16. *Decay of Plutonium* Repeat **Exercise 15** for 500 g of plutonium-241, which decays according to the function $A(t) = A_0e^{-0.053t}$, where t is time in years.

17. *Decay of Radium* Find the half-life of radium-226, which decays according to the function $A(t) = A_0e^{-0.00043t}$, where t is time in years.

18. *Decay of Tritium* Find the half-life of tritium, a radioactive isotope of hydrogen, which decays according to the function $A(t) = A_0e^{-0.056t}$, where t is time in years.

19. *Radioactive Decay* If 12 g of a radioactive substance are present initially and 4 yr later only 6.0 g remain, how much of the substance will be present after 7 yr?

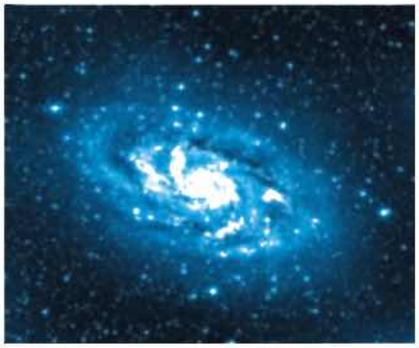
20. *Radioactive Decay* If 1 g of strontium-90 is present initially, and 2 yr later 0.95 g remains, how much strontium-90 will be present after 5 yr?

21. *Decay of Iodine* How long will it take any quantity of iodine-131 to decay to 25% of its initial amount, knowing that it decays according to the exponential function $A(t) = A_0e^{-0.087t}$, where t is time in days?

22. *Magnitude of a Star* The magnitude M of a star is modeled by

$$M = 6 - \frac{5}{2}\log\frac{I}{I_0},$$

where I_0 is the intensity of a just-visible star and I is the actual intensity of the star being measured. The dimmest stars are of magnitude 6, and the brightest are of magnitude 1. Determine the ratio of light intensities between a star of magnitude 1 and a star of magnitude 3.



23. *Carbon-14 Dating* Suppose an Egyptian mummy is discovered in which the amount of carbon-14 present is only about one-third the amount found in living human beings. How long ago did the Egyptian die?

24. *Carbon-14 Dating* A sample from a refuse deposit near the Strait of Magellan had 60% of the carbon-14 of a contemporary sample. How old was the sample?

25. *Carbon-14 Dating* Paint from the Lascaux caves of France contains 15% of the normal amount of carbon-14. Estimate the age of the paintings.

26. *Dissolving a Chemical* The amount of a chemical that will dissolve in a solution increases exponentially as the (Celsius) temperature t is increased according to the model

$$A(t) = 10e^{0.0095t}.$$

At what temperature will 15 g dissolve?

27. *Newton's Law of Cooling* Boiling water, at 100°C, is placed in a freezer at 0°C. The temperature of the water is 50°C after 24 min. Find the temperature of the water to the nearest hundredth after 96 min. (*Hint:* Change minutes to hours.)

28. *Newton's Law of Cooling* A piece of metal is heated to 300°C and then placed in a cooling liquid at 50°C. After 4 min, the metal has cooled to 175°C. Find its temperature to the nearest hundredth after 12 min. (*Hint:* Change minutes to hours.)

Finance (Exercises 29–34)

29. *Comparing Investments* Russ, who is self-employed, wants to invest \$60,000 in a pension plan. One investment offers 3% compounded quarterly. Another offers 2.75% compounded continuously.

(a) Which investment will earn more interest in 5 yr?

(b) How much more will the better plan earn?

30. *Growth of an Account* If Russ (see **Exercise 29**) chooses the plan with continuous compounding, how long will it take for his \$60,000 to grow to \$70,000?

31. *Doubling Time* Find the doubling time of an investment earning 2.5% interest if interest is compounded continuously.

32. *Doubling Time* If interest is compounded continuously and the interest rate is tripled, what effect will this have on the time required for an investment to double?

33. *Growth of an Account* How long will it take an investment to triple if interest is compounded continuously at 3%?

34. *Growth of an Account* Use the Table feature of a graphing calculator to find how long it will take \$1500 invested at 2.75% compounded daily to triple in value. Zoom in on the solution by systematically decreasing the increment for x. Find the answer to the nearest day. (Find the answer to the nearest day by eventually letting the increment of x equal $\frac{1}{365}$. The decimal part of the solution can be multiplied by 365 to determine the number of days greater than the nearest year. For example, if the solution is determined to be 16.2027 yr, then multiply 0.2027 by 365 to get 73.9855. The solution is then, to the nearest day, 16 yr, 74 days.) Confirm the answer algebraically.

Social Sciences *(Exercises 35–44)*

35. *Legislative Turnover* The turnover of legislators is a problem of interest to political scientists. It was found that one model of legislative turnover in a particular body was

$$M(t) = 434e^{-0.08t},$$

where $M(t)$ represents the number of continuously serving members at time t. Here, $t = 0$ represents 1965, $t = 1$ represents 1966, and so on. Use this model to approximate the number of continuously serving members in each year.

(a) 1969 **(b)** 1973 **(c)** 1979

36. *Legislative Turnover* Use the model in **Exercise 35** to determine the year in which the number of continuously serving members was 338.

37. *Population Growth* In 2000 India's population reached 1 billion, and it is projected to be 1.4 billion in 2025. (*Source:* U.S. Census Bureau.)

(a) Find values for P_0 and a so that $P(x) = P_0a^{x-2000}$ models the population of India in year x. Round a to five decimal places.

(b) Predict India's population in 2020 to the nearest tenth of a billion.

(c) In what year is India's population expected to reach 1.5 billion?

38. *Population Decline* A midwestern city finds its residents moving to the suburbs. Its population is declining according to the function

$$P(t) = P_0e^{-0.04t},$$

where t is time measured in years and P_0 is the population at time $t = 0$. Assume that $P_0 = 1{,}000{,}000$.

(a) Find the population at time $t = 1$ to the nearest thousand.

(b) How long, to the nearest tenth of a year, will it take for the population to decline to 750,000?

(c) How long, to the nearest tenth of a year, will it take for the population to decline to half the initial number?

39. *Health Care Spending* Out-of-pocket spending in the United States for health care increased between 2008 and 2012. The function

$$f(x) = 7446e^{0.0305x}$$

models average annual expenditures per household, in dollars. In this model, x represents the year, where $x = 0$ corresponds to 2008. (*Source*: U.S. Bureau of Labor Statistics.)

(a) Estimate out-of-pocket household spending on health care in 2012 to the nearest dollar.

(b) In what year did spending reach \$7915 per household?

40. *Recreational Expenditures* Personal consumption expenditures for recreation in billions of dollars in the United States during the years 2000–2013 can be approximated by the function

$$A(t) = 632.37e^{0.0351t},$$

where $t = 0$ corresponds to the year 2000. Based on this model, how much were personal consumption expenditures in 2013 to the nearest billion? (*Source:* U.S. Bureau of Economic Analysis.)

41. *Housing Costs* Average annual per-household spending on housing over the years 2000–2012 is approximated by

$$H = 12{,}744e^{0.0264t},$$

where t is the number of years since 2000. Find H to the nearest dollar for each year. (*Source:* U.S. Bureau of Labor Statistics.)

(a) 2005 (b) 2009 (c) 2012

42. *Evolution of Language* The number of years, n, since two independently evolving languages split off from a common ancestral language is approximated by

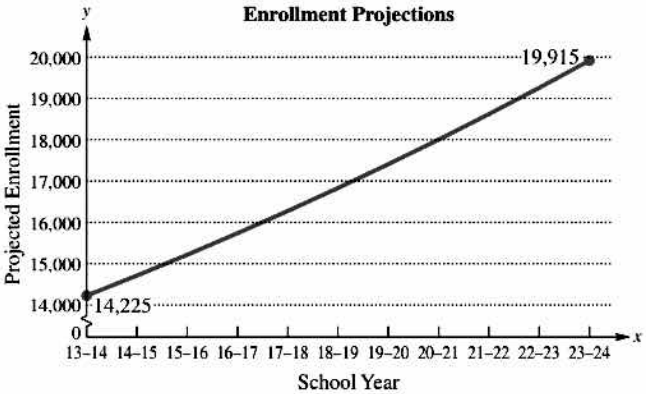
$$n \approx -7600 \log r,$$

where r is the proportion of words from the ancestral language common to both languages. Find each of the following to the nearest year.

(a) Find n if $r = 0.9$. (b) Find n if $r = 0.3$.

(c) How many years have elapsed since the split if half of the words of the ancestral language are common to both languages?

43. *School District Growth* Student enrollment in the Wentzville School District, one of the fastest-growing school districts in the state of Missouri, has projected growth as shown in the graph.



Source: Wentzville School District.

4.6 Applications and Models of Exponential Growth and Decay

- The Exponential Growth or Decay Function
- Growth Function Models
- Decay Function Models

LOOKING AHEAD TO CALCULUS

The exponential growth and decay function formulas are studied in calculus in conjunction with the topic known as **differential equations.**

The Exponential Growth or Decay Function

In many situations in ecology, biology, economics, and the social sciences, a quantity changes at a rate proportional to the amount present. The amount present at time t is a special function of t called an **exponential growth or decay function.**

Exponential Growth or Decay Function

Let y_0 be the amount or number present at time $t = 0$. Then, under certain conditions, the amount y present at any time t is modeled by

$$\boldsymbol{y = y_0 e^{kt}}, \quad \text{where } k \text{ is a constant.}$$

The constant k determines the type of function.

- When $k > 0$, the function describes growth. Examples of exponential growth include compound interest and atmospheric carbon dioxide.
- When $k < 0$, the function describes decay. One example of exponential decay is radioactive decay.

Growth Function Models

The amount of time it takes for a quantity that grows exponentially to become twice its initial amount is its **doubling time.**

EXAMPLE 1 Determining a Function to Model Exponential Growth

Earlier in this chapter, we discussed the growth of atmospheric carbon dioxide over time using a function based on the data from the table. Now we determine such a function from the data.

Year	Carbon Dioxide (ppm)
1990	353
2000	375
2075	590
2175	1090
2275	2000

Source: International Panel on Climate Change (IPCC).

(a) Find an exponential function that gives the amount of carbon dioxide y in year x.

(b) Estimate the year when future levels of carbon dioxide will be double the preindustrial level of 280 ppm.

SOLUTION

(a) The data points exhibit exponential growth, so the equation will take the form

$$y = y_0 e^{kx}.$$

We must find the values of y_0 and k. The data begin with the year 1990, so to simplify our work we let 1990 correspond to $x = 0$, 1991 correspond to $x = 1$, and so on. Here y_0 is the initial amount and $y_0 = 353$ in 1990 when $x = 0$. Thus the equation is

$$y = 353e^{kx}. \quad \text{Let } y_0 = 353.$$

From the last pair of values in the table, we know that in 2275 the carbon dioxide level is expected to be 2000 ppm. The year 2275 corresponds to $2275 - 1990 = 285$. Substitute 2000 for y and 285 for x, and solve for k.

$$y = 353e^{kx} \quad \text{Solve for } k.$$

$$2000 = 353e^{k(285)} \quad \text{Substitute 2000 for } y \text{ and 285 for } x.$$

$$\frac{2000}{353} = e^{285k} \quad \text{Divide by 353.}$$

$$\ln \frac{2000}{353} = \ln e^{285k} \quad \text{Take the natural logarithm on each side.}$$

$$\ln \frac{2000}{353} = 285k \quad \ln e^x = x, \text{ for all } x.$$

$$k = \frac{1}{285} \cdot \ln \frac{2000}{353} \quad \text{Multiply by } \tfrac{1}{285} \text{ and rewrite.}$$

$$k \approx 0.00609 \quad \text{Use a calculator.}$$

A function that models the data is

$$y = 353e^{0.00609x}.$$

(b)

$$y = 353e^{0.00609x} \quad \text{Solve the model from part (a) for the year } x.$$

$$560 = 353e^{0.00609x} \quad \text{To double the level 280, let } y = 2(280) = 560.$$

$$\frac{560}{353} = e^{0.00609x} \quad \text{Divide by 353.}$$

$$\ln \frac{560}{353} = \ln e^{0.00609x} \quad \text{Take the natural logarithm on each side.}$$

$$\ln \frac{560}{353} = 0.00609x \quad \ln e^x = x, \text{ for all } x.$$

$$x = \frac{1}{0.00609} \cdot \ln \frac{560}{353} \quad \text{Multiply by } \tfrac{1}{0.00609} \text{ and rewrite.}$$

$$x \approx 75.8 \quad \text{Use a calculator.}$$

Since $x = 0$ corresponds to 1990, the preindustrial carbon dioxide level will double in the 75th year after 1990, or during 2065, according to this model.

✔ **Now Try Exercise 43.**

EXAMPLE 2 Finding Doubling Time for Money

How long will it take for money in an account that accrues interest at a rate of 3%, compounded continuously, to double?

SOLUTION

$$A = Pe^{rt} \quad \text{Continuous compounding formula}$$

$$2P = Pe^{0.03t} \quad \text{Let } A = 2P \text{ and } r = 0.03.$$

$$2 = e^{0.03t} \quad \text{Divide by } P.$$

$$\ln 2 = \ln e^{0.03t} \quad \text{Take the natural logarithm on each side.}$$

$$\ln 2 = 0.03t \quad \ln e^x = x$$

$$\frac{\ln 2}{0.03} = t \quad \text{Divide by 0.03.}$$

$$23.10 \approx t \quad \text{Use a calculator.}$$

It will take about 23 yr for the amount to double. ✔ **Now Try Exercise 31.**

EXAMPLE 3 Using an Exponential Function to Model Population Growth

According to the U.S. Census Bureau, the world population reached 6 billion people during 1999 and was growing exponentially. By the end of 2010, the population had grown to 6.947 billion. The projected world population (in billions of people) t years after 2010 is given by the function

$$f(t) = 6.947e^{0.00745t}.$$

(a) Based on this model, what will the world population be in 2025?

(b) If this trend continues, approximately when will the world population reach 9 billion?

SOLUTION

(a) Since $t = 0$ represents the year 2010, in 2025, t would be $2025 - 2010 = 15$ yr. We must find $f(t)$ when t is 15.

$$\begin{aligned} f(t) &= 6.947e^{0.00745t} && \text{Given function} \\ f(15) &= 6.947e^{0.00745(15)} && \text{Let } t = 15. \\ f(15) &\approx 7.768 && \text{Use a calculator.} \end{aligned}$$

The population will be 7.768 billion at the end of 2025.

(b)

$$\begin{aligned} f(t) &= 6.947e^{0.00745t} && \text{Given function} \\ 9 &= 6.947e^{0.00745t} && \text{Let } f(t) = 9. \\ \frac{9}{6.947} &= e^{0.00745t} && \text{Divide by 6.947.} \\ \ln \frac{9}{6.947} &= \ln e^{0.00745t} && \text{Take the natural logarithm on each side.} \\ \ln \frac{9}{6.947} &= 0.00745t && \ln e^x = x, \text{ for all } x. \\ t &= \frac{\ln \frac{9}{6.947}}{0.00745} && \text{Divide by 0.00745 and rewrite.} \\ t &\approx 34.8 && \text{Use a calculator.} \end{aligned}$$

Thus, 34.8 yr after 2010, during the year 2044, world population will reach 9 billion.

✔ **Now Try Exercise 39.**

Decay Function Models **Half-life** is the amount of time it takes for a quantity that decays exponentially to become half its initial amount.

NOTE In **Example 4** on the next page, the initial amount of substance is given as 600 g. Because half-life is constant over the lifetime of a decaying quantity, starting with any initial amount, y_0, and substituting $\frac{1}{2}y_0$ for y in $y = y_0e^{kt}$ would allow the common factor y_0 to be divided out. The rest of the work would be the same.

EXAMPLE 4 Determining an Exponential Function to Model Radioactive Decay

Suppose 600 g of a radioactive substance are present initially and 3 yr later only 300 g remain.

(a) Determine an exponential function that models this decay.

(b) How much of the substance will be present after 6 yr?

SOLUTION

(a) We use the given values to find k in the exponential equation

$$y = y_0 e^{kt}.$$

Because the initial amount is 600 g, $y_0 = 600$, which gives $y = 600e^{kt}$. The initial amount (600 g) decays to half that amount (300 g) in 3 yr, so its half-life is 3 yr. Now we solve this exponential equation for k.

$$\begin{aligned}
y &= 600e^{kt} && \text{Let } y_0 = 600. \\
300 &= 600e^{3k} && \text{Let } y = 300 \text{ and } t = 3. \\
0.5 &= e^{3k} && \text{Divide by 600.} \\
\ln 0.5 &= \ln e^{3k} && \text{Take the natural logarithm on each side.} \\
\ln 0.5 &= 3k && \ln e^x = x, \text{ for all } x. \\
\frac{\ln 0.5}{3} &= k && \text{Divide by 3.} \\
k &\approx -0.231 && \text{Use a calculator.}
\end{aligned}$$

A function that models the situation is

$$y = 600e^{-0.231t}.$$

(b) To find the amount present after 6 yr, let $t = 6$.

$$\begin{aligned}
y &= 600e^{-0.231t} && \text{Model from part (a)} \\
y &= 600e^{-0.231(6)} && \text{Let } t = 6. \\
y &= 600e^{-1.386} && \text{Multiply.} \\
y &\approx 150 && \text{Use a calculator.}
\end{aligned}$$

After 6 yr, 150 g of the substance will remain. ✔ **Now Try Exercise 19.**

EXAMPLE 5 Solving a Carbon Dating Problem

Carbon-14, also known as radiocarbon, is a radioactive form of carbon that is found in all living plants and animals. After a plant or animal dies, the radiocarbon disintegrates. Scientists can determine the age of the remains by comparing the amount of radiocarbon with the amount present in living plants and animals. This technique is called **carbon dating.** The amount of radiocarbon present after t years is given by

$$y = y_0 e^{-0.0001216t},$$

where y_0 is the amount present in living plants and animals.

(a) Find the half-life of carbon-14.

(b) Charcoal from an ancient fire pit on Java contained $\frac{1}{4}$ the carbon-14 of a living sample of the same size. Estimate the age of the charcoal.

SOLUTION

(a) If y_0 is the amount of radiocarbon present in a living thing, then $\frac{1}{2}y_0$ is half this initial amount. We substitute and solve the given equation for t.

$$y = y_0e^{-0.0001216t} \quad \text{Given equation}$$

$$\frac{1}{2}y_0 = y_0e^{-0.0001216t} \quad \text{Let } y = \tfrac{1}{2}y_0.$$

$$\frac{1}{2} = e^{-0.0001216t} \quad \text{Divide by } y_0.$$

$$\ln\frac{1}{2} = \ln e^{-0.0001216t} \quad \text{Take the natural logarithm on each side.}$$

$$\ln\frac{1}{2} = -0.0001216t \quad \ln e^x = x, \text{ for all } x.$$

$$\frac{\ln\frac{1}{2}}{-0.0001216} = t \quad \text{Divide by } -0.0001216.$$

$$5700 \approx t \quad \text{Use a calculator.}$$

The half-life is 5700 yr.

(b) Solve again for t, this time letting the amount $y = \frac{1}{4}y_0$.

$$y = y_0e^{-0.0001216t} \quad \text{Given equation}$$

$$\frac{1}{4}y_0 = y_0e^{-0.0001216t} \quad \text{Let } y = \tfrac{1}{4}y_0.$$

$$\frac{1}{4} = e^{-0.0001216t} \quad \text{Divide by } y_0.$$

$$\ln\frac{1}{4} = \ln e^{-0.0001216t} \quad \text{Take the natural logarithm on each side.}$$

$$\frac{\ln\frac{1}{4}}{-0.0001216} = t \quad \ln e^x = x; \text{ Divide by } -0.0001216.$$

$$t \approx 11{,}400 \quad \text{Use a calculator.}$$

The charcoal is 11,400 yr old. ✔ **Now Try Exercise 23.**

EXAMPLE 6 Modeling Newton's Law of Cooling

Newton's law of cooling says that the rate at which a body cools is proportional to the difference in temperature between the body and the environment around it. The temperature $f(t)$ of the body at time t in appropriate units after being introduced into an environment having constant temperature T_0 is

$$f(t) = T_0 + Ce^{-kt}, \quad \text{where } C \text{ and } k \text{ are constants.}$$

A pot of coffee with a temperature of 100°C is set down in a room with a temperature of 20°C. The coffee cools to 60°C after 1 hr.

(a) Write an equation to model the data.

(b) Find the temperature after half an hour.

(c) How long will it take for the coffee to cool to 50°C?

SOLUTION

(a) We must find values for C and k in the given formula. As given, when $t = 0$, $T_0 = 20$, and the temperature of the coffee is $f(0) = 100$.

$$\begin{aligned}
f(t) &= T_0 + Ce^{-kt} && \text{Given function}\\
100 &= 20 + Ce^{-0k} && \text{Let } t = 0,\ f(0) = 100, \text{ and } T_0 = 20.\\
100 &= 20 + C && e^0 = 1\\
80 &= C && \text{Subtract 20.}
\end{aligned}$$

The following function models the data.

$$f(t) = 20 + 80e^{-kt} \qquad \text{Let } T_0 = 20 \text{ and } C = 80.$$

The coffee cools to 60°C after 1 hr, so when $t = 1$, $f(1) = 60$.

$$\begin{aligned}
f(t) &= 20 + 80e^{-kt} && \text{Above function with } T_0 = 20 \text{ and } C = 80\\
60 &= 20 + 80e^{-1k} && \text{Let } t = 1 \text{ and } f(1) = 60.\\
40 &= 80e^{-k} && \text{Subtract 20.}\\
\frac{1}{2} &= e^{-k} && \text{Divide by 80.}\\
\ln\frac{1}{2} &= \ln e^{-k} && \text{Take the natural logarithm on each side.}\\
\ln\frac{1}{2} &= -k && \ln e^x = x, \text{ for all } x.\\
k &\approx 0.693 && \text{Multiply by } -1, \text{ rewrite, and use a calculator.}
\end{aligned}$$

Thus, the model is $f(t) = 20 + 80e^{-0.693t}$.

(b) To find the temperature after $\frac{1}{2}$ hr, let $t = \frac{1}{2}$ in the model from part (a).

$$\begin{aligned}
f(t) &= 20 + 80e^{-0.693t} && \text{Model from part (a)}\\
f\left(\frac{1}{2}\right) &= 20 + 80e^{(-0.693)(1/2)} && \text{Let } t = \tfrac{1}{2}.\\
f\left(\frac{1}{2}\right) &\approx 76.6°\text{C} && \text{Use a calculator.}
\end{aligned}$$

(c) To find how long it will take for the coffee to cool to 50°C, let $f(t) = 50$.

$$\begin{aligned}
f(t) &= 20 + 80e^{-0.693t} && \text{Model from part (a)}\\
50 &= 20 + 80e^{-0.693t} && \text{Let } f(t) = 50.\\
30 &= 80e^{-0.693t} && \text{Subtract 20.}\\
\frac{3}{8} &= e^{-0.693t} && \text{Divide by 80.}\\
\ln\frac{3}{8} &= \ln e^{-0.693t} && \text{Take the natural logarithm on each side.}\\
\ln\frac{3}{8} &= -0.693t && \ln e^x = x, \text{ for all } x.\\
t &= \frac{\ln\frac{3}{8}}{-0.693} && \text{Divide by } -0.693 \text{ and rewrite.}\\
t &\approx 1.415 \text{ hr}, \quad \text{or} \quad \text{about 1 hr, 25 min}
\end{aligned}$$

✔ **Now Try Exercise 27.**