Step 3 Substitute for k to find the relationship among the variables.

$$n = \frac{2\sqrt{T}}{L}$$

Step 4 Now use the second set of values for T and L to find n.

$$n = \frac{2\sqrt{196}}{0.65} \approx 43 \quad \text{Let } T = 196,\ L = 0.65.$$

The number of vibrations per second is approximately 43.

Now Try Exercise 43.

3.6 Exercises

CONCEPT PREVIEW *Fill in the blank(s) to correctly complete each sentence, or answer the question as appropriate.*

1. For $k > 0$, if y varies directly as x, then when x increases, y ______, and when x decreases, y ______.

2. For $k > 0$, if y varies inversely as x, then when x increases, y ______, and when x decreases, y ______.

3. In the equation $y = 6x$, y varies directly as x. When $x = 5$, $y = 30$. What is the value of y when $x = 10$?

4. In the equation $y = \frac{12}{x}$, y varies inversely as x. When $x = 3$, $y = 4$. What is the value of y when $x = 6$?

5. Consider the two ordered pairs (x, y) from **Exercise 3.** Divide the y-value by the x-value. What is the result in each case?

6. Consider the two ordered pairs (x, y) from **Exercise 4.** Multiply the y-value by the x-value. What is the result in each case?

Solve each problem. ***See Examples 1–4.***

7. If y varies directly as x, and $y = 20$ when $x = 4$, find y when $x = -6$.

8. If y varies directly as x, and $y = 9$ when $x = 30$, find y when $x = 40$.

9. If m varies jointly as x and y, and $m = 10$ when $x = 2$ and $y = 14$, find m when $x = 21$ and $y = 8$.

10. If m varies jointly as z and p, and $m = 10$ when $z = 2$ and $p = 7.5$, find m when $z = 6$ and $p = 9$.

11. If y varies inversely as x, and $y = 10$ when $x = 3$, find y when $x = 20$.

12. If y varies inversely as x, and $y = 20$ when $x = \frac{1}{4}$, find y when $x = 15$.

13. Suppose r varies directly as the square of m, and inversely as s. If $r = 12$ when $m = 6$ and $s = 4$, find r when $m = 6$ and $s = 20$.

14. Suppose p varies directly as the square of z, and inversely as r. If $p = \frac{32}{5}$ when $z = 4$ and $r = 10$, find p when $z = 3$ and $r = 32$.

15. Let a be directly proportional to m and n^2, and inversely proportional to y^3. If $a = 9$ when $m = 4$, $n = 9$, and $y = 3$, find a when $m = 6$, $n = 2$, and $y = 5$.

16. Let y vary directly as x, and inversely as m^2 and r^2. If $y = \frac{5}{3}$ when $x = 1$, $m = 2$, and $r = 3$, find y when $x = 3$, $m = 1$, and $r = 8$.

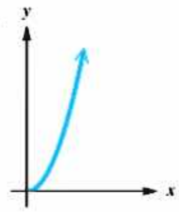
Concept Check *Match each statement with its corresponding graph in choices A–D. In each case, $k > 0$.*

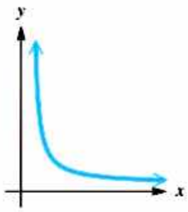
17. y varies directly as x. $(y = kx)$

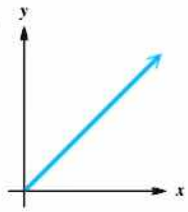
18. y varies inversely as x. $\left(y = \frac{k}{x}\right)$

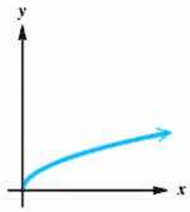
19. y varies directly as the second power of x. $(y = kx^2)$

20. x varies directly as the second power of y. $(x = ky^2)$

A. 

B. 

C. 

D. 

Concept Check *Write each formula as an English phrase using the word* varies *or* proportional.

21. $C = 2\pi r$, where C is the circumference of a circle of radius r

22. $d = \frac{1}{5}s$, where d is the approximate distance (in miles) from a storm, and s is the number of seconds between seeing lightning and hearing thunder

23. $r = \frac{d}{t}$, where r is the speed when traveling d miles in t hours

24. $d = \dfrac{1}{4\pi n r^2}$, where d is the distance a gas atom of radius r travels between collisions, and n is the number of atoms per unit volume

25. $s = kx^3$, where s is the strength of a muscle that has length x

26. $f = \dfrac{mv^2}{r}$, where f is the centripetal force of an object of mass m moving along a circle of radius r at velocity v

Solve each problem. **See Examples 1–4.**

27. ***Circumference of a Circle*** The circumference of a circle varies directly as the radius. A circle with radius 7 in. has circumference 43.96 in. Find the circumference of the circle if the radius changes to 11 in.

28. ***Pressure Exerted by a Liquid*** The pressure exerted by a certain liquid at a given point varies directly as the depth of the point beneath the surface of the liquid. The pressure at 10 ft is 50 pounds per square inch (psi). What is the pressure at 15 ft?

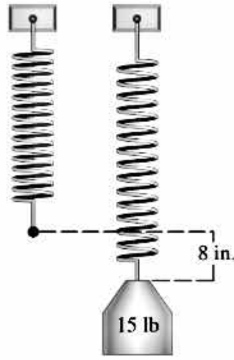
29. ***Resistance of a Wire*** The resistance in ohms of a platinum wire temperature sensor varies directly as the temperature in kelvins (K). If the resistance is 646 ohms at a temperature of 190 K, find the resistance at a temperature of 250 K.

30. ***Weight on the Moon*** The weight of an object on Earth is directly proportional to the weight of that same object on the moon. A 200-lb astronaut would weigh 32 lb on the moon. How much would a 50-lb dog weigh on the moon?

31. ***Distance to the Horizon*** The distance that a person can see to the horizon on a clear day from a point above the surface of Earth varies directly as the square root of the height at that point. If a person 144 m above the surface of Earth can see 18 km to the horizon, how far can a person see to the horizon from a point 64 m above the surface?

32. *Water Emptied by a Pipe* The amount of water emptied by a pipe varies directly as the square of the diameter of the pipe. For a certain constant water flow, a pipe emptying into a canal will allow 200 gal of water to escape in an hour. The diameter of the pipe is 6 in. How much water would a 12-in. pipe empty into the canal in an hour, assuming the same water flow?

33. *Hooke's Law for a Spring* Hooke's law for an elastic spring states that the distance a spring stretches varies directly as the force applied. If a force of 15 lb stretches a certain spring 8 in., how much will a force of 30 lb stretch the spring?



34. *Current in a Circuit* The current in a simple electrical circuit varies inversely as the resistance. If the current is 50 amps when the resistance is 10 ohms, find the current if the resistance is 5 ohms.

35. *Speed of a Pulley* The speed of a pulley varies inversely as its diameter. One kind of pulley, with diameter 3 in., turns at 150 revolutions per minute. Find the speed of a similar pulley with diameter 5 in.

36. *Weight of an Object* The weight of an object varies inversely as the square of its distance from the center of Earth. If an object 8000 mi from the center of Earth weighs 90 lb, find its weight when it is 12,000 mi from the center of Earth.

37. *Current Flow* In electric current flow, it is found that the resistance offered by a fixed length of wire of a given material varies inversely as the square of the diameter of the wire. If a wire 0.01 in. in diameter has a resistance of 0.4 ohm, what is the resistance of a wire of the same length and material with diameter 0.03 in., to the nearest ten-thousandth of an ohm?

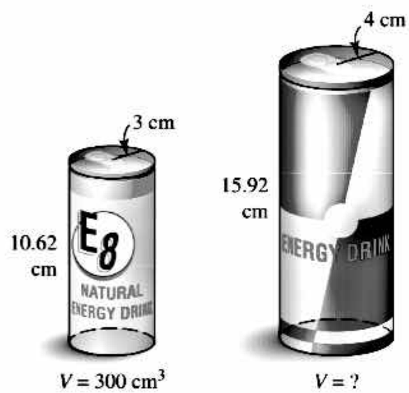
38. *Illumination* The illumination produced by a light source varies inversely as the square of the distance from the source. The illumination of a light source at 5 m is 70 candelas. What is the illumination 12 m from the source?

39. *Simple Interest* Simple interest varies jointly as principal and time. If \$1000 invested for 2 yr earned \$70, find the amount of interest earned by \$5000 for 5 yr.

40. *Volume of a Gas* The volume of a gas varies inversely as the pressure and directly as the temperature in kelvins (K). If a certain gas occupies a volume of 1.3 L at 300 K and a pressure of 18 newtons, find the volume at 340 K and a pressure of 24 newtons.

41. *Force of Wind* The force of the wind blowing on a vertical surface varies jointly as the area of the surface and the square of the velocity. If a wind of 40 mph exerts a force of 50 lb on a surface of $\frac{1}{2}$ ft^2, how much force will a wind of 80 mph place on a surface of 2 ft^2?

42. *Volume of a Cylinder* The volume of a right circular cylinder is jointly proportional to the square of the radius of the circular base and to the height. If the volume is 300 cm^3 when the height is 10.62 cm and the radius is 3 cm, find the volume, to the nearest tenth, of a cylinder with radius 4 cm and height 15.92 cm.



Solving a Variation Problem

Step 1 Write the general relationship among the variables as an equation. Use the constant k.

Step 2 Substitute given values of the variables and find the value of k.

Step 3 Substitute this value of k into the equation from Step 1, obtaining a specific formula.

Step 4 Substitute the remaining values and solve for the required unknown.

EXAMPLE 1 Solving a Direct Variation Problem

The area of a rectangle varies directly as its length. If the area is 50 m² when the length is 10 m, find the area when the length is 25 m. (See **Figure 59.**)

SOLUTION

Step 1 The area varies directly as the length, so

$$\mathcal{A} = kL,$$

where $\mathcal{A}$ represents the area of the rectangle, L is the length, and k is a nonzero constant.

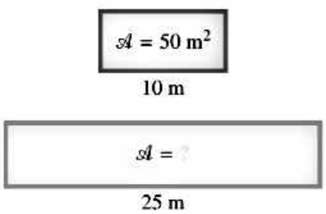


Figure 59

Step 2 Because $\mathcal{A} = 50$ when $L = 10$, we can solve the equation $\mathcal{A} = kL$ for k.

$$50 = 10k \quad \text{Substitute for } \mathcal{A} \text{ and } L.$$

$$k = 5 \quad \text{Divide by 10. Interchange sides.}$$

Step 3 Using this value of k, we can express the relationship between the area and the length as follows.

$$\mathcal{A} = 5L \quad \text{Direct variation equation}$$

Step 4 To find the area when the length is 25, we replace L with 25.

$$\mathcal{A} = 5L$$

$$\mathcal{A} = 5(25) \quad \text{Substitute for } L.$$

$$\mathcal{A} = 125 \quad \text{Multiply.}$$

The area of the rectangle is 125 m² when the length is 25 m.

✔ Now Try Exercise 27.

Sometimes y varies as a power of x. If n is a positive integer greater than or equal to 2, then y is a greater-power polynomial function of x.

Direct Variation as *n*th Power

Let n be a positive real number. Then y **varies directly as the *n*th power** of x, or y is **directly proportional to the *n*th power** of x, if for all x there exists a nonzero real number k such that

$$y = kx^n.$$

For example, the area of a square of side x is given by the formula $\mathcal{A} = x^2$, so the area varies directly as the square of the length of a side. Here $k = 1$.

Inverse Variation Another type of variation is *inverse variation*. With inverse variation, where $k > 0$, as the value of one variable increases, the value of the other decreases. This relationship can be expressed as a rational function.

Inverse Variation as *n*th Power

Let n be a positive real number. Then y **varies inversely as the nth power** of x, or y is **inversely proportional to the nth power** of x, if for all x there exists a nonzero real number k such that

$$y = \frac{k}{x^n}.$$

If $n = 1$, then $y = \frac{k}{x}$, and y **varies inversely** as x.

EXAMPLE 2 Solving an Inverse Variation Problem

In a certain manufacturing process, the cost of producing a single item varies inversely as the square of the number of items produced. If 100 items are produced, each costs \$2. Find the cost per item if 400 items are produced.

SOLUTION

Step 1 Let x represent the number of items produced and y represent the cost per item. Then, for some nonzero constant k, the following holds.

$$y = \frac{k}{x^2} \quad \text{y varies inversely as the square of x.}$$

Step 2

$$2 = \frac{k}{100^2} \quad \text{Substitute; $y = 2$ when $x = 100$.}$$

$$k = 20{,}000 \quad \text{Solve for k.}$$

Step 3 The relationship between x and y is $y = \frac{20{,}000}{x^2}$.

Step 4 When 400 items are produced, the cost per item is found as follows.

$$y = \frac{20{,}000}{x^2} = \frac{20{,}000}{400^2} = 0.125$$

The cost per item is \$0.125, or 12.5 cents.

✔ **Now Try Exercise 37.**

Combined and Joint Variation In **combined variation,** one variable depends on more than one other variable. Specifically, when a variable depends on the *product* of two or more other variables, it is referred to as *joint variation.*

Joint Variation

Let m and n be real numbers. Then y **varies jointly** as the nth power of x and the mth power of z if for all x and z, there exists a nonzero real number k such that

$$y = kx^nz^m.$$

CAUTION Note that *and* in the expression "y varies jointly as x and z" translates as the product $y = kxz$. The word "and" does not indicate addition here.

EXAMPLE 3 Solving a Joint Variation Problem

The area of a triangle varies jointly as the lengths of the base and the height. A triangle with base 10 ft and height 4 ft has area 20 ft^2. Find the area of a triangle with base 3 ft and height 8 ft. (See **Figure 60.**)

SOLUTION

Step 1 Let $\mathcal{A}$ represent the area, b the base, and h the height of the triangle. Then, for some number k,

$$\mathcal{A} = kbh. \quad \mathcal{A} \text{ varies jointly as } b \text{ and } h.$$

Step 2 $\mathcal{A}$ is 20 when b is 10 and h is 4, so substitute and solve for k.

$$20 = k(10)(4) \quad \text{Substitute for } \mathcal{A}, b, \text{ and } h.$$

$$\frac{1}{2} = k \quad \text{Solve for } k.$$

Step 3 The relationship among the variables is the familiar formula for the area of a triangle,

$$\mathcal{A} = \frac{1}{2}bh.$$

Step 4 To find $\mathcal{A}$ when $b = 3$ ft and $h = 8$ ft, substitute into the formula.

$$\mathcal{A} = \frac{1}{2}(3)(8) = 12 \text{ ft}^2$$

✔ **Now Try Exercise 39.**

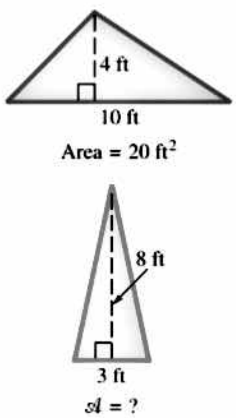


Figure 60

EXAMPLE 4 Solving a Combined Variation Problem

The number of vibrations per second (the pitch) of a steel guitar string varies directly as the square root of the tension and inversely as the length of the string. If the number of vibrations per second is 50 when the tension is 225 newtons and the length is 0.60 m, find the number of vibrations per second when the tension is 196 newtons and the length is 0.65 m.

SOLUTION

Step 1 Let n represent the number of vibrations per second, T represent the tension, and L represent the length of the string. Then, from the information in the problem, write the variation equation.

$$n = \frac{k\sqrt{T}}{L} \quad n \text{ varies directly as the square root of } T \text{ and inversely as } L.$$

Step 2 Substitute the given values for n, T, and L and solve for k.

$$50 = \frac{k\sqrt{225}}{0.60} \quad \text{Let } n = 50,\ T = 225,\ L = 0.60.$$

$$30 = k\sqrt{225} \quad \text{Multiply by 0.60.}$$

$$30 = 15k \quad \sqrt{225} = 15$$

$$k = 2 \quad \text{Divide by 15. Interchange sides.}$$