}{=} 3$
$11 = 11$ ✓ True		$3 = 3$ ✓ True

True statements result when the solution is substituted in both equations, confirming that the solution set is $\{(1, 4)\}$.

✔ **Now Try Exercise 7.**

Elimination Method The **elimination method** for solving a system of two equations uses multiplication and addition to eliminate a variable from one equation. To eliminate a variable, the coefficients of that variable in the two equations must be additive inverses. We use properties of algebra to change the system to an **equivalent system,** one with the same solution set.

The three transformations that produce an equivalent system are listed here.

Transformations of a Linear System

1. Interchange any two equations of the system.
2. Multiply or divide any equation of the system by a nonzero real number.
3. Replace any equation of the system by the sum of that equation and a multiple of another equation in the system.

EXAMPLE 2 Solving a System (Elimination Method)

Solve the system.

$$\begin{aligned} 3x - 4y &= 1 \quad (1) \\ 2x + 3y &= 12 \quad (2) \end{aligned}$$

SOLUTION One way to eliminate a variable is to use the second transformation and multiply each side of equation (2) by -3, to obtain an equivalent system.

$$\begin{aligned} 3x - 4y &= 1 && (1) \\ -6x - 9y &= -36 && \text{Multiply (2) by } -3. \quad (3) \end{aligned}$$

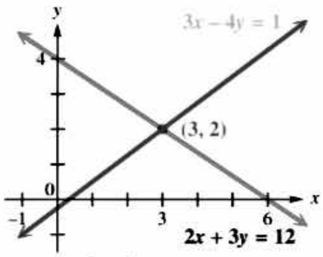
Now multiply each side of equation (1) by 2, and use the third transformation to add the result to equation (3), eliminating x. Solve the result for y.

$$\begin{aligned} 6x - 8y &= 2 && \text{Multiply (1) by 2.} \\ \underline{-6x - 9y} &\underline{= -36} && (3) \\ -17y &= -34 && \text{Add.} \\ y &= 2 && \text{Solve for } y. \end{aligned}$$

Substitute 2 for y in either of the original equations and solve for x.

$$\begin{aligned} 3x - 4y &= 1 && (1) \\ 3x - 4(2) &= 1 && \text{Let } y = 2 \text{ in (1).} \\ 3x - 8 &= 1 && \text{Multiply.} \\ 3x &= 9 && \text{Add 8.} \\ x &= 3 && \text{Divide by 3.} \end{aligned}$$

Write the x-value first.



Consistent system with solution set $\{(3, 2)\}$

Figure 2

A check shows that $(3, 2)$ satisfies both equations (1) and (2). Therefore, the solution set is $\{(3, 2)\}$. The graph in **Figure 2** confirms this.

✔ Now Try Exercise 21.

Special Systems The systems in **Examples 1 and 2** were both consistent, having a single solution. This is not always the case.

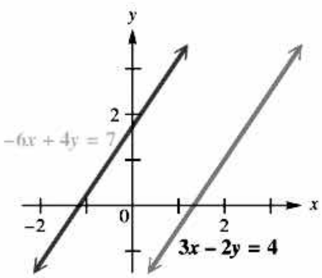
EXAMPLE 3 Solving an Inconsistent System

Solve the system.

$$\begin{aligned} 3x - 2y &= 4 \quad (1) \\ -6x + 4y &= 7 \quad (2) \end{aligned}$$

SOLUTION To eliminate the variable x, multiply each side of equation (1) by 2, and add the result to equation (2).

$$\begin{aligned} 6x - 4y &= 8 && \text{Multiply (1) by 2.} \\ \underline{-6x + 4y} &\underline{= 7} && (2) \\ 0 &= 15 && \text{False} \end{aligned}$$



Inconsistent system with solution set $\emptyset$

Figure 3

Since $0 = 15$ is false, the system is inconsistent and has no solution. As suggested by **Figure 3,** this means that the graphs of the equations of the system never intersect. (The lines are parallel.) The solution set is $\emptyset$.

✔ Now Try Exercise 31.

EXAMPLE 4 Solving a System with Infinitely Many Solutions

Solve the system.

$$8x - 2y = -4 \quad (1)$$
$$-4x + y = 2 \quad (2)$$

ALGEBRAIC SOLUTION

Divide each side of equation (1) by 2, and add the result to equation (2).

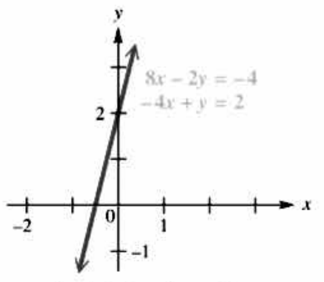
$$\begin{aligned} 4x - y &= -2 \quad &&\text{Divide (1) by 2.} \\ -4x + y &= 2 \quad &&(2) \\ \hline 0 &= 0 \quad &&\text{True} \end{aligned}$$

The result, $0 = 0$, is a true statement, which indicates that the equations are equivalent. Any ordered pair (x, y) that satisfies either equation will satisfy the system. Solve equation (2) for y.

$$\begin{aligned} -4x + y &= 2 \quad (2) \\ y &= 4x + 2 \end{aligned}$$

The solutions of the system can be written in the form of a set of ordered pairs $(x, 4x + 2)$, for any real number x. Some ordered pairs in the solution set are $(0, 4 \cdot 0 + 2)$, or $(0, 2)$, and $(1, 4 \cdot 1 + 2)$, or $(1, 6)$, as well as $(3, 14)$, and $(-2, -6)$.

As shown in **Figure 4,** the equations of the original system are dependent and lead to the same straight-line graph. Using this method, the solution set can be written $\{(x, 4x + 2)\}$.



Consistent system with infinitely many solutions

Figure 4

GRAPHING CALCULATOR SOLUTION

Solving the equations for y gives

$$y_1 = 4x + 2 \quad (1)$$

and

$$y_2 = 4x + 2. \quad (2)$$

When written in this form, we can immediately determine that the equations are identical. Each has slope 4 and y-intercept $(0, 2)$.

As expected, the graphs coincide. See the top screen in **Figure 5.** The table indicates that

$$y_1 = y_2 \text{ for selected values of } x,$$

providing another way to show that the two equations lead to the same graph.

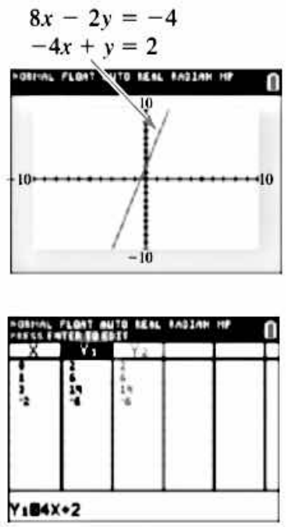


Figure 5

Refer to the algebraic solution to see how the solution set can be written using an arbitrary variable.

✔ Now Try Exercise 33.

NOTE In the algebraic solution for **Example 4,** we wrote the solution set with the variable x arbitrary. We could write the solution set with y arbitrary.

$$\left\{\left(\frac{y-2}{4}, y\right)\right\} \quad \text{Solve } -4x + y = 2 \text{ for } x.$$

By selecting values for y and solving for x in this ordered pair, we can find individual solutions. Verify again that $(0, 2)$ is a solution by letting $y = 2$ and solving for x to obtain $\frac{2-2}{4} = 0$.

Application of Systems of Equations Many applied problems involve more than one unknown quantity. Although some problems with two unknowns can be solved using just one variable, it is often easier to use two variables.

To solve a problem with two unknowns, we must write two equations that relate the unknown quantities. The system formed by the pair of equations can then be solved using the methods of this chapter. The following steps, based on the six-step problem-solving method introduced earlier, give a strategy for solving such applied problems.

Solving an Applied Problem by Writing a System of Equations

Step 1 **Read** the problem carefully until you understand what is given and what is to be found.

Step 2 **Assign variables** to represent the unknown values, using diagrams or tables as needed. *Write down* what each variable represents.

Step 3 **Write a system of equations** that relates the unknowns.

Step 4 **Solve** the system of equations.

Step 5 **State the answer** to the problem. Does it seem reasonable?

Step 6 **Check** the answer in the words of the original problem.

EXAMPLE 5 **Using a Linear System to Solve an Application**

Salaries for the same position can vary depending on the location. In 2015, the average of the median salaries for the position of Accountant I in San Diego, California, and Salt Lake City, Utah, was \$47,449.50. The median salary in San Diego, however, exceeded the median salary in Salt Lake City by \$5333. Determine the median salary for the Accountant I position in San Diego and in Salt Lake City. (*Source:* www.salary.com)

SOLUTION

Step 1 **Read** the problem. We must find the median salary of the Accountant I position in San Diego and in Salt Lake City.

Step 2 **Assign variables.** Let x represent the median salary of the Accountant I position in San Diego and y represent the median salary for the same position in Salt Lake City.

Step 3 **Write a system of equations.** Since the average of the two medians for the Accountant I position in San Diego and Salt Lake City was \$47,449.50, one equation is as follows.

$$\frac{x+y}{2} = 47{,}449.50$$

Multiply each side of this equation by 2 to clear the fraction and obtain an equivalent equation.

$$x + y = 94{,}899 \quad (1)$$

The median salary in San Diego exceeded the median salary in Salt Lake City by \$5333. Thus, $x - y = 5333$, which gives the following system of equations.

$$x + y = 94{,}899 \quad (1)$$
$$x - y = 5333 \quad (2)$$

Step 4 **Solve** the system. To eliminate y, add the two equations.

$$\begin{aligned} x + y &= 94{,}899 && (1) \\ x - y &= 5333 && (2) \\ \hline 2x \qquad &= 100{,}232 && \text{Add.} \\ x &= 50{,}116 && \text{Solve for } x. \end{aligned}$$

To find y, substitute 50,116 for x in equation (2).

$$\begin{aligned} x - y &= 5333 && (2) \\ 50{,}116 - y &= 5333 && \text{Let } x = 50{,}116. \\ -y &= -44{,}783 && \text{Subtract } 50{,}116. \\ y &= 44{,}783 && \text{Multiply by } -1. \end{aligned}$$

Step 5 **State the answer.** The median salary for the position of Accountant I was \$50,116 in San Diego and \$44,783 in Salt Lake City.

Step 6 **Check.** The average of \$50,116 and \$44,783 is

$$\frac{\$50{,}116 + \$44{,}783}{2} = \$47{,}449.50.$$

Also, $\quad \$50{,}116 - \$44{,}783 = \$5333, \quad$ as required.

✔ Now Try Exercise 101.

Linear Systems with Three Unknowns (Variables) We have seen that the graph of a linear equation in two unknowns is a straight line. The graph of a linear equation in three unknowns requires a three-dimensional coordinate system. The three number lines are placed at right angles. The graph of a linear equation in three unknowns is a plane. Some possible intersections of planes representing three equations in three variables are shown in **Figure 6.**

In two dimensions we customarily label the axes x and y. When working in three dimensions they are usually labeled x, y, and z.

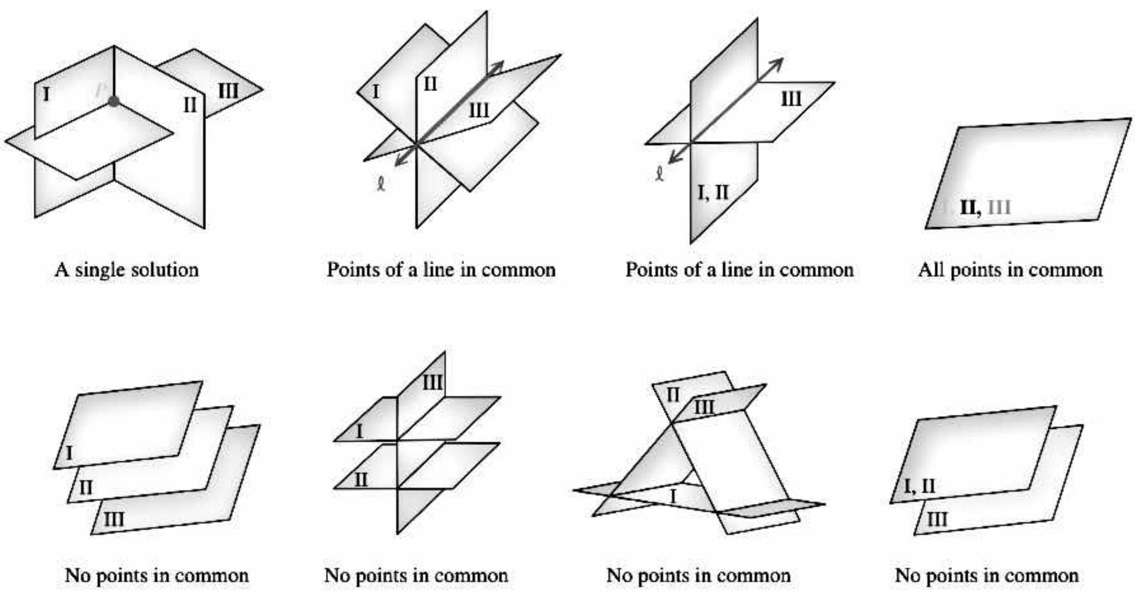


Figure 6

Solving a Linear System with Three Unknowns

Step 1 Eliminate a variable from any two of the equations.

Step 2 Eliminate the ***same variable*** from a different pair of equations.

Step 3 Eliminate a second variable using the resulting two equations in two variables to obtain an equation with just one variable whose value we can now determine.

Step 4 Find the values of the remaining variables by substitution. Write the solution of the system as an **ordered triple.**

EXAMPLE 6 **Solving a System of Three Equations with Three Variables**

Solve the system.

$$\begin{aligned} 3x + 9y + 6z &= 3 \quad (1) \\ 2x + y - z &= 2 \quad (2) \\ x + y + z &= 2 \quad (3) \end{aligned}$$

SOLUTION

Step 1 Eliminate z by adding equations (2) and (3).

$$3x + 2y = 4 \quad (4)$$

Step 2 To eliminate z from another pair of equations, multiply each side of equation (2) by 6 and add the result to equation (1).

$$\begin{aligned} 12x + 6y - 6z &= 12 \quad \text{Multiply (2) by 6.} \\ 3x + 9y + 6z &= 3 \quad (1) \\ \hline 15x + 15y \quad &= 15 \quad (5) \end{aligned}$$

Make sure equation (5) has the *same* two variables as equation (4).

Step 3 To eliminate x from equations (4) and (5), multiply each side of equation (4) by -5 and add the result to equation (5). Solve the resulting equation for y.

$$\begin{aligned} -15x - 10y &= -20 \quad \text{Multiply (4) by } -5. \\ 15x + 15y &= 15 \quad (5) \\ \hline 5y &= -5 \quad \text{Add.} \\ y &= -1 \quad \text{Divide by 5.} \end{aligned}$$

Step 4 Use $y = -1$ to find x from equation (4) by substitution.

$$\begin{aligned} 3x + 2y &= 4 \quad (4) \\ 3x + 2(-1) &= 4 \quad \text{Let } y = -1. \\ x &= 2 \quad \text{Solve for } x. \end{aligned}$$

Substitute 2 for x and -1 for y in equation (3) to find z.

$$\begin{aligned} x + y + z &= 2 \quad (3) \\ 2 + (-1) + z &= 2 \quad \text{Let } x = 2,\ y = -1. \\ z &= 1 \quad \text{Solve for } z. \end{aligned}$$

Write the values of x, y, and z in the correct order.

Verify that the ordered triple $(2, -1, 1)$ satisfies all three equations in the *original* system. The solution set is $\{(2, -1, 1)\}$.

Now Try Exercise 47.