Concept Check Match the rational function in Column I with the appropriate description in Column II. Choices in Column II can be used only once.

I	II
29. $f(x) = \frac{x+7}{x+1}$	**A.** The x-intercept is $(-3, 0)$.
30. $f(x) = \frac{x+10}{x+2}$	**B.** The y-intercept is $(0, 5)$.
31. $f(x) = \frac{1}{x+4}$	**C.** The horizontal asymptote is $y = 4$.
32. $f(x) = \frac{-3}{x^2}$	**D.** The vertical asymptote is $x = -1$.
33. $f(x) = \frac{x^2-16}{x+4}$	**E.** There is a hole in its graph at $(-4, -8)$.
34. $f(x) = \frac{4x+3}{x-7}$	**F.** The graph has an oblique asymptote.
35. $f(x) = \frac{x^2+3x+4}{x-5}$	**G.** The x-axis is its horizontal asymptote, and the y-axis is not its vertical asymptote.
36. $f(x) = \frac{x+3}{x-6}$	**H.** The x-axis is its horizontal asymptote, and the y-axis is its vertical asymptote.

Give the equations of any vertical, horizontal, or oblique asymptotes for the graph of each rational function. ***See Example 4.***

37. $f(x) = \frac{3}{x-5}$

38. $f(x) = \frac{-6}{x+9}$

39. $f(x) = \frac{4-3x}{2x+1}$

40. $f(x) = \frac{2x+6}{x-4}$

41. $f(x) = \frac{x^2-1}{x+3}$

42. $f(x) = \frac{x^2+4}{x-1}$

43. $f(x) = \frac{x^2-2x-3}{2x^2-x-10}$

44. $f(x) = \frac{3x^2-6x-24}{5x^2-26x+5}$

45. $f(x) = \frac{x^2+1}{x^2+9}$

46. $f(x) = \frac{4x^2+25}{x^2+9}$

Concept Check Work each problem.

47. Let f be the function whose graph is obtained by translating the graph of $y = \frac{1}{x}$ to the right 3 units and up 2 units.

(a) Write an equation for $f(x)$ as a quotient of two polynomials.

(b) Determine the zero(s) of f.

(c) Identify the asymptotes of the graph of $f(x)$.

48. Repeat **Exercise 47** if f is the function whose graph is obtained by translating the graph of $y = -\frac{1}{x^2}$ to the left 3 units and up 1 unit.

49. After the numerator is divided by the denominator,

$$f(x) = \frac{x^5+x^4+x^2+1}{x^4+1} \quad \text{becomes} \quad f(x) = x + 1 + \frac{x^2-x}{x^4+1}.$$

(a) What is the oblique asymptote of the graph of the function?

(b) Where does the graph of the function intersect its asymptote?

(c) As $x \to \infty$, does the graph of the function approach its asymptote from above or below?

The graph of $f(x) = \frac{1}{x}$ is shown in **Figure 43.**

Reciprocal Function $f(x) = \frac{1}{x}$

Domain: $(-\infty, 0) \cup (0, \infty)$ Range: $(-\infty, 0) \cup (0, \infty)$

x	y
-2	$-\frac{1}{2}$
-1	-1
$-\frac{1}{2}$	-2
0	undefined
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$

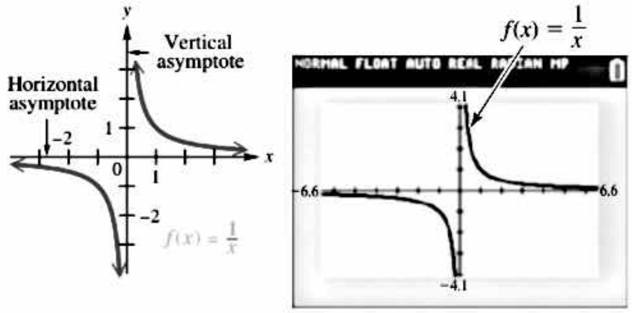


Figure 43

- $f(x) = \frac{1}{x}$ decreases on the open intervals $(-\infty, 0)$ and $(0, \infty)$.
- It is discontinuous at $x = 0$.
- The y-axis is a vertical asymptote, and the x-axis is a horizontal asymptote.
- It is an odd function, and its graph is symmetric with respect to the origin.

The graph of $y = \frac{1}{x}$ can be translated and/or reflected.

EXAMPLE 1 Graphing a Rational Function

Graph $y = -\frac{2}{x}$. Give the domain and range and the largest open intervals of the domain over which the function is increasing or decreasing.

SOLUTION The expression $-\frac{2}{x}$ can be written as $-2\left(\frac{1}{x}\right)$ or $2\left(\frac{1}{-x}\right)$, indicating that the graph may be obtained by stretching the graph of $y = \frac{1}{x}$ vertically by a factor of 2 and reflecting it across either the x-axis or the y-axis. The x- and y-axes remain the horizontal and vertical asymptotes. The domain and range are both still $(-\infty, 0) \cup (0, \infty)$. See **Figure 44.**

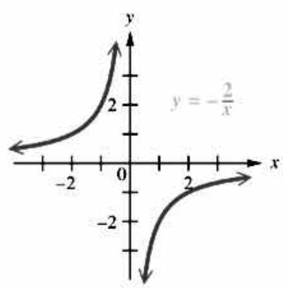
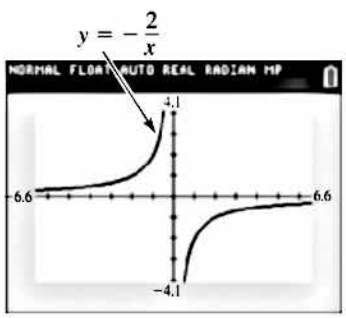


Figure 44



The graph in **Figure 44** is shown here using a **decimal window.** Using a nondecimal window *may* produce an extraneous vertical line that is not part of the graph.

The graph shows that $f(x)$ is increasing on both sides of its vertical asymptote. Thus, it is increasing on $(-\infty, 0)$ and $(0, \infty)$.

✔ **Now Try Exercise 17.**

EXAMPLE 2 Graphing a Rational Function

Graph $f(x) = \frac{2}{x+1}$. Give the domain and range and the largest open intervals of the domain over which the function is increasing or decreasing.

ALGEBRAIC SOLUTION

The expression $\frac{2}{x+1}$ can be written as $2\left(\frac{1}{x+1}\right)$, indicating that the graph may be obtained by shifting the graph of $y = \frac{1}{x}$ to the left 1 unit and stretching it vertically by a factor of 2. See **Figure 45.**

The horizontal shift affects the domain, which is now $(-\infty, -1) \cup (-1, \infty)$. The line $x = -1$ is the vertical asymptote, and the line $y = 0$ (the x-axis) remains the horizontal asymptote. The range is still $(-\infty, 0) \cup (0, \infty)$. The graph shows that $f(x)$ is decreasing on both sides of its vertical asymptote. Thus, it is decreasing on $(-\infty, -1)$ and $(-1, \infty)$.

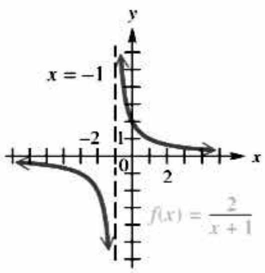


Figure 45

GRAPHING CALCULATOR SOLUTION

When entering this rational function into the function editor of a calculator, make sure that the numerator is 2 and the denominator is the entire expression $(x + 1)$.

The graph of this function has a vertical asymptote at $x = -1$ and a horizontal asymptote at $y = 0$, so it is reasonable to choose a viewing window that contains the locations of both asymptotes as well as enough of the graph to determine its basic characteristics. See **Figure 46.**

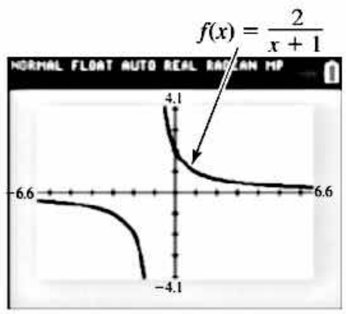


Figure 46

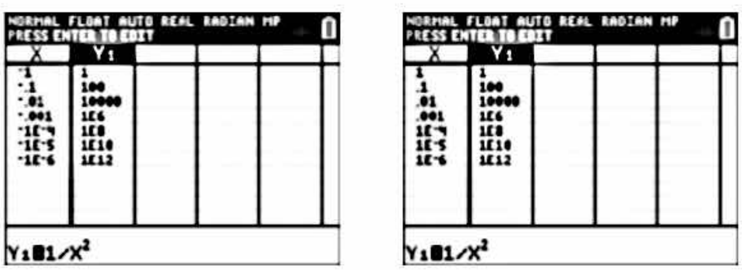
✔ **Now Try Exercise 19.**

The Function $f(x) = \frac{1}{x^2}$

The rational function

$$f(x) = \frac{1}{x^2} \qquad \text{Rational function}$$

also has domain $(-\infty, 0) \cup (0, \infty)$. We can use the table feature of a graphing calculator to examine values of $f(x)$ for some x-values close to 0. See **Figure 47.**

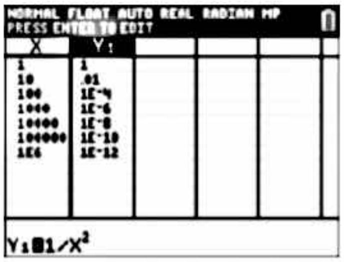


As x approaches 0 from the left, $y_1 = \frac{1}{x^2}$ approaches ∞.

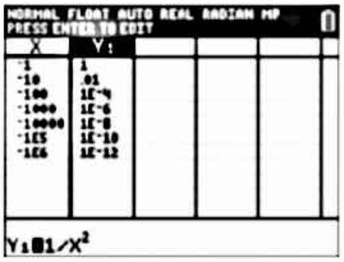
As x approaches 0 from the right, $y_1 = \frac{1}{x^2}$ approaches ∞.

Figure 47

The tables suggest that $f(x)$ increases without bound as x gets closer and closer to 0. Notice that as x approaches 0 from *either* side, function values are all positive and there is symmetry with respect to the y-axis. Thus, $f(x) \to \infty$ as $x \to 0$. The y-axis $(x = 0)$ is the vertical asymptote.



As x approaches ∞, $y_1 = \frac{1}{x^2}$ approaches 0 through positive values.



As x approaches $-\infty$, $y_1 = \frac{1}{x^2}$ approaches 0 through positive values.

Figure 48

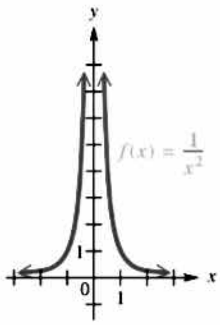
As $|x|$ increases without bound, $f(x)$ approaches 0, as suggested by the tables in **Figure 48.** Again, function values are all positive. The x-axis is the horizontal asymptote of the graph.

The graph of $f(x) = \frac{1}{x^2}$ is shown in **Figure 49.**

Rational Function $f(x) = \frac{1}{x^2}$

Domain: $(-\infty, 0) \cup (0, \infty)$ Range: $(0, \infty)$

x	y
± 3	$\frac{1}{9}$
± 2	$\frac{1}{4}$
± 1	1
$\pm \frac{1}{2}$	4
$\pm \frac{1}{4}$	16
0	undefined



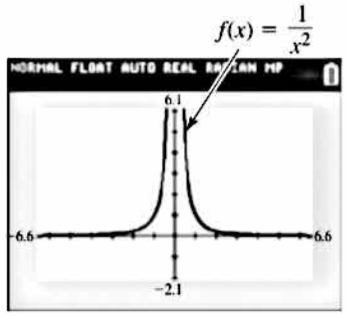


Figure 49

- $f(x) = \frac{1}{x^2}$ increases on the open interval $(-\infty, 0)$ and decreases on the open interval $(0, \infty)$.
- It is discontinuous at $x = 0$.
- The y-axis is a vertical asymptote, and the x-axis is a horizontal asymptote.
- It is an even function, and its graph is symmetric with respect to the y-axis.

EXAMPLE 3 Graphing a Rational Function

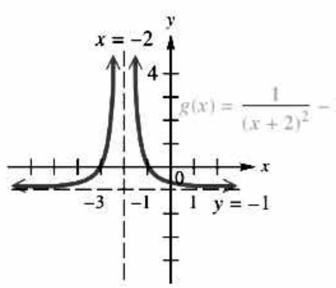
Graph $g(x) = \frac{1}{(x+2)^2} - 1$. Give the domain and range and the largest open intervals of the domain over which the function is increasing or decreasing.

SOLUTION The function $g(x) = \frac{1}{(x+2)^2} - 1$ is equivalent to

$$g(x) = f(x+2) - 1, \quad \text{where} \quad f(x) = \frac{1}{x^2}.$$

This indicates that the graph will be shifted 2 units to the left and 1 unit down. The horizontal shift affects the domain, now $(-\infty, -2) \cup (-2, \infty)$. The vertical shift affects the range, now $(-1, \infty)$.

The vertical asymptote has equation $x = -2$, and the horizontal asymptote has equation $y = -1$. A traditional graph is shown in **Figure 50,** with a calculator graph in **Figure 51.** Both graphs show that this function is increasing on $(-\infty, -2)$ and decreasing on $(-2, \infty)$.



This is the graph of $y = \frac{1}{x^2}$ shifted 2 units to the left and 1 unit down.

Figure 50

✔ **Now Try Exercise 27.**

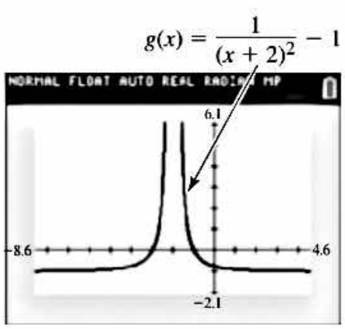


Figure 51

LOOKING AHEAD TO CALCULUS

The rational function

$$f(x) = \frac{2}{x+1}$$

in **Example 2** has a vertical asymptote at $x = -1$. In calculus, the behavior of the graph of this function for values close to -1 is described using **one-sided limits.** As x approaches -1 from the *left*, the function values decrease without bound. This is written

$$\lim_{x \to -1^-} f(x) = -\infty.$$

As x approaches -1 from the *right*, the function values increase without bound. This is written

$$\lim_{x \to -1^+} f(x) = \infty.$$

Asymptotes The preceding examples suggest the following definitions of vertical and horizontal asymptotes.

Asymptotes

Let $p(x)$ and $q(x)$ define polynomial functions. Consider the rational function $f(x) = \frac{p(x)}{q(x)}$, written in lowest terms, and real numbers a and b.

1. If $|f(x)| \to \infty$ as $x \to a$, then the line $x = a$ is a **vertical asymptote.**

2. If $f(x) \to b$ as $|x| \to \infty$, then the line $y = b$ is a **horizontal asymptote.**

Locating asymptotes is important when graphing rational functions.

- We find *vertical asymptotes* by determining the values of x that make the denominator equal to 0.
- We find *horizontal asymptotes* (and, in some cases, *oblique asymptotes*), by considering what happens to $f(x)$ as $|x| \to \infty$. These asymptotes determine the end behavior of the graph.

Determining Asymptotes

To find the asymptotes of a rational function defined by a rational expression in *lowest terms*, use the following procedures.

1. Vertical Asymptotes
Find any vertical asymptotes by setting the denominator equal to 0 and solving for x. If a is a zero of the denominator, then **the line $x = a$ is a vertical asymptote.**

2. Other Asymptotes
Determine any other asymptotes by considering three possibilities:

(a) If the numerator has lesser degree than the denominator, then there is a **horizontal asymptote $y = 0$** (the x-axis).

(b) If the numerator and denominator have the same degree, and the function is of the form

$$f(x) = \frac{a_n x^n + \cdots + a_0}{b_n x^n + \cdots + b_0}, \quad \text{where} \quad a_n, b_n \neq 0,$$

then the **horizontal asymptote has equation $y = \frac{a_n}{b_n}$.**

(c) If the numerator is of degree exactly one more than the denominator, then there will be an **oblique (slanted) asymptote.** To find it, divide the numerator by the denominator and disregard the remainder. Set the rest of the quotient equal to y to obtain the equation of the asymptote.

NOTE The graph of a rational function may have more than one vertical asymptote, or it may have none at all.

The graph cannot intersect any vertical asymptote. There can be at most one other (nonvertical) asymptote, and the graph may intersect that asymptote. **(See Example 7.)**

EXAMPLE 4 Finding Asymptotes of Rational Functions

Give the equations of any vertical, horizontal, or oblique asymptotes for the graph of each rational function.

(a) $f(x) = \dfrac{x+1}{(2x-1)(x+3)}$ **(b)** $f(x) = \dfrac{2x+1}{x-3}$ **(c)** $f(x) = \dfrac{x^2+1}{x-2}$

SOLUTION

(a) To find the vertical asymptotes, set the denominator equal to 0 and solve.

$$(2x-1)(x+3) = 0$$

$$2x - 1 = 0 \quad \text{or} \quad x + 3 = 0 \qquad \text{Zero-factor property}$$

$$x = \frac{1}{2} \quad \text{or} \quad x = -3 \qquad \text{Solve each equation.}$$

The equations of the vertical asymptotes are $x = \frac{1}{2}$ and $x = -3$.

To find the equation of the horizontal asymptote, begin by multiplying the factors in the denominator.

$$f(x) = \frac{x+1}{(2x-1)(x+3)} = \frac{x+1}{2x^2+5x-3}$$

Now divide each term in the numerator and denominator by x^2. We choose the exponent 2 because it is the greatest power of x in the entire expression.

$$f(x) = \frac{\frac{x}{x^2} + \frac{1}{x^2}}{\frac{2x^2}{x^2} + \frac{5x}{x^2} - \frac{3}{x^2}} = \frac{\frac{1}{x} + \frac{1}{x^2}}{2 + \frac{5}{x} - \frac{3}{x^2}}$$

Stop here. Leave the expression in complex form.

As $|x|$ increases without bound, the quotients $\frac{1}{x}$, $\frac{1}{x^2}$, $\frac{5}{x}$, and $\frac{3}{x^2}$ all approach 0, and the value of $f(x)$ approaches

$$\frac{0+0}{2+0-0} = 0. \qquad \frac{0}{2} = 0$$

The line $y = 0$ (that is, the x-axis) is therefore the horizontal asymptote. This supports procedure 2(a) of determining asymptotes on the previous page.

(b) Set the denominator $x - 3$ equal to 0 to find that the vertical asymptote has equation $x = 3$. To find the horizontal asymptote, divide each term in the rational expression by x since the greatest power of x in the expression is 1.

$$f(x) = \frac{2x+1}{x-3} = \frac{\frac{2x}{x} + \frac{1}{x}}{\frac{x}{x} - \frac{3}{x}} = \frac{2 + \frac{1}{x}}{1 - \frac{3}{x}}$$

As $|x|$ increases without bound, the quotients $\frac{1}{x}$ and $\frac{3}{x}$ both approach 0, and the value of $f(x)$ approaches

$$\frac{2+0}{1-0} = 2.$$

The line $y = 2$ is the horizontal asymptote. This supports procedure 2(b) of determining asymptotes on the previous page.

(c) Setting the denominator $x - 2$ equal to 0 shows that the vertical asymptote has equation $x = 2$. If we divide by the greatest power of x as before (x^2 in this case), we see that there is no horizontal asymptote because

$$f(x) = \frac{x^2 + 1}{x - 2} = \frac{\frac{x^2}{x^2} + \frac{1}{x^2}}{\frac{x}{x^2} - \frac{2}{x^2}} = \frac{1 + \frac{1}{x^2}}{\frac{1}{x} - \frac{2}{x^2}}$$

does not approach any real number as $|x| \to \infty$, due to the fact that $\frac{1 + 0}{0 - 0} = \frac{1}{0}$ is undefined. This happens whenever the degree of the numerator is greater than the degree of the denominator.

$$\begin{array}{r|rrr} 2 & 1 & 0 & 1 \\ & & 2 & 4 \\ \hline & 1 & 2 & 5 \end{array}$$

Setup for synthetic division

In such cases, divide the denominator into the numerator to write the expression in another form. We use synthetic division, as shown in the margin. The result enables us to write the function as follows.

$$f(x) = x + 2 + \frac{5}{x - 2}$$

For very large values of $|x|$, $\frac{5}{x-2}$ is close to 0, and the graph approaches the line $y = x + 2$. This line is an **oblique asymptote** (slanted, neither vertical nor horizontal) for the graph of the function. This supports procedure 2(c) of determining asymptotes.

✔ **Now Try Exercises 37, 39, and 41.**

Graphing Techniques A comprehensive graph of a rational function will show the following characteristics.

- all x- and y-intercepts
- all asymptotes: vertical, horizontal, and/or oblique
- the point at which the graph intersects its nonvertical asymptote (if there is any such point)
- the behavior of the function on each domain interval determined by the vertical asymptotes and x-intercepts

Graphing a Rational Function

Let $f(x) = \frac{p(x)}{q(x)}$ define a function where $p(x)$ and $q(x)$ are polynomial functions and the rational expression is written in lowest terms. To sketch its graph, follow these steps.

Step 1 Find any vertical asymptotes.

Step 2 Find any horizontal or oblique asymptotes.

Step 3 If $q(0) \neq 0$, plot the y-intercept by evaluating $f(0)$.

Step 4 Plot the x-intercepts, if any, by solving $f(x) = 0$. (These will correspond to the zeros of the numerator, $p(x)$.)

Step 5 Determine whether the graph will intersect its nonvertical asymptote $y = b$ or $y = mx + b$ by solving $f(x) = b$ or $f(x) = mx + b$.

Step 6 Plot selected points, as necessary. Choose an x-value in each domain interval determined by the vertical asymptotes and x-intercepts.

Step 7 Complete the sketch.