NOTE The values in **Example 7** are approximations of logarithms, so the final digit may differ from the actual 4-decimal-place approximation after properties of logarithms are applied.

Recall that for inverse functions f and g, $(f \circ g)(x) = (g \circ f)(x) = x$. We can use this property with exponential and logarithmic functions to state two more properties. If $f(x) = a^x$ and $g(x) = \log_a x$, then

$$(f \circ g)(x) = a^{\log_a x} = x \quad \text{and} \quad (g \circ f)(x) = \log_a(a^x) = x.$$

Theorem on Inverses

For $a > 0$, $a \neq 1$, the following properties hold.

$$a^{\log_a x} = x \text{ (for } x > 0) \quad \text{and} \quad \log_a a^x = x$$

Examples: $7^{\log_7 10} = 10$, $\log_5 5^3 = 3$, and $\log_r r^{k+1} = k + 1$

The second statement in the theorem will be useful when we solve logarithmic and exponential equations.

4.3 Exercises

CONCEPT PREVIEW *Match the logarithm in Column I with its value in Column II. Remember that $\log_a x$ is the exponent to which a must be raised in order to obtain x.*

	I	II
1.	**(a)** $\log_2 16$	**A.** 0
	(b) $\log_3 1$	**B.** $\frac{1}{2}$
	(c) $\log_{10} 0.1$	**C.** 4
	(d) $\log_2 \sqrt{2}$	**D.** -3
	(e) $\log_e \frac{1}{e^2}$	**E.** -1
	(f) $\log_{1/2} 8$	**F.** -2

	I	II
2.	**(a)** $\log_3 81$	**A.** -2
	(b) $\log_3 \frac{1}{3}$	**B.** -1
	(c) $\log_{10} 0.01$	**C.** 0
	(d) $\log_6 \sqrt{6}$	**D.** $\frac{1}{2}$
	(e) $\log_e 1$	**E.** $\frac{9}{2}$
	(f) $\log_3 27^{3/2}$	**F.** 4

CONCEPT PREVIEW *Write each equivalent form.*

3. Write $\log_2 8 = 3$ in exponential form.

4. Write $10^3 = 1000$ in logarithmic form.

CONCEPT PREVIEW *Solve each logarithmic equation.*

5. $\log_x \frac{16}{81} = 2$

6. $\log_{36} \sqrt[3]{6} = x$

CONCEPT PREVIEW *Sketch the graph of each function. Give the domain and range.*

7. $f(x) = \log_5 x$

8. $g(x) = \log_{1/5} x$

CONCEPT PREVIEW *Use the properties of logarithms to rewrite each expression. Assume all variables represent positive real numbers.*

9. $\log_{10} \frac{2x}{7}$ **10.** $3 \log_4 x - 5 \log_4 y$

If the statement is in exponential form, write it in an equivalent logarithmic form. If the statement is in logarithmic form, write it in exponential form. ***See Example 1.***

11. $3^4 = 81$ **12.** $2^5 = 32$ **13.** $\left(\frac{2}{3}\right)^{-3} = \frac{27}{8}$ **14.** $10^{-4} = 0.0001$

15. $\log_6 36 = 2$ **16.** $\log_5 5 = 1$ **17.** $\log_{\sqrt{3}} 81 = 8$ **18.** $\log_4 \frac{1}{64} = -3$

Solve each equation. ***See Example 2.***

19. $x = \log_5 \frac{1}{625}$ **20.** $x = \log_3 \frac{1}{81}$ **21.** $\log_x \frac{1}{32} = 5$ **22.** $\log_x \frac{27}{64} = 3$

23. $x = \log_8 \sqrt[4]{8}$ **24.** $x = \log_7 \sqrt[5]{7}$ **25.** $x = 3^{\log_3 8}$ **26.** $x = 12^{\log_{12} 5}$

27. $x = 2^{\log_2 9}$ **28.** $x = 8^{\log_8 11}$ **29.** $\log_x 25 = -2$ **30.** $\log_x 16 = -2$

31. $\log_4 x = 3$ **32.** $\log_2 x = 3$ **33.** $x = \log_4 \sqrt[3]{16}$ **34.** $x = \log_5 \sqrt[4]{25}$

35. $\log_9 x = \frac{5}{2}$ **36.** $\log_4 x = \frac{7}{2}$

37. $\log_{1/2} (x + 3) = -4$ **38.** $\log_{1/3} (x + 6) = -2$

39. $\log_{(x+3)} 6 = 1$ **40.** $\log_{(x-4)} 19 = 1$

41. $3x - 15 = \log_x 1 \quad (x > 0, x \neq 1)$ **42.** $4x - 24 = \log_x 1 \quad (x > 0, x \neq 1)$

Graph each function. Give the domain and range. ***See Example 4.***

43. $f(x) = (\log_2 x) + 3$ **44.** $f(x) = \log_2 (x + 3)$ **45.** $f(x) = |\log_2 (x + 3)|$

Graph each function. Give the domain and range. ***See Example 4.***

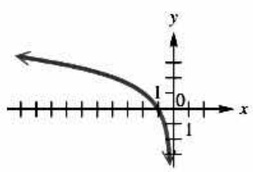
46. $f(x) = (\log_{1/2} x) - 2$ **47.** $f(x) = \log_{1/2} (x - 2)$ **48.** $f(x) = |\log_{1/2} (x - 2)|$

Concept Check In Exercises 49–54, match the function with its graph from choices A–F.

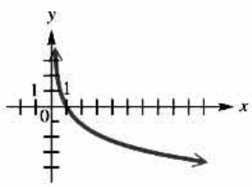
49. $f(x) = \log_2 x$ **50.** $f(x) = \log_2 2x$ **51.** $f(x) = \log_2 \frac{1}{x}$

52. $f(x) = \log_2 \left(\frac{1}{2}x\right)$ **53.** $f(x) = \log_2 (x - 1)$ **54.** $f(x) = \log_2 (-x)$

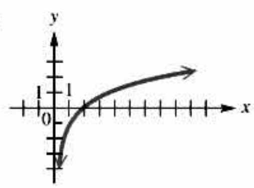
A.



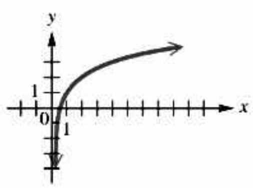
B.



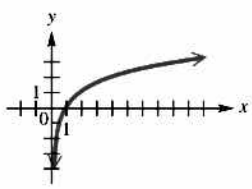
C.



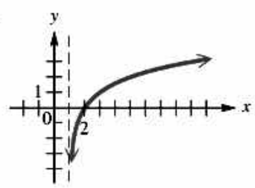
D.



E.



F.



Graph each function. ***See Examples 3 and 4.***

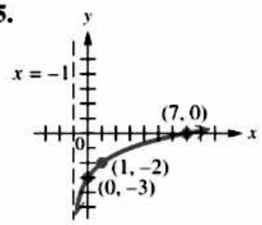
55. $f(x) = \log_5 x$ **56.** $f(x) = \log_{10} x$ **57.** $f(x) = \log_5 (x + 1)$

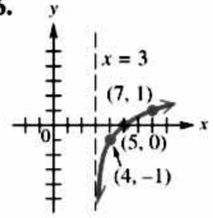
58. $f(x) = \log_6 (x - 2)$ **59.** $f(x) = \log_{1/2} (1 - x)$ **60.** $f(x) = \log_{1/3} (3 - x)$

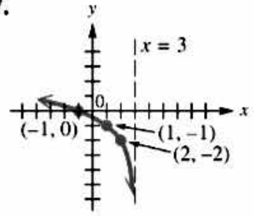
61. $f(x) = \log_3 (x - 1) + 2$ **62.** $f(x) = \log_2 (x + 2) - 3$ **63.** $f(x) = \log_{1/2} (x + 3) - 2$

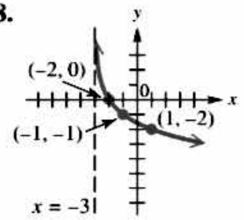
64. *Concept Check* To graph the function $f(x) = -\log_5 (x - 7) - 4$, reflect the graph of $y = \log_5 x$ across the ____-axis, then shift the graph ____ units to the right and ____ units down.

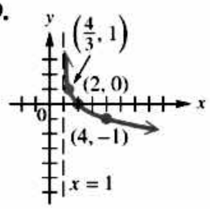
Connecting Graphs with Equations Write an equation for the graph given. Each represents a logarithmic function f with base 2 or 3, translated and/or reflected. ***See the Note following Example 4.***

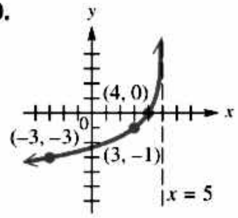
65.


66.


67.


68.


69.


70.


Use the properties of logarithms to rewrite each expression. Simplify the result if possible. Assume all variables represent positive real numbers. ***See Example 5.***

71. $\log_2 \frac{6x}{y}$ **72.** $\log_3 \frac{4p}{q}$ **73.** $\log_5 \frac{5\sqrt{7}}{3}$

74. $\log_2 \frac{2\sqrt{3}}{5}$ **75.** $\log_4 (2x + 5y)$ **76.** $\log_6 (7m + 3q)$

77. $\log_2 \sqrt{\frac{5r^3}{z^5}}$ **78.** $\log_3 \sqrt[3]{\frac{m^5 n^4}{t^2}}$ **79.** $\log_2 \frac{ab}{cd}$

80. $\log_2 \frac{xy}{tqr}$ **81.** $\log_3 \frac{\sqrt{x} \cdot \sqrt[3]{y}}{w^2 \sqrt{z}}$ **82.** $\log_4 \frac{\sqrt[3]{a} \cdot \sqrt[4]{b}}{\sqrt{c} \cdot \sqrt[3]{d^2}}$

Write each expression as a single logarithm with coefficient 1. *Assume all variables represent positive real numbers, with a* $\neq$ 1 *and b* $\neq$ 1. ***See Example 6.***

83. $\log_a x + \log_a y - \log_a m$ **84.** $\log_b k + \log_b m - \log_b a$

85. $\log_a m - \log_a n - \log_a t$ **86.** $\log_b p - \log_b q - \log_b r$

87. $\frac{1}{3} \log_b x^4 y^5 - \frac{3}{4} \log_b x^2 y$ **88.** $\frac{1}{2} \log_a p^3 q^4 - \frac{2}{3} \log_a p^4 q^3$

89. $2 \log_a (z + 1) + \log_a (3z + 2)$ **90.** $5 \log_a (z + 7) + \log_a (2z + 9)$

91. $-\frac{2}{3} \log_5 5m^2 + \frac{1}{2} \log_5 25m^2$ **92.** $-\frac{3}{4} \log_3 16p^4 - \frac{2}{3} \log_3 8p^3$

Given that $\log_{10} 2 \approx 0.3010$ *and* $\log_{10} 3 \approx 0.4771$, *find each logarithm without using a calculator.* ***See Example 7.***

93. $\log_{10} 6$ **94.** $\log_{10} 12$ **95.** $\log_{10} \frac{3}{2}$ **96.** $\log_{10} \frac{2}{9}$

97. $\log_{10} \frac{9}{4}$ **98.** $\log_{10} \frac{20}{27}$ **99.** $\log_{10} \sqrt{30}$ **100.** $\log_{10} 36^{1/3}$

Solve each problem.

101. ***(Modeling) Interest Rates of Treasury Securities*** The table gives interest rates for various U.S. Treasury Securities on January 2, 2015.

(a) Make a scatter diagram of the data.

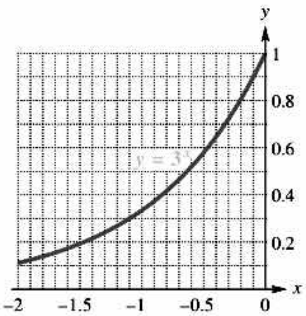
(b) Which type of function will model this data best: linear, exponential, or logarithmic?

Time	Yield
3-month	0.02%
6-month	0.10%
2-year	0.66%
5-year	1.61%
10-year	2.11%
30-year	2.60%

Source: www.federalreserve.gov

102. ***Concept Check*** Use the graph to estimate each logarithm.

(a) $\log_3 0.3$ **(b)** $\log_3 0.8$



103. ***Concept Check*** Suppose $f(x) = \log_a x$ and $f(3) = 2$. Determine each function value.

(a) $f\left(\frac{1}{9}\right)$ **(b)** $f(27)$ **(c)** $f(9)$ **(d)** $f\left(\frac{\sqrt{3}}{3}\right)$

104. Use properties of logarithms to evaluate each expression.

(a) $100^{\log_{10} 3}$ **(b)** $\log_{10}(0.01)^3$ **(c)** $\log_{10}(0.0001)^5$ **(d)** $1000^{\log_{10} 5}$

105. Using the compound interest formula $A = P\left(1 + \frac{r}{n}\right)^{tn}$, show that the amount of time required for a deposit to double is

$$\frac{1}{\log_2\left(1 + \frac{r}{n}\right)^n}.$$

106. ***Concept Check*** If $(5, 4)$ is on the graph of the logarithmic function with base a, which of the following statements is true:

$$5 = \log_a 4 \quad \text{or} \quad 4 = \log_a 5?$$

Use a graphing calculator to find the solution set of each equation. Give solutions to the nearest hundredth.

107. $\log_{10} x = x - 2$ **108.** $2^{-x} = \log_{10} x$

109. Prove the quotient property of logarithms: $\log_a \frac{x}{y} = \log_a x - \log_a y$.

110. Prove the power property of logarithms: $\log_a x^r = r \log_a x$.

In calculus, the following can be shown.

$$e^x = 1 + x + \frac{x^2}{2 \cdot 1} + \frac{x^3}{3 \cdot 2 \cdot 1} + \frac{x^4}{4 \cdot 3 \cdot 2 \cdot 1} + \frac{x^5}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} + \cdots$$

Using more terms, one can obtain a more accurate approximation for e^x.

125. Use the terms shown, and replace x with 1 to approximate $e^1 = e$ to three decimal places. Check the result with a calculator.

126. Use the terms shown, and replace x with -0.05 to approximate $e^{-0.05}$ to four decimal places. Check the result with a calculator.

Relating Concepts

For individual or collaborative investigation *(Exercises 127–132)*

Consider $f(x) = a^x$, where $a > 1$. ***Work these exercises in order.***

127. Is f a one-to-one function? If so, what kind of related function exists for f?

128. If f has an inverse function f^{-1}, sketch f and f^{-1} on the same set of axes.

129. If f^{-1} exists, find an equation for $y = f^{-1}(x)$. (You need not solve for y.)

130. If $a = 10$, what is the equation for $y = f^{-1}(x)$? (You need not solve for y.)

131. If $a = e$, what is the equation for $y = f^{-1}(x)$? (You need not solve for y.)

132. If the point (p, q) is on the graph of f, then the point ______ is on the graph of f^{-1}.

4.3 Logarithmic Functions

- Logarithms
- Logarithmic Equations
- Logarithmic Functions
- Properties of Logarithms

Logarithms The previous section dealt with exponential functions of the form $y = a^x$ for all positive values of a, where $a \neq 1$. The horizontal line test shows that exponential functions are one-to-one and thus have inverse functions. The equation defining the inverse of a function is found by interchanging x and y in the equation that defines the function. Starting with $y = a^x$ and interchanging x and y yields

$$x = a^y.$$

Here y is the exponent to which a must be raised in order to obtain x. We call this exponent a **logarithm,** symbolized by the abbreviation **"log."** The expression $\mathbf{\log_a x}$ represents the logarithm in this discussion. The number a is the **base** of the logarithm, and x is the **argument** of the expression. It is read **"logarithm with base a of x,"** or **"logarithm of x with base a,"** or **"base a logarithm of x."**

Logarithm

For all real numbers y and all positive numbers a and x, where $a \neq 1$,

$$\mathbf{y = \log_a x} \quad \text{is equivalent to} \quad \mathbf{x = a^y}.$$

The expression $\log_a x$ represents the exponent to which the base a must be raised in order to obtain x.

EXAMPLE 1 Writing Equivalent Logarithmic and Exponential Forms

The table shows several pairs of equivalent statements, written in both logarithmic and exponential forms.

SOLUTION

Logarithmic Form	Exponential Form
$\log_2 8 = 3$	$2^3 = 8$
$\log_{1/2} 16 = -4$	$\left(\frac{1}{2}\right)^{-4} = 16$
$\log_{10} 100{,}000 = 5$	$10^5 = 100{,}000$
$\log_3 \frac{1}{81} = -4$	$3^{-4} = \frac{1}{81}$
$\log_5 5 = 1$	$5^1 = 5$
$\log_{3/4} 1 = 0$	$\left(\frac{3}{4}\right)^0 = 1$

To remember the relationships among a, x, and y in the two equivalent forms $y = \log_a x$ and $x = a^y$, refer to these diagrams.

A logarithm is an exponent.

Exponent
↓
Logarithmic form: $y = \log_a x$
↑
Base

Exponent
↓
Exponential form: $a^y = x$
↑
Base

✔ **Now Try Exercises 11, 13, 15, and 17.**

Logarithmic Equations

The definition of logarithm can be used to solve a **logarithmic equation,** which is an equation with a logarithm in at least one term.

EXAMPLE 2 Solving Logarithmic Equations

Solve each equation.

(a) $\log_x \dfrac{8}{27} = 3$ **(b)** $\log_4 x = \dfrac{5}{2}$ **(c)** $\log_{49} \sqrt[3]{7} = x$

SOLUTION Many logarithmic equations can be solved by first writing the equation in exponential form.

(a)

$$\log_x \frac{8}{27} = 3$$

$$x^3 = \frac{8}{27} \quad \text{Write in exponential form.}$$

$$x^3 = \left(\frac{2}{3}\right)^3 \quad \frac{8}{27} = \left(\frac{2}{3}\right)^3$$

$$x = \frac{2}{3} \quad \text{Take cube roots.}$$

CHECK

$$\log_x \frac{8}{27} = 3 \quad \text{Original equation}$$

$$\log_{2/3} \frac{8}{27} \stackrel{?}{=} 3 \quad \text{Let } x = \tfrac{2}{3}.$$

$$\left(\frac{2}{3}\right)^3 \stackrel{?}{=} \frac{8}{27} \quad \text{Write in exponential form.}$$

$$\frac{8}{27} = \frac{8}{27} \ ✓ \quad \text{True}$$

The solution set is $\left\{\frac{2}{3}\right\}$.

(b)

$$\log_4 x = \frac{5}{2}$$

$4^{5/2} = x$ Write in exponential form.

$(4^{1/2})^5 = x$ $a^{mn} = (a^m)^n$

$2^5 = x$ $4^{1/2} = (2^2)^{1/2} = 2$

$32 = x$ Apply the exponent.

CHECK $\log_4 32 \stackrel{?}{=} \frac{5}{2}$ Let $x = 32$.

$4^{5/2} \stackrel{?}{=} 32$

$2^5 \stackrel{?}{=} 32$ $4^{5/2} = \left(\sqrt{4}\right)^5 = 2^5$

$32 = 32$ ✓ True

The solution set is $\{32\}$.

(c) $\log_{49} \sqrt[3]{7} = x$

$49^x = \sqrt[3]{7}$ Write in exponential form.

$(7^2)^x = 7^{1/3}$ Write with the same base.

$7^{2x} = 7^{1/3}$ Power rule for exponents

$2x = \frac{1}{3}$ Set exponents equal.

$x = \frac{1}{6}$ Divide by 2.

A check shows that the solution set is $\left\{\frac{1}{6}\right\}$.

✔ **Now Try Exercises 19, 29, and 35.**

Logarithmic Functions We define the logarithmic function with base a.

Logarithmic Function

If $a > 0$, $a \neq 1$, and $x > 0$, then the **logarithmic function with base a** is

$$f(x) = \log_a x.$$

Exponential and logarithmic functions are inverses of each other. To show this, we use the three steps for finding the inverse of a function.

	$f(x) = 2^x$	Exponential function with base 2
	$y = 2^x$	Let $y = f(x)$.
Step 1	$x = 2^y$	Interchange x and y.
Step 2	$y = \log_2 x$	Solve for y by writing in equivalent logarithmic form.
Step 3	$f^{-1}(x) = \log_2 x$	Replace y with $f^{-1}(x)$.

The graph of $f(x) = 2^x$ has the x-axis as horizontal asymptote and is shown in red in **Figure 25.** Its inverse, $f^{-1}(x) = \log_2 x$, has the y-axis as vertical asymptote and is shown in blue. The graphs are reflections of each other across the line $y = x$. As a result, their domains and ranges are interchanged.

x	$f(x) = 2^x$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4

x	$f^{-1}(x) = \log_2 x$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2

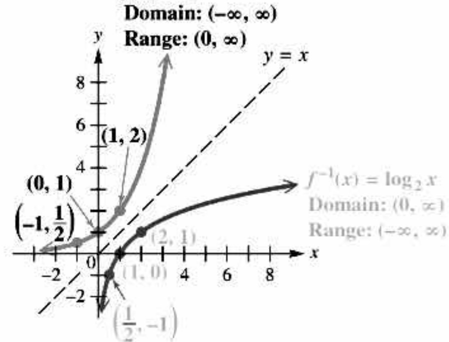


Figure 25

The domain of an exponential function is the set of all real numbers, so the range of a logarithmic function also will be the set of all real numbers. In the same way, both the range of an exponential function and the domain of a logarithmic function are the set of all positive real numbers.

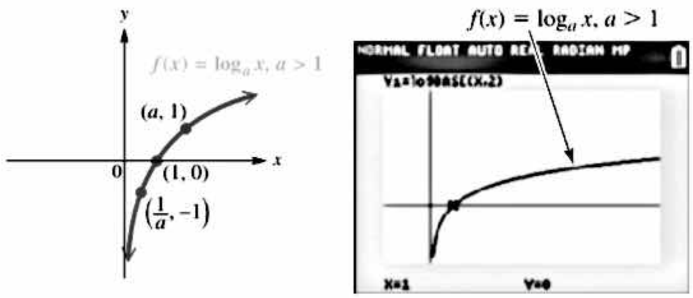
Thus, logarithms can be found for positive numbers only.

Logarithmic Function $f(x) = \log_a x$

Domain: $(0, \infty)$ Range: $(-\infty, \infty)$

For $f(x) = \log_2 x$:

x	$f(x)$
$\frac{1}{4}$	-2
$\frac{1}{2}$	-1
1	0
2	1
4	2
8	3



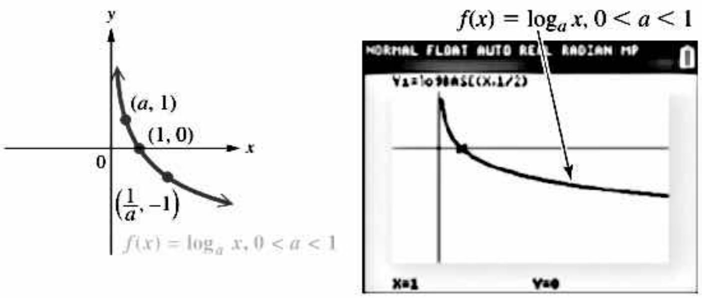
This is the general behavior seen on a calculator graph for **any base *a*, for *a* > 1.**

Figure 26

- $\boldsymbol{f(x) = \log_a x}$**, for $\boldsymbol{a > 1}$,** is increasing and continuous on its entire domain, $(0, \infty)$.
- The y-axis is a vertical asymptote as $x \to 0$ from the right.
- The graph passes through the points $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, and $(a, 1)$.

For $f(x) = \log_{1/2} x$:

x	$f(x)$
$\frac{1}{4}$	2
$\frac{1}{2}$	1
1	0
2	-1
4	-2
8	-3



This is the general behavior seen on a calculator graph for **any base *a*, for 0 < *a* < 1.**

Figure 27

- $\boldsymbol{f(x) = \log_a x}$**, for $\boldsymbol{0 < a < 1}$,** is decreasing and continuous on its entire domain, $(0, \infty)$.
- The y-axis is a vertical asymptote as $x \to 0$ from the right.
- The graph passes through the points $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, and $(a, 1)$.

Calculator graphs of logarithmic functions sometimes do not give an accurate picture of the behavior of the graphs near the vertical asymptotes. While it may seem as if the graph has an endpoint, this is not the case. The resolution of the calculator screen is not precise enough to indicate that the graph approaches the vertical asymptote as the value of x gets closer to it. Do not draw incorrect conclusions just because the calculator does not show this behavior. ■

The graphs in **Figures 26 and 27** and the information with them suggest the following generalizations about the graphs of logarithmic functions of the form $f(x) = \log_a x$.

Characteristics of the Graph of $f(x) = \log_a x$

1. The points $\left(\frac{1}{a}, -1\right)$, $(1, 0)$, and $(a, 1)$ are on the graph.
2. If $a > 1$, then f is an increasing function.
   If $0 < a < 1$, then f is a decreasing function.
3. The y-axis is a vertical asymptote.
4. The domain is $(0, \infty)$, and the range is $(-\infty, \infty)$.

EXAMPLE 3 Graphing Logarithmic Functions

Graph each function.

(a) $f(x) = \log_{1/2} x$ **(b)** $f(x) = \log_3 x$

SOLUTION

(a) One approach is to first graph $y = \left(\frac{1}{2}\right)^x$, which defines the inverse function of f, by plotting points. Some ordered pairs are given in the table with the graph shown in red in **Figure 28.**

The graph of $f(x) = \log_{1/2} x$ is the reflection of the graph of $y = \left(\frac{1}{2}\right)^x$ across the line $y = x$. The ordered pairs for $y = \log_{1/2} x$ are found by interchanging the x- and y-values in the ordered pairs for $y = \left(\frac{1}{2}\right)^x$. See the graph in blue in **Figure 28.**

x	$y = \left(\frac{1}{2}\right)^x$
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$
4	$\frac{1}{16}$

x	$f(x) = \log_{1/2} x$
4	-2
2	-1
1	0
$\frac{1}{2}$	1
$\frac{1}{4}$	2
$\frac{1}{16}$	4

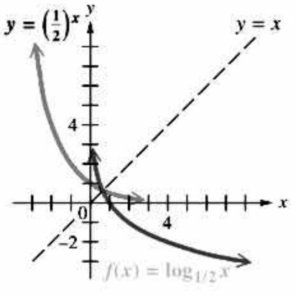


Figure 28

x	$f(x) = \log_3 x$
$\frac{1}{3}$	-1
1	0
3	1
9	2

Think: $x = 3^y$

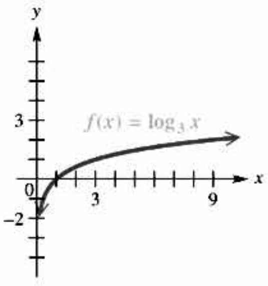


Figure 29

(b) Another way to graph a logarithmic function is to write $f(x) = y = \log_3 x$ in exponential form as $x = 3^y$, and then select y-values and calculate corresponding x-values. Several selected ordered pairs are shown in the table for the graph in **Figure 29.**

✔ **Now Try Exercise 55.**

CAUTION If we write a logarithmic function in exponential form in order to graph it, as in **Example 3(b),** we start *first* with y-values to calculate corresponding x-values. ***Be careful to write the values in the ordered pairs in the correct order.***

More general logarithmic functions can be obtained by forming the composition of $f(x) = \log_a x$ with a function $g(x)$. For example, if $f(x) = \log_2 x$ and $g(x) = x - 1$, then

$$(f \circ g)(x) = f(g(x)) = \log_2(x - 1).$$

The next example shows how to graph such functions.

EXAMPLE 4 Graphing Translated Logarithmic Functions

Graph each function. Give the domain and range.

(a) $f(x) = \log_2(x - 1)$ **(b)** $f(x) = (\log_3 x) - 1$

(c) $f(x) = \log_4(x + 2) + 1$

SOLUTION

(a) The graph of $f(x) = \log_2(x - 1)$ is the graph of $g(x) = \log_2 x$ translated 1 unit to the right. The vertical asymptote has equation $x = 1$. Because logarithms can be found only for positive numbers, we solve $x - 1 > 0$ to find the domain, $(1, \infty)$. To determine ordered pairs to plot, use the equivalent exponential form of the equation $y = \log_2(x - 1)$.

$$y = \log_2(x - 1)$$
$$x - 1 = 2^y \quad \text{Write in exponential form.}$$
$$x = 2^y + 1 \quad \text{Add 1.}$$

We first choose values for y and then calculate each of the corresponding x-values. The range is $(-\infty, \infty)$. See **Figure 30.**

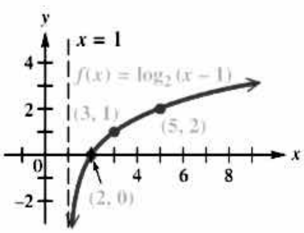


Figure 30

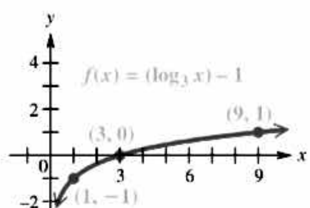


Figure 31

(b) The function $f(x) = (\log_3 x) - 1$ has the same graph as $g(x) = \log_3 x$ translated 1 unit down. We find ordered pairs to plot by writing the equation $y = (\log_3 x) - 1$ in exponential form.

$$y = (\log_3 x) - 1$$
$$y + 1 = \log_3 x \quad \text{Add 1.}$$
$$x = 3^{y+1} \quad \text{Write in exponential form.}$$

Again, choose y-values and calculate the corresponding x-values. The graph is shown in **Figure 31.** The domain is $(0, \infty)$, and the range is $(-\infty, \infty)$.

(c) The graph of $f(x) = \log_4(x + 2) + 1$ is obtained by shifting the graph of $y = \log_4 x$ to the left 2 units and up 1 unit. The domain is found by solving

$$x + 2 > 0,$$

which yields $(-2, \infty)$. The vertical asymptote has been shifted to the left 2 units as well, and it has equation $x = -2$. The range is unaffected by the vertical shift and remains $(-\infty, \infty)$. See **Figure 32.**

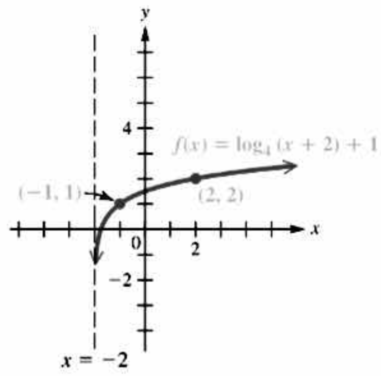


Figure 32

✔ **Now Try Exercises 43, 47, and 61.**

NOTE If we are given a graph such as the one in **Figure 31** and asked to find its equation, we could reason as follows: The point $(1, 0)$ on the basic logarithmic graph has been shifted *down* 1 unit, and the point $(3, 0)$ on the given graph is 1 unit lower than $(3, 1)$, which is on the graph of $y = \log_3 x$. Thus, the equation will be

$$y = (\log_3 x) - 1.$$

Properties of Logarithms The properties of logarithms enable us to change the form of logarithmic statements so that products can be converted to sums, quotients can be converted to differences, and powers can be converted to products.

Properties of Logarithms

For $x > 0$, $y > 0$, $a > 0$, $a \neq 1$, and any real number r, the following properties hold.

Property	Description
Product Property $\log_a xy = \log_a x + \log_a y$	The logarithm of the product of two numbers is equal to the sum of the logarithms of the numbers.
Quotient Property $\log_a \frac{x}{y} = \log_a x - \log_a y$	The logarithm of the quotient of two numbers is equal to the difference between the logarithms of the numbers.
Power Property $\log_a x^r = r \log_a x$	The logarithm of a number raised to a power is equal to the exponent multiplied by the logarithm of the number.
Logarithm of 1 $\log_a 1 = 0$	The base a logarithm of 1 is 0.
Base a Logarithm of a $\log_a a = 1$	The base a logarithm of a is 1.

Proof To prove the product property, let $m = \log_a x$ and $n = \log_a y$.

$\log_a x = m$ means $a^m = x$
$\log_a y = n$ means $a^n = y$ Write in exponential form.

LOOKING AHEAD TO CALCULUS

A technique called **logarithmic differentiation,** which uses the properties of logarithms, can often be used to differentiate complicated functions.

Now consider the product xy.

$$xy = a^m \cdot a^n \qquad x = a^m \text{ and } y = a^n\text{; Substitute.}$$

$$xy = a^{m+n} \qquad \text{Product rule for exponents}$$

$$\log_a xy = m + n \qquad \text{Write in logarithmic form.}$$

$$\log_a xy = \log_a x + \log_a y \qquad \text{Substitute.}$$

The last statement is the result we wished to prove. The quotient and power properties are proved similarly and are left as exercises.

EXAMPLE 5 Using Properties of Logarithms

Use the properties of logarithms to rewrite each expression. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(a) $\log_6(7 \cdot 9)$ **(b)** $\log_9 \frac{15}{7}$ **(c)** $\log_5 \sqrt{8}$

(d) $\log_a \sqrt[3]{m^2}$ **(e)** $\log_a \frac{mnq}{p^2t^4}$ **(f)** $\log_b \sqrt[n]{\frac{x^3y^5}{z^m}}$

SOLUTION

(a) $\log_6(7 \cdot 9)$

$= \log_6 7 + \log_6 9$ Product property

(b) $\log_9 \frac{15}{7}$

$= \log_9 15 - \log_9 7$ Quotient property

(c) $\log_5 \sqrt{8}$

$= \log_5(8^{1/2})$ $\sqrt{a} = a^{1/2}$

$= \frac{1}{2}\log_5 8$ Power property

(d) $\log_a \sqrt[3]{m^2}$

$= \log_a m^{2/3}$ $\sqrt[n]{a^m} = a^{m/n}$

$= \frac{2}{3}\log_a m$ Power property

(e) $\log_a \frac{mnq}{p^2t^4}$

Use parentheses to avoid errors.

$= \log_a m + \log_a n + \log_a q - (\log_a p^2 + \log_a t^4)$ Product and quotient properties

$= \log_a m + \log_a n + \log_a q - (2 \log_a p + 4 \log_a t)$ Power property

$= \log_a m + \log_a n + \log_a q - 2 \log_a p - 4 \log_a t$ Distributive property

Be careful with signs.

(f) $\log_b \sqrt[n]{\frac{x^3y^5}{z^m}}$

$= \log_b \left(\frac{x^3y^5}{z^m}\right)^{1/n}$ $\sqrt[n]{a} = a^{1/n}$

$= \frac{1}{n}\log_b \frac{x^3y^5}{z^m}$ Power property

$= \frac{1}{n}(\log_b x^3 + \log_b y^5 - \log_b z^m)$ Product and quotient properties

$= \frac{1}{n}(3 \log_b x + 5 \log_b y - m \log_b z)$ Power property

$= \frac{3}{n}\log_b x + \frac{5}{n}\log_b y - \frac{m}{n}\log_b z$ Distributive property

✔ **Now Try Exercises 71, 73, and 77.**

EXAMPLE 6 Using Properties of Logarithms

Write each expression as a single logarithm with coefficient 1. Assume all variables represent positive real numbers, with $a \neq 1$ and $b \neq 1$.

(a) $\log_3(x+2) + \log_3 x - \log_3 2$

(b) $2\log_a m - 3\log_a n$

(c) $\frac{1}{2}\log_b m + \frac{3}{2}\log_b 2n - \log_b m^2 n$

SOLUTION

(a) $\log_3(x+2) + \log_3 x - \log_3 2$

$$= \log_3 \frac{(x+2)x}{2} \quad \text{Product and quotient properties}$$

(b) $2\log_a m - 3\log_a n$

$$= \log_a m^2 - \log_a n^3 \quad \text{Power property}$$

$$= \log_a \frac{m^2}{n^3} \quad \text{Quotient property}$$

(c) $\frac{1}{2}\log_b m + \frac{3}{2}\log_b 2n - \log_b m^2 n$

$$= \log_b m^{1/2} + \log_b (2n)^{3/2} - \log_b m^2 n \quad \text{Power property}$$

Use parentheses around $2n$.

$$= \log_b \frac{m^{1/2}(2n)^{3/2}}{m^2 n} \quad \text{Product and quotient properties}$$

$$= \log_b \frac{2^{3/2} n^{1/2}}{m^{3/2}} \quad \text{Rules for exponents}$$

$$= \log_b \left(\frac{2^3 n}{m^3}\right)^{1/2} \quad \text{Rules for exponents}$$

$$= \log_b \sqrt{\frac{8n}{m^3}} \quad \text{Definition of } a^{1/n}$$

✔ **Now Try Exercises 83, 87, and 89.**

CAUTION ***There is no property of logarithms to rewrite a logarithm of a sum or difference.*** That is why, in **Example 6(a),**

$$\log_3(x+2) \quad \text{cannot be written as} \quad \log_3 x + \log_3 2.$$

The distributive property does not apply here because $\log_3(x+y)$ is one term. The abbreviation "log" is a function name, *not* a factor.

Napier's Rods

The search for ways to make calculations easier has been a long, ongoing process. Machines built by Charles Babbage and Blaise Pascal, a system of "rods" used by John Napier, and slide rules were the forerunners of today's calculators and computers. The invention of logarithms by John Napier in the 16th century was a great breakthrough in the search for easier calculation methods.

Source: IBM Corporate Archives.

EXAMPLE 7 Using Properties of Logarithms with Numerical Values

Given that $\log_{10} 2 \approx 0.3010$, find each logarithm without using a calculator.

(a) $\log_{10} 4$

(b) $\log_{10} 5$

SOLUTION

(a) $\log_{10} 4$

$$= \log_{10} 2^2$$

$$= 2\log_{10} 2$$

$$\approx 2(0.3010)$$

$$\approx 0.6020$$

(b) $\log_{10} 5$

$$= \log_{10} \frac{10}{2}$$

$$= \log_{10} 10 - \log_{10} 2$$

$$\approx 1 - 0.3010$$

$$\approx 0.6990$$

✔ **Now Try Exercises 93 and 95.**