4.2 Exercises

CONCEPT PREVIEW *Fill in the blank(s) to correctly complete each sentence.*

1. If $f(x) = 4^x$, then $f(2) =$ ____ and $f(-2) =$ ____.

2. If $a > 1$, then the graph of $f(x) = a^x$ ________ (rises/falls) from left to right.

3. If $0 < a < 1$, then the graph of $f(x) = a^x$ ________ (rises/falls) from left to right.

4. The domain of $f(x) = 4^x$ is ______ and the range is ______.

5. The graph of $f(x) = 8^x$ passes through the points $(-1,$ ___), $(0,$ ___), and $(1,$ ___).

6. The graph of $f(x) = -\left(\frac{1}{3}\right)^{x+4} - 5$ is that of $f(x) = \left(\frac{1}{3}\right)^x$ reflected across the ____-axis, translated ____ units to the left and ____ units down.

CONCEPT PREVIEW *Solve each equation. Round answers to the nearest hundredth as needed.*

7. $\left(\frac{1}{4}\right)^x = 64$

8. $x^{2/3} = 36$

9. $A = 2000\left(1 + \frac{0.03}{4}\right)^{8(4)}$

10. $10{,}000 = 5000(1 + r)^{25}$

For $f(x) = 3^x$ *and* $g(x) = \left(\frac{1}{4}\right)^x$, *find each of the following. Round answers to the nearest thousandth as needed.* ***See Example 1.***

11. $f(2)$ **12.** $f(3)$ **13.** $f(-2)$ **14.** $f(-3)$

15. $g(2)$ **16.** $g(3)$ **17.** $g(-2)$ **18.** $g(-3)$

19. $f\left(\frac{3}{2}\right)$ **20.** $f\left(-\frac{5}{2}\right)$ **21.** $g\left(\frac{3}{2}\right)$ **22.** $g\left(-\frac{5}{2}\right)$

23. $f(2.34)$ **24.** $f(-1.68)$ **25.** $g(-1.68)$ **26.** $g(2.34)$

Graph each function. ***See Example 2.***

27. $f(x) = 3^x$ **28.** $f(x) = 4^x$ **29.** $f(x) = \left(\frac{1}{3}\right)^x$

30. $f(x) = \left(\frac{1}{4}\right)^x$ **31.** $f(x) = \left(\frac{3}{2}\right)^x$ **32.** $f(x) = \left(\frac{5}{3}\right)^x$

33. $f(x) = \left(\frac{1}{10}\right)^{-x}$ **34.** $f(x) = \left(\frac{1}{6}\right)^{-x}$ **35.** $f(x) = 4^{-x}$

36. $f(x) = 10^{-x}$ **37.** $f(x) = 2^{|x|}$ **38.** $f(x) = 2^{-|x|}$

Graph each function. Give the domain and range. ***See Example 3.***

39. $f(x) = 2^x + 1$ **40.** $f(x) = 2^x - 4$ **41.** $f(x) = 2^{x+1}$

42. $f(x) = 2^{x-4}$ **43.** $f(x) = -2^{x+2}$ **44.** $f(x) = -2^{x-3}$

45. $f(x) = 2^{-x}$ **46.** $f(x) = -2^{-x}$ **47.** $f(x) = 2^{x-1} + 2$

48. $f(x) = 2^{x+3} + 1$ **49.** $f(x) = 2^{x+2} - 4$ **50.** $f(x) = 2^{x-3} - 1$

Graph each function. Give the domain and range. ***See Example 3.***

51. $f(x)=\left(\frac{1}{3}\right)^{x}-2$ **52.** $f(x)=\left(\frac{1}{3}\right)^{x}+4$ **53.** $f(x)=\left(\frac{1}{3}\right)^{x+2}$

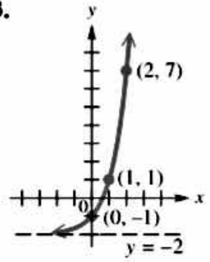
54. $f(x)=\left(\frac{1}{3}\right)^{x-4}$ **55.** $f(x)=\left(\frac{1}{3}\right)^{-x+1}$ **56.** $f(x)=\left(\frac{1}{3}\right)^{-x-2}$

57. $f(x)=\left(\frac{1}{3}\right)^{-x}$ **58.** $f(x)=-\left(\frac{1}{3}\right)^{-x}$ **59.** $f(x)=\left(\frac{1}{3}\right)^{x-2}+2$

60. $f(x)=\left(\frac{1}{3}\right)^{x-1}+3$ **61.** $f(x)=\left(\frac{1}{3}\right)^{x+2}-1$ **62.** $f(x)=\left(\frac{1}{3}\right)^{x+3}-2$

Connecting Graphs with Equations *Write an equation for the graph given. Each represents an exponential function f with base 2 or 3, translated and/or reflected.*

63.



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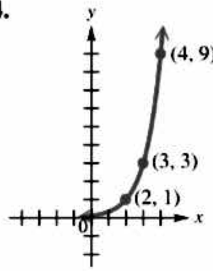

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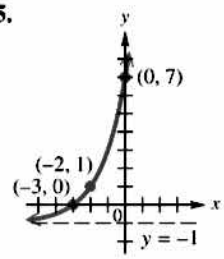

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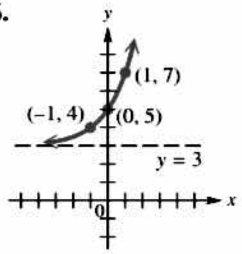

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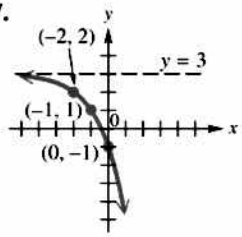

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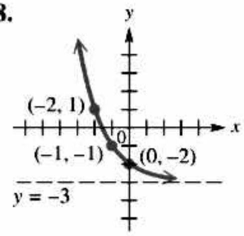

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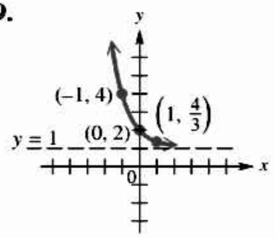
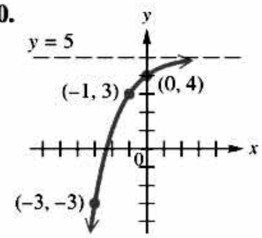


70.



Solve each equation. ***See Examples 4–6.***

71. $4^x=2$ **72.** $125^x=5$ **73.** $\left(\frac{5}{2}\right)^x=\frac{4}{25}$ **74.** $\left(\frac{2}{3}\right)^x=\frac{9}{4}$

75. $2^{3-2x}=8$ **76.** $5^{2+2x}=25$ **77.** $e^{4x-1}=(e^2)^x$ **78.** $e^{3-x}=(e^3)^{-x}$

79. $27^{4x}=9^{x+1}$ **80.** $32^{2x}=16^{x-1}$ **81.** $4^{x-2}=2^{3x+3}$ **82.** $2^{6-3x}=8^{x+1}$

83. $x^{2/3}=4$ **84.** $x^{2/5}=16$ **85.** $x^{5/2}=32$ **86.** $x^{3/2}=27$

87. $x^{-6}=\frac{1}{64}$ **88.** $x^{-4}=\frac{1}{256}$ **89.** $x^{5/3}=-243$ **90.** $x^{7/5}=-128$

91. $\left(\frac{1}{e}\right)^{-x}=\left(\frac{1}{e^2}\right)^{x+1}$ **92.** $e^{x-1}=\left(\frac{1}{e^4}\right)^{x+1}$ **93.** $\left(\sqrt{2}\right)^{x+4}=4^x$

94. $\left(\sqrt[3]{5}\right)^{-x}=\left(\frac{1}{5}\right)^{x+2}$ **95.** $\frac{1}{27}=x^{-3}$ **96.** $\frac{1}{32}=x^{-5}$

Use a graphing calculator to graph each function defined as follows, using the given viewing window. Use the graph to decide which functions are one-to-one. If a function is one-to-one, give the equation of its inverse.

93. $f(x) = 6x^3 + 11x^2 - 6$; $[-3, 2]$ by $[-10, 10]$

94. $f(x) = x^4 - 5x^2$; $[-3, 3]$ by $[-8, 8]$

95. $f(x) = \dfrac{x-5}{x+3}$, $x \neq -3$; $[-8, 8]$ by $[-6, 8]$

96. $f(x) = \dfrac{-x}{x-4}$, $x \neq 4$; $[-1, 8]$ by $[-6, 6]$

Use the following alphabet coding assignment to work each problem. ***See Example 9.***

A	1	**H**	8	**O**	15	**V**	22
B	2	**I**	9	**P**	16	**W**	23
C	3	**J**	10	**Q**	17	**X**	24
D	4	**K**	11	**R**	18	**Y**	25
E	5	**L**	12	**S**	19	**Z**	26
F	6	**M**	13	**T**	20		
G	7	**N**	14	**U**	21		

97. The function $f(x) = 3x - 2$ was used to encode a message as

37 25 19 61 13 34 22 1 55 1 52 52 25 64 13 10.

Find the inverse function and determine the message.

98. The function $f(x) = 2x - 9$ was used to encode a message as

-5 9 5 5 9 27 15 29 -1 21 19 31 -3 27 41.

Find the inverse function and determine the message.

99. Encode the message SEND HELP, using the one-to-one function

$$f(x) = x^3 - 1.$$

Give the inverse function that the decoder will need when the message is received.

100. Encode the message SAILOR BEWARE, using the one-to-one function

$$f(x) = (x + 1)^3.$$

Give the inverse function that the decoder will need when the message is received.

4.2 Exponential Functions

- Exponents and Properties
- Exponential Functions
- Exponential Equations
- Compound Interest
- The Number e and Continuous Compounding
- Exponential Models

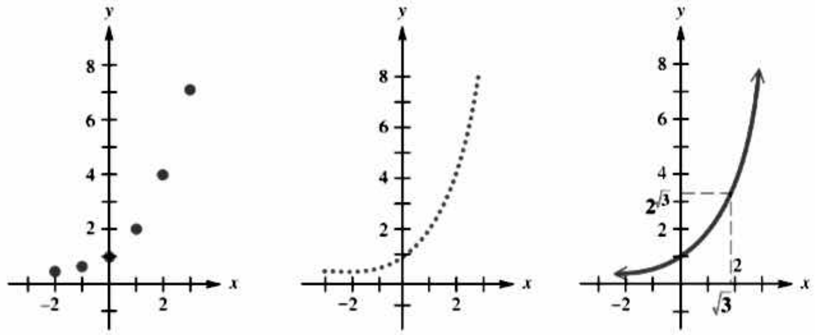
Exponents and Properties Recall the definition of $a^{m/n}$: If a is a real number, m is an integer, n is a positive integer, and $\sqrt[n]{a}$ is a real number, then

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m.$$

For example, $16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8,$

$$27^{-1/3} = \frac{1}{27^{1/3}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3}, \quad \text{and} \quad 64^{-1/2} = \frac{1}{64^{1/2}} = \frac{1}{\sqrt{64}} = \frac{1}{8}.$$

In this section, we extend the definition of a^r to include all *real* (not just rational) values of the exponent r. Consider the graphs of $y = 2^x$ for different domains in **Figure 13.**



$y = 2^x$; integers as domain

$y = 2^x$; selected rational numbers as domain

$y = 2^x$; real numbers as domain

Figure 13

The equations that use just integers or selected rational numbers as domain in **Figure 13** leave holes in the graphs. In order for the graph to be continuous, we must extend the domain to include irrational numbers such as $\sqrt{3}$. We might evaluate $2^{\sqrt{3}}$ by approximating the exponent with the rational numbers 1.7, 1.73, 1.732, and so on. Because these values approach the value of $\sqrt{3}$ more and more closely, it is reasonable that $2^{\sqrt{3}}$ should be approximated more and more closely by the numbers $2^{1.7}$, $2^{1.73}$, $2^{1.732}$, and so on. These expressions can be evaluated using rational exponents as follows.

$$2^{1.7} = 2^{17/10} = \left(\sqrt[10]{2}\right)^{17} \approx 3.249009585$$

Because any irrational number may be approximated more and more closely using rational numbers, we can extend the definition of a^r to include all real number exponents and apply all previous theorems for exponents. In addition to the rules for exponents presented earlier, we use several new properties in this chapter.

Additional Properties of Exponents

For any real number $a > 0$, $a \neq 1$, the following statements hold.

Property	Description
(a) a^x **is a unique real number for all real numbers** x.	$y = a^x$ can be considered a function $f(x) = a^x$ with domain $(-\infty, \infty)$.
(b) $a^b = a^c$ **if and only if** $b = c$.	The function $f(x) = a^x$ is one-to-one.
(c) If $a > 1$ **and** $m < n$, **then** $a^m < a^n$.	*Example:* $2^3 < 2^4 \quad (a > 1)$ Increasing the exponent leads to a *greater* number. The function $f(x) = 2^x$ is an *increasing* function.
(d) If $0 < a < 1$ **and** $m < n$, **then** $a^m > a^n$.	*Example:* $\left(\frac{1}{2}\right)^2 > \left(\frac{1}{2}\right)^3 \quad (0 < a < 1)$ Increasing the exponent leads to a *lesser* number. The function $f(x) = \left(\frac{1}{2}\right)^x$ is a *decreasing* function.

Exponential Functions We now define a function $f(x) = a^x$ whose domain is the set of all real numbers. Notice how the independent variable x appears in the exponent in this function. In earlier chapters, this was not the case.

Exponential Function

If $a > 0$ and $a \neq 1$, then the **exponential function with base a** is

$$f(x) = a^x.$$

NOTE The restrictions on a in the definition of an exponential function are important. Consider the outcome of breaking each restriction.

If $a < 0$, say $a = -2$, and we let $x = \frac{1}{2}$, then $f\left(\frac{1}{2}\right) = (-2)^{1/2} = \sqrt{-2}$, which is *not* a real number.

If $a = 1$, then the function becomes the constant function $f(x) = 1^x = 1$, which is *not* an exponential function.

EXAMPLE 1 Evaluating an Exponential Function

For $f(x) = 2^x$, find each of the following.

(a) $f(-1)$ **(b)** $f(3)$ **(c)** $f\left(\frac{5}{2}\right)$ **(d)** $f(4.92)$

SOLUTION

(a) $f(-1) = 2^{-1} = \frac{1}{2}$ Replace x with -1. **(b)** $f(3) = 2^3 = 8$

(c) $f\left(\frac{5}{2}\right) = 2^{5/2} = (2^5)^{1/2} = 32^{1/2} = \sqrt{32} = \sqrt{16 \cdot 2} = 4\sqrt{2}$

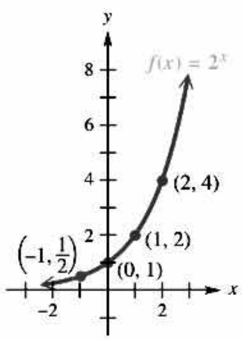
(d) $f(4.92) = 2^{4.92} \approx 30.2738447$ Use a calculator.

✔ **Now Try Exercises 13, 19, and 23.**

We repeat the final graph of $y = 2^x$ (with real numbers as domain) from **Figure 13** and summarize important details of the function $f(x) = 2^x$ here.

- The y-intercept is $(0, 1)$.
- Because $2^x > 0$ for all x and $2^x \to 0$ as $x \to -\infty$, the x-axis is a horizontal asymptote.
- As the graph suggests, the domain of the function is $(-\infty, \infty)$ and the range is $(0, \infty)$.
- The function is increasing on its entire domain. Therefore, it is one-to-one.

These observations lead to the following generalizations about the graphs of exponential functions.



Graph of $f(x) = 2^x$ with domain $(-\infty, \infty)$

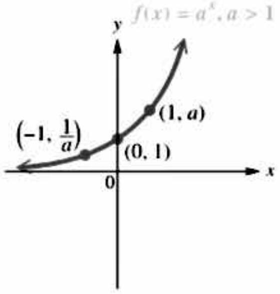
Figure 13 (repeated)

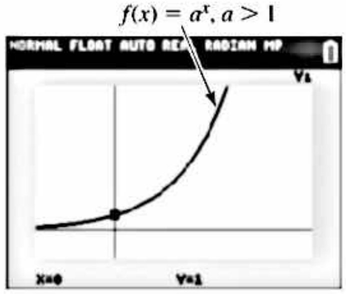
Exponential Function $f(x) = a^x$

Domain: $(-\infty, \infty)$ Range: $(0, \infty)$

For $f(x) = 2^x$:

x	$f(x)$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8





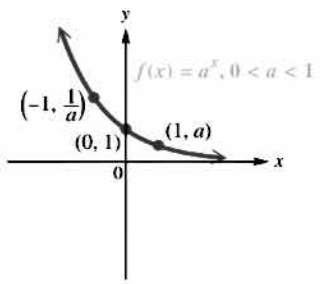
This is the general behavior seen on a calculator graph for **any base *a*, for *a* > 1.**

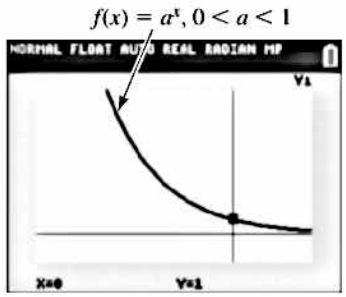
Figure 14

- **$f(x) = a^x$, for $a > 1$,** is increasing and continuous on its entire domain, $(-\infty, \infty)$.
- The x-axis is a horizontal asymptote as $x \to -\infty$.
- The graph passes through the points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.

For $f(x) = \left(\frac{1}{2}\right)^x$:

x	$f(x)$
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$





This is the general behavior seen on a calculator graph for **any base *a*, for 0 < *a* < 1.**

Figure 15

- **$f(x) = a^x$, for $0 < a < 1$,** is decreasing and continuous on its entire domain, $(-\infty, \infty)$.
- The x-axis is a horizontal asymptote as $x \to \infty$.
- The graph passes through the points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.

Recall that the graph of $y = f(-x)$ is the graph of $y = f(x)$ reflected across the y-axis. Thus, we have the following.

$$\text{If } f(x) = 2^x, \quad \text{then} \quad f(-x) = 2^{-x} = 2^{-1 \cdot x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x.$$

This is supported by the graphs in **Figures 14 and 15.**

The graph of $f(x) = 2^x$ is typical of graphs of $f(x) = a^x$ where $a > 1$. For larger values of a, the graphs rise more steeply, but the general shape is similar to the graph in **Figure 14.** When $0 < a < 1$, the graph decreases in a manner similar to the graph of $f(x) = \left(\frac{1}{2}\right)^x$ in **Figure 15.**

In **Figure 16,** the graphs of several typical exponential functions illustrate these facts.

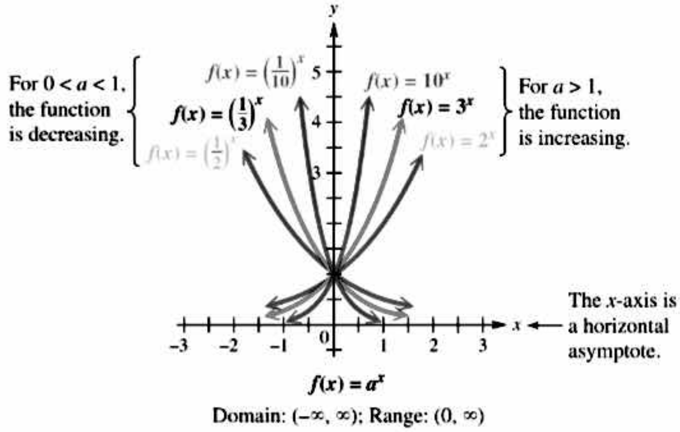


Figure 16

In summary, the graph of a function of the form $f(x) = a^x$ has the following features.

Characteristics of the Graph of $f(x) = a^x$

1. The points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$ are on the graph.
2. If $a > 1$, then f is an increasing function.

   If $0 < a < 1$, then f is a decreasing function.
3. The x-axis is a horizontal asymptote.
4. The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

EXAMPLE 2 Graphing an Exponential Function

Graph $f(x) = \left(\frac{1}{5}\right)^x$. Give the domain and range.

SOLUTION The y-intercept is $(0, 1)$, and the x-axis is a horizontal asymptote. Plot a few ordered pairs, and draw a smooth curve through them as shown in **Figure 17.**

x	$f(x)$
-2	25
-1	5
0	1
1	$\frac{1}{5}$
2	$\frac{1}{25}$

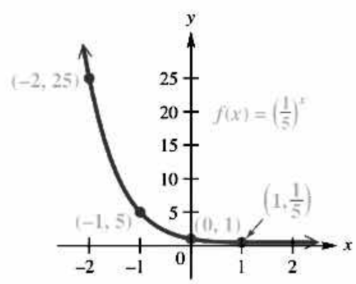


Figure 17

This function has domain $(-\infty, \infty)$, range $(0, \infty)$, and is one-to-one. It is decreasing on its entire domain.

✔ **Now Try Exercise 29.**

EXAMPLE 3 Graphing Reflections and Translations

Graph each function. Show the graph of $y = 2^x$ for comparison. Give the domain and range.

(a) $f(x) = -2^x$ **(b)** $f(x) = 2^{x+3}$ **(c)** $f(x) = 2^{x-2} - 1$

SOLUTION In each graph, we show in particular how the point $(0, 1)$ on the graph of $y = 2^x$ has been translated.

(a) The graph of $f(x) = -2^x$ is that of $f(x) = 2^x$ reflected across the x-axis. See **Figure 18.** The domain is $(-\infty, \infty)$, and the range is $(-\infty, 0)$.

(b) The graph of $f(x) = 2^{x+3}$ is the graph of $f(x) = 2^x$ translated 3 units to the left, as shown in **Figure 19.** The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

(c) The graph of $f(x) = 2^{x-2} - 1$ is that of $f(x) = 2^x$ translated 2 units to the right and 1 unit down. See **Figure 20.** The domain is $(-\infty, \infty)$, and the range is $(-1, \infty)$.

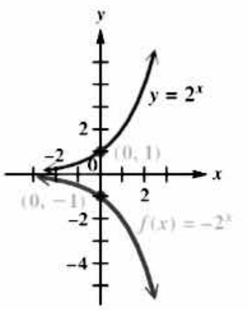


Figure 18

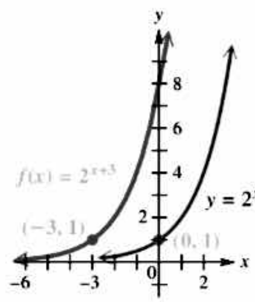


Figure 19

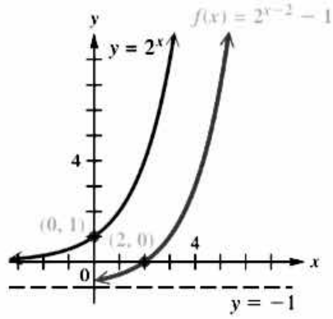


Figure 20

✔ **Now Try Exercises 39, 41, and 47.**

Exponential Equations Because the graph of $f(x) = a^x$ is that of a one-to-one function, to solve $a^{x_1} = a^{x_2}$, we need only show that $x_1 = x_2$. This property is used to solve an **exponential equation,** which is an equation with a variable as exponent.

EXAMPLE 4 Solving an Exponential Equation

Solve $\left(\frac{1}{3}\right)^x = 81$.

SOLUTION ***Write each side of the equation using a common base.***

$$\left(\frac{1}{3}\right)^x = 81$$

$$(3^{-1})^x = 81 \qquad \text{Definition of negative exponent}$$

$$3^{-x} = 81 \qquad (a^m)^n = a^{mn}$$

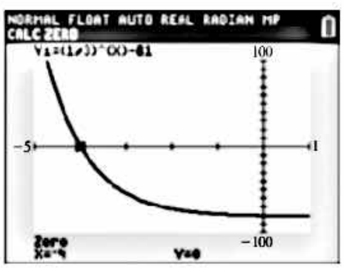
$$3^{-x} = 3^4 \qquad \text{Write 81 as a power of 3.}$$

$$-x = 4 \qquad \text{Set exponents equal (Property (b) given earlier).}$$

$$x = -4 \qquad \text{Multiply by } -1.$$

Check by substituting -4 for x in the original equation. The solution set is $\{-4\}$.

✔ **Now Try Exercise 73.**



The x-intercept of the graph of $y = \left(\frac{1}{3}\right)^x - 81$ can be used to verify the solution in **Example 4.**

EXAMPLE 5 Solving an Exponential Equation

Solve $2^{x+4} = 8^{x-6}$.

SOLUTION Write each side of the equation using a common base.

$$2^{x+4} = 8^{x-6}$$

$$2^{x+4} = (2^3)^{x-6} \quad \text{Write 8 as a power of 2.}$$

$$2^{x+4} = 2^{3x-18} \quad (a^m)^n = a^{mn}$$

$$x + 4 = 3x - 18 \quad \text{Set exponents equal (Property (b)).}$$

$$-2x = -22 \quad \text{Subtract } 3x \text{ and } 4.$$

$$x = 11 \quad \text{Divide by } -2.$$

Check by substituting 11 for x in the original equation. The solution set is $\{11\}$.

✔ **Now Try Exercise 81.**

Later in this chapter, we describe a general method for solving exponential equations where the approach used in **Examples 4 and 5** is not possible. For instance, the above method could not be used to solve an equation like

$$7^x = 12$$

because it is not easy to express both sides as exponential expressions with the same base.

In **Example 6,** we solve an equation that has the variable as the base of an exponential expression.

EXAMPLE 6 Solving an Equation with a Fractional Exponent

Solve $x^{4/3} = 81$.

SOLUTION Notice that the variable is in the base rather than in the exponent.

$$x^{4/3} = 81$$

$$\left(\sqrt[3]{x}\right)^4 = 81 \quad \text{Radical notation for } a^{m/n}$$

$$\sqrt[3]{x} = \pm 3 \quad \text{Take fourth roots on each side. Remember to use } \pm.$$

$$x = \pm 27 \quad \text{Cube each side.}$$

Check *both* solutions in the original equation. Both check, so the solution set is $\{\pm 27\}$.

Alternative Method There may be more than one way to solve an exponential equation, as shown here.

$$x^{4/3} = 81$$

$$(x^{4/3})^3 = 81^3 \quad \text{Cube each side.}$$

$$x^4 = (3^4)^3 \quad \text{Write 81 as } 3^4.$$

$$x^4 = 3^{12} \quad (a^m)^n = a^{mn}$$

$$x = \pm\sqrt[4]{3^{12}} \quad \text{Take fourth roots on each side.}$$

$$x = \pm 3^3 \quad \text{Simplify the radical.}$$

$$x = \pm 27 \quad \text{Apply the exponent.}$$

The same solution set, $\{\pm 27\}$, results.

✔ **Now Try Exercise 83.**

Compound Interest Recall the formula for simple interest, $I = Prt$, where P is principal (amount deposited), r is annual rate of interest expressed as a decimal, and t is time in years that the principal earns interest. Suppose $t = 1$ yr. Then at the end of the year, the amount has grown to the following.

$$P + Pr = P(1 + r) \quad \text{Original principal plus interest}$$

If this balance earns interest at the same interest rate for another year, the balance at the end of *that* year will increase as follows.

$$\begin{aligned} [P(1 + r)] + [P(1 + r)]r &= [P(1 + r)](1 + r) && \text{Factor.} \\ &= P(1 + r)^2 && a \cdot a = a^2 \end{aligned}$$

After the third year, the balance will grow in a similar pattern.

$$\begin{aligned} [P(1 + r)^2] + [P(1 + r)^2]r &= [P(1 + r)^2](1 + r) && \text{Factor.} \\ &= P(1 + r)^3 && a^2 \cdot a = a^3 \end{aligned}$$

Continuing in this way produces a formula for interest compounded annually.

$$A = P(1 + r)^t$$

The general formula for compound interest can be derived in the same way.

Compound Interest

If P dollars are deposited in an account paying an annual rate of interest r compounded (paid) n times per year, then after t years the account will contain A dollars, according to the following formula.

$$A = P\left(1 + \frac{r}{n}\right)^{tn}$$

EXAMPLE 7 Using the Compound Interest Formula

Suppose $1000 is deposited in an account paying 4% interest per year compounded quarterly (four times per year).

(a) Find the amount in the account after 10 yr with no withdrawals.

(b) How much interest is earned over the 10-yr period?

SOLUTION

(a)

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{tn} && \text{Compound interest formula} \\ A &= 1000\left(1 + \frac{0.04}{4}\right)^{10(4)} && \text{Let } P = 1000,\ r = 0.04,\ n = 4, \text{ and } t = 10. \\ A &= 1000(1 + 0.01)^{40} && \text{Simplify.} \\ A &= 1488.86 && \text{Round to the nearest cent.} \end{aligned}$$

Thus, $1488.86 is in the account after 10 yr.

(b) The interest earned for that period is

$$\$1488.86 - \$1000 = \$488.86.$$

✔ **Now Try Exercise 97(a).**